Proceedings of the International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2009 30 June, 1–3 July 2009.

An algorithm for Bang-Bang control of fixed-head hydroplants

L. Bayón[1], J.M. Grau[1], M.M. Ruiz[1] and P.M. Suárez[1]

[1] *Department of Mathematics, University of Oviedo. Spain*

emails: bayon@uniovi.es, grau@uniovi.es, mruiz@uniovi.es, pedrosr@uniovi.es

Abstract

This paper deals with the optimal control problem that arise when a hydraulic system with fixed-head hydroplants is considered. In the frame of a deregulated electricity market the resulting Hamiltonian for such systems is linear in the control variable and results in an optimal singular/bang-bang control policy. To avoid difficulties associated with the computation of optimal singular/bang-bang controls, an efficient and simple optimization algorithm is proposed. The computational technique is illustrated on one example.

Key words: Optimal Control, Singular/Bang-Bang Problems, Hydroplants
MSC 2000: 49J30

1 Introduction

The computation of optimal singular/bang-bang controls is of particular interest to researchers because of the difficulty in obtaining the optimal solution. Several engineering control problems, such as chemical reactor start-up or hydrothermal optimization problems, are known to have optimal singular/bang-bang problems. This paper deals with the optimal control (OC) problem that arises when addressing the new short-term problems that are faced by a generation company in a deregulated electricity market. Our model of the spot market explicitly represents the price of electricity as a known exogenous variable and we consider a system with fixed-head hydroplants. These plants, with a large capacity reservoir, are the most important in the electricity market. The resulting Hamiltonian for such systems, H, is linear in the control variable, u, and results in an optimal singular/bang-bang control policy.

In general, the application of Pontryagin's Maximum Principle (PMP) is not well suited for computing singular control problems as it fails to yield a unique value for the control. Different methods for determining optimal controls with a possibly singular part have already been developed. In [1], the switching function is used as a constraint and the resulting problem is solved as a differential algebraic equation (DAE) problem. Other popular approaches are the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm and other decay methods taken from nonlinear optimization [2], Maurer's Method [3], which converts the problem into a two point boundary value problem (TPBVP) that can be solved by the multi shooting method, and the Method by Fraser-Andrews [4], which determines the structure using orthogonal functions.

Another method that has been used by a number of researcher is the ε-method by Bell and Jacobson. This method [5] involves solving the singular/bang-bang optimal control problem as the limit of a series of nonsingular problems. The problem then becomes well defined so that methods based on PMP can be used. However, some existing numerical methods for handling such problems behave poorly. One alternative, Iterative Dynamic Programming (IDP) [6], has been used and applications to different types of problems have been reported. Recently [7] Maurer et al. presented a numerical scheme for computing optimal bang-bang controls on problems with a larger number of switchings. They assume that every component of the optimal control is bang-bang and that there are only finitely many switching times. Such a bang-bang control can be computed by solving an induced optimization problem, using the durations of the bang-bang arcs as optimization variables instead of the switching times.

In this paper we propose a simple and efficient optimization algorithm that avoids all the difficulties that the above methods present. The algorithm has been specifically developed for a hydraulic problem and we remark that no approach has yet been developed to find the bang-bang solution to our hydro-problem. The paper is organized as follows. In Section 2, we present the mathematical environment of our work: the singular optimal control problem with control appearing linearly. In Section 3, we present the mathematical models of our fixed-head hydroplant. In Section 4 we formulate our optimization problem: *profit maximization of fixed-head hydroplants in a deregulated electricity market* and prove that singular controls can be excluded. In Section 5 we describe the algorithm that provides the structure of bang-bang arcs. The results of the application of the method to a numerical example are presented in Section 6. Finally, the main conclusions of our research are summarized in Section 7.

2 General statement of the singular OC problem

Let us assume a system given by: a state $x(t) \in \mathbb{R}^n$ at time $t \in [0,T]$, a control $u(t) \in U(t) \subset \mathbb{R}^m$, where u is piecewise continuous and $U(t)$ is compact for every $t \in [0,T]$, a state equation $x'(t) = f(t, x(t), u(t))$ almost everywhere, an initial condition $x(0) = x_0$ and final condition $x(T) \in Z \neq \varnothing$, where $[0,T]$ is fixed, and the scalar functions g and L with a suitable domain. The following problem is called the Bolza problem **(P)**:

Find an admissible pair (x, u) *on* $[0,T]$ *such that the functional*

$$J(u) = g(x(T)) + \int_0^T L(t, x(t), u(t))dt$$

becomes maximal. If $g \equiv 0$, we call **(P)** a Lagrange problem, while **(P)** is called a Mayer problem if $L \equiv 0$. We define the Hamiltonian:

$$H(u, x, \lambda, t) := L(t, x, u) + \lambda^T f(t, x, u)$$

where $\lambda^T \in \mathbb{R}^n$ holds. We assume that every f_i $(i = 1, ..., n)$ is continuous in (t, x, u), that the derivatives $\frac{\partial}{\partial t} f_i$ and $\nabla_x f_i$ exist and are continuous in (t, x, u) for every i. Furthermore we assume that $g \in C^1$ and that **(P)** has a solution (x^*, u^*) with $Z = \mathbb{R}^n$. The following theorem is often very useful in solving Bolza problems:

Theorem 1 (PMP). *Under the above hypothesis, there thus exists an absolutely continuous function* $\lambda : [0, T] \to \mathbb{R}^n$ *with the following properties*:

a) $x' = H_\lambda$ *and* $\lambda' = -H_x$ *along* (x^*, u^*)
b) $H(u^*(t), x^*(t), \lambda(t), t) = \max\{H(u, x^*(t), \lambda(t), t) \mid u \subset U(t)\}$ *for every* $t \in [0, T]$
c) $\lambda \neq 0$ *on* $[0, T]$
d) $\lambda(T)dx(T) - dg = 0$ (*transversality condition*)

In the usual case, the optimality condition

$$H(u^*(t), x^*(t), \lambda(t), t) = \max\{H(u, x^*(t), \lambda(t), t) \mid u \subset U(t)\} \tag{1}$$

is used to solve for the extremal control in terms of the state and adjoint (x, λ). Normally, the optimality condition is imposed as $H_u = 0$ and this system of equations is solved for the control vector $u(t)$. Additionally, since u^* is to maximize H, the Hessian must be positive definite: $H_{uu} < 0$ (Legendre-Clebsch (LC) condition).

We now consider the case of scalar control appearing linearly (H_{uu} is singular):

$$\begin{aligned} &\max \int_0^T [f_1(t, x) + u f_2(t, x)] dt \\ &x' = g_1(t, x) + u g_2(t, x); \quad x(0) = x_0 \\ &u_{\min} \leq u(t) \leq u_{\max} \end{aligned} \tag{2}$$

The variational Hamiltonian is linear in u and can be written as

$$H(u, x, \lambda, t) := f_1(t, x) + \lambda g_1(t, x) + [f_2(t, x) + \lambda g_2(t, x)]u$$

The optimality condition (maximize H w.r.t. u) leads to:

$$u^*(t) = \begin{cases} u_{\max} & \text{if} \quad f_2(t, x) + \lambda g_2(t, x) > 0 \\ u_{\text{sing}} & \text{if} \quad f_2(t, x) + \lambda g_2(t, x) = 0 \\ u_{\min} & \text{if} \quad f_2(t, x) + \lambda g_2(t, x) < 0 \end{cases}$$

and u^* is undetermined if $\Phi(x, \lambda) \equiv H_u = f_2(t, x) + \lambda g_2(t, x) = 0$. The function Φ is called the switching function. If $\Phi(x^*(t), \lambda(t)) = 0$ only at isolated time points, then the optimal control switches between its upper and lower bounds, which is said to be a bang-bang type control (i.e. the problem is not singular). The times when the OC switches from $u_{\max}$ to $u_{\min}$ or vice-versa are called switch times.

If $\Phi(x^*(t), \lambda(t)) = 0$ for every t in some subinterval $[t', t'']$ of $[0, T]$, then the original problem is called a singular control problem and the corresponding trajectory for $[t', t'']$, a singular arc. The case when Φ vanishes over an interval is more troublesome, because the optimality condition is vacuous, since $H(u, x^*(t), \lambda(t), t)$ is independent of u. In the singular case, PMP yields no information on the extremal (or stationary) control.

In order to find the control on a singular arc, we use the fact that H_u remains zero along the whole arc. Hence, all the time derivatives are zero along such an arc. By successive differentiation of the switching function, one of the time derivatives may contain the control u, in which case u can be obtained as a function of x and λ. The next result (see [8]) is important.

Proposition 1. *If H_u is successively differentiated with respect to time, then u cannot first appear in an odd-order derivative.*

As u first appears in an even-order derivative, we denote this by $\frac{d^{2q}(H_u)}{dt^{2q}}$ and q is the order of the singular arc. An important theorem (see [8]) is the necessary condition for a singular arc to be optimal: the Generalized Legendre-Clebsch (GLC) condition.

Theorem 2 (GLC Condition). *If $x^*(t), u^*(t)$ are optimal on a singular arc, then, for scalar u,*

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}(H_u)}{dt^{2q}}\right] \leq 0$$

3 Hydroplant performance models

Conventional hydroplants are classified as run-of-river plants and storage plants. Run-of-river plants have little storage capacity and use water as it becomes available. The water not utilized is spilled. Storage plants are associated with reservoirs that have significant storage capacity. During periods of low power requirements, water can be stored and then released when demand is high.

A basic physically-based relationship between the active power generated P (in MW) by a hydroplant and the rate of water discharge, q (in m^3/s), and the effective head, h (in m), is given by

$$P = 0.0085\ q.h.\eta(q, h)$$

where η is a function of q and h. A variety of models have been proposed in the literature [9], [10] due to the diversity of plant types and their characteristics (see Table I). The appropriate choice of mathematical models for representing the physical system is a crucial aspect when addressing any optimization problem. In this paper we consider the approximation presented by El-Hawary [9] to be the most appropriate on account of its precision and flexibility.

Table I. Hydroplant models.

Glimn-Kirchmayer	$q = K\psi(h)\phi(P)$	$\psi(h) = \alpha h^2 + \beta h + \gamma$ $\phi(P) = xP^2 + yP + z$
Hildebrand	$q = \sum_{i=0}^{L}\sum_{j=0}^{K} C_{ij}P^i h^j$	(L and K are usually taken to be 2)
Hamilton-Lamont	$q = \psi(h)\frac{\phi(P)}{h}$	$\psi(h) = \alpha h^2 + \beta h + \gamma$ $\phi(P) = xP^3 + yP + z$
Arvanitidis-Rosing	$P = qh[\beta - e^{-\alpha S}]$	(S is reservoir storage)

El-Hawary's Model. In this model the output power P (MW) is given by

$$P = \frac{qh}{G}$$

where q is the rate of water discharge (m^3/h), h is the effective water head (m), and G is the efficiency (m^4/h.MW). For the sake of simplicity, we assume the rate of water spillage and the penstock head losses to be negligible. Thus, we have $h = y - y_T$, where y is the forebay elevation and y_T the tailrace elevation. In most cases, a typical linear variation between y_T and the discharge, q, and a typical linear elevation-storage curve may be assumed:

$$y(t) = [y_0 + B_y s(t)] - [y_{T_0} + B_T q(t)]$$

where $s(t)$ is the reservoir storage. The reservoir's dynamics may be suitably described by the equation

$$\frac{ds(t)}{dt} = i(t) - q(t) \rightarrow s(t) = S_0 + i \cdot t - Q(t)$$

being i the natural inflow (that is, in general, assumed constant), $Q(t)$ being the volume discharged up to the instant t by the plant and S_0 the initial volume. So, we have that

$$\begin{aligned} &P(t, Q(t), q(t)) := A(t) \cdot q(t) - B \cdot Q(t) \cdot q(t) - C \cdot q^2(t) \\ &A(t) = \frac{(y_0 - y_{T_0}) + B_y(S_0 + i \cdot t)}{G};\ B = \frac{B_y}{G};\ C = \frac{B_T}{G} \end{aligned} \tag{3}$$

This is a *variable-head* model and the hydroplant's hydraulic generation, P, is a function of $Q(t)$ and $q(t)$. According to El-Hawary's model, power output is a function of discharge and the head. For a large capacity reservoir, it is practical to assume that the effective head is constant over the optimization interval. Here the *fixed-head* hydroplant model is defined and P is represented by the linear equation:

$$P(t) = \frac{(y_0 - y_{T_0}) + B_y(S_0)}{G} q(t) = Aq(t) \tag{4}$$

4 Structure of the solution of the optimization problem

For convenience of formulation, in this section we introduce this new notation: $q(t) \equiv z'(t); Q(t) \equiv z(t)$. Let $P(t, z(t), z'(t))$ be the function of the hydroplant's hydraulic generation, where $z(t)$ is the volume that is discharged up to the instant t by the plant, and $z'(t)$ the rate of water discharge of the plant at the instant t. If we assume that b is the volume of water that must be discharged during the entire optimization interval $[0, T]$, the following boundary conditions will have to be fulfilled:

$$z(0) = 0, z(T) = b$$

Throughout the paper we assume that $P(t, z, z') : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$; that is, we shall only admit non-negative volumes, $z(t)$, and rates of water discharge, $z'(t)$ (pumped-storage plants will be not considered). Besides the previous statement, we consider $z'(t)$ to be bounded by technical constraints

$$q_{\min} \leq z'(t) \leq q_{\max}, \quad \forall t \in [0, T]$$

No transmission losses will be considered in our study; this is a crucial aspect when addressing the optimization problem from a centralized viewpoint. From the perspective of a generation company and within the framework of the new electricity market, said losses are not relevant, as power generators currently receive payment for all the energy they generate in power plant bars.

This study constitutes a modification of previous papers by the authors [11], [12], where a variable-head model (3) was considered. When the term $-C \cdot q^2(t)$ is considered, the Hamiltonian is not linear in u and the control is not singular/bang-bang. The Hamiltonian is also not linear in u when transmission losses are considered using the classic Kirchmayer model: $P_L = BP(t)^2$; P_L being the losses.

In our problem, the objective function is given by revenue during the optimization interval $[0, T]$

$$F(z) = \int_0^T L(t, z(t), z'(t))dt = \int_0^T \pi(t)P(t, z(t), z'(t))dt$$

Revenue is obtained by multiplying the hydraulic production of the hydroplant by the clearing price $\pi(t)$ at each hour t. Our model of the spot market explicitly represents the price of electricity as a known exogenous variable. Here the fixed-head hydroplant model (4) for P is used. In keeping with the previous statement, our objective functional in continuous time form is

$$\max_z F(z) = \max_z \int_0^T \pi(t) \; A \; z'(t)dt$$

$$\text{on} \quad \Omega = \left\{ z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b; \; q_{\min} \leq z'(t) \leq q_{\max}, \forall t \in [0, T] \right\}$$

where $\widehat{C}^1$ is the set of piecewise C^1 functions. A standard Lagrange type OC problem of type (2) can be mathematically formulated as follows:

$$\begin{aligned}
&\max \int_0^T A\pi(t)u dt = \max \int_0^T f(t)u dt \\
&z' = u; \quad z(0) = 0, z(T) = b \\
&u_{\min} \le u(t) \le u_{\max}
\end{aligned}$$

With the aim of obtaining a solution numerically, we first attempt to determine the structure of the solution; that is, the sequence of the bang-bang and the singular parts. We define the Hamiltonian:

$$H(u, x, \lambda, t) := f(t)u + \lambda u = [f(t) + \lambda]u$$

The switching function is $\Phi(x, \lambda) \equiv H_u = f(t) + \lambda$. The optimality condition (1) leads to:

$$u^*(t) = \begin{cases} u_{\max} & \text{if} \quad f(t) + \lambda > 0 \\ u_{\text{sing}} & \text{if} \quad f(t) + \lambda = 0 \\ u_{\min} & \text{if} \quad f(t) + \lambda < 0 \end{cases} \tag{5}$$

On the other hand, the co-state equation of PMP allows us to obtain:

$$\lambda' = -H_z = 0 \rightarrow \lambda = \lambda_0 \text{ (cte)} \tag{6}$$

To find the control on a singular arc, we use the fact that H_u remains zero along the whole arc. By differentiation of the switching function, we obtain

$$\begin{aligned}
&\frac{d}{dt}H_u = \frac{d}{dt}[f(t) + \lambda] = f'(t) = A\pi'(t) = 0 \\
&\qquad \cdots \\
&\frac{d^n}{dt^n}H_u = A\pi^{(n)}(t) = 0
\end{aligned}$$

We can see that in the successive derivatives of H_u w.r.t. t, doesn't appear the control u. We have only derivatives of the clearing price $\pi(t)$. The presence of singular arcs in the solution are thus ruled out.

5 Algorithm for the Bang-Bang solution

Having ruled out the presence of singular arcs, we now determine the bang-bang segments and the boundary on which the solution is situated. To obtain the optimal solution, we apply (5) and (6), obtaining

$$u^*(t) = \begin{cases} u_{\max} & \text{if} \quad f(t) > -\lambda_0 \\ u_{\min} & \text{if} \quad f(t) < -\lambda_0 \end{cases} \tag{7}$$

The algorithm that leads to the optimal solution (7) comprises the following steps:

(i) First, $f(t)$ must be interpolated to obtain a continuous function. Note that in real electricity markets, the clearing price $\pi(t)$ is only known at each hour ($t = 1, 2, ..., 24$). In this paper we have used linear interpolation with good results.

(ii) Second, we have to determine the switch times: $t_1, t_2, ...$ These instants are calculated solving the equation

$$f(t) = -\lambda$$

(iii) Third, the optimal value λ_0 must be determined in order for:

$$z_\lambda(T) = \sum_{i=1}^{N_s} \delta_i \cdot q_{\max} + (T - \sum_{i=1}^{N_s} \delta_i) \cdot q_{\min} = b$$

δ_i being the duration of the i-th bang-bang segment in the upper bound $u_{\max}$, N_s the number of such segments, and $z_\lambda(T)$ the final volume obtained for each λ. Figure 1 illustrates the proposed method.

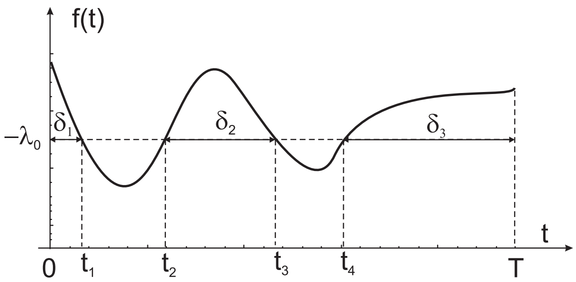

Figure 1. Illustration of the method.

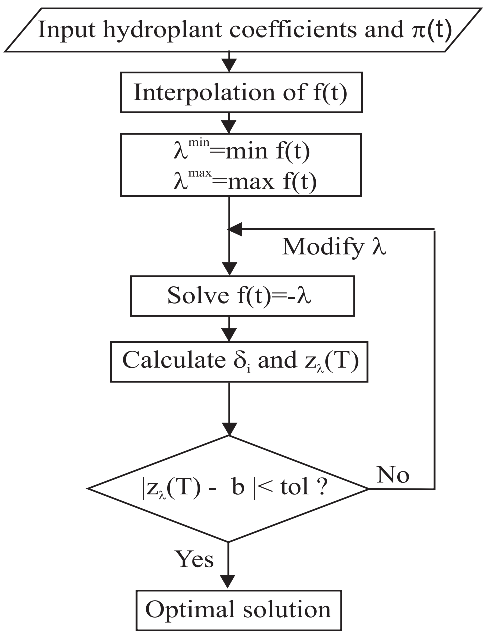

Figure 2. Computational flow of the proposed algorithm.

(iv) To calculate an approximate value of λ_0, we propose an iterative method (like, for example, bisection or the secant method) using this condition

$$Error = |z_\lambda(T) - b| < tol$$

to finalize the algorithm. As we shall see in the next section, the secant method has provided satisfactory results using these initial values:

$$\lambda^{\min} = \min f(t); \quad \lambda^{\max} = \max f(t)$$

6 Example

A program was written using the Mathematica package to apply the results obtained in this paper to an example of a hydraulic system made up of one fixed-head hydroplant. The hydroplant data are summarized in Table II.

Table II: Hydroplant coefficients.

$G(m^4/\mathrm{h}.\mathrm{MW})$	$b(m^3)$	$S_0(m^3)$	$y_0(m)$	$y_{T_0}(m)$	$B_y(m^{-2})$
319840	$45\ 10^6$	$2.395\ 10^8$	6.18166	5	$2.89386\ 10^{-8}$

We shall also consider the technical constraints: $q_{\min} = 0$; $q_{\max} = 3.94258\ 10^6\ (m^3/\mathrm{h})$, which correspond, respectively, to $P_{\min} = 0$; $P_{\max} = 100$ (MW). With these coefficients, the hydraulic model is:

$$P(t) = 0.0000253641\ q(t)$$

In this paper, we focus on the problem that a generation company faces when preparing its offers for the day-ahead market. Thus, the classic optimization interval of $T = 24$ h. was considered. The clearing price $\pi(t)$ $(euros/h \cdot \mathrm{MW})$ corresponding to one day was taken from the Spanish electricity market [13]. The known values of $\pi(t) : t = 1, 2, ..., 24$ were linearly interpolated (see Figure 3).

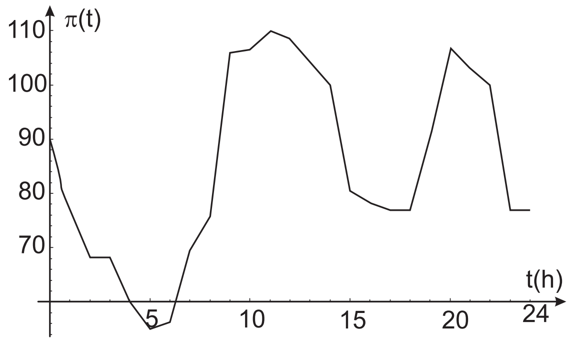

Figure 3. Clearing price $\pi(t)$.

The solution may be constructed in a simple way by taking into account the above algorithm. In this example we have:

$$f(t) = 0.0000253641\ \pi(t)$$
$$\lambda^{\min} = \min f(t) = 0.00139528$$
$$\lambda^{\max} = \max f(t) = 0.00279005$$

The secant method was used to calculate the approximate value of λ for which

$$Error = |z_\lambda(T) - b| < tol$$

with $tol = 50$ (m^3). The optimal value obtained is $\lambda_0 = 0.002107617885177008$ and the switch times are:

$$t_1 = 0.528346,\ t_2 = 8.24259,\ t_3 = 14.8669,\ t_4 = 18.4717,\ t_5 = 22.7328$$

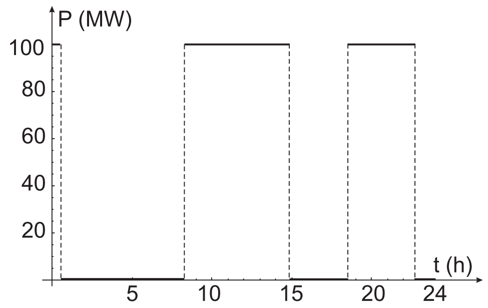

Figure 4. Optimal hydro-power $P(t)$.

Figure 4 presents the optimal hydro-power, P. The profits from the optimal solution are 130908 *euros*.

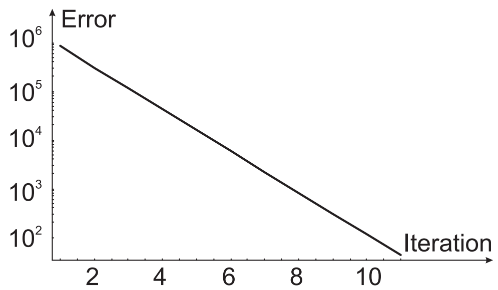

Figure 5. Convergence of the algorithm.

The algorithm runs very quickly (see Figure 5). In the example, 11 iterations were needed and the CPU time required by the program was 0.188 sec on a personal computer (Pentium IV/$2GHz$).

7 Conclusions and future perspectives

This paper presents a novel method for developing the optimal control problem faced by a fixed-head hydroplant in a deregulated electricity market (no transmission losses). We have proved that singular controls do not exist and, for the first time, a simple and very efficient algorithm has been specifically developed for the resulting bang-bang problem. In spite of its hydraulic origin, it should be noted that our method may be applied to other problems with the same characteristics. As far as future perspectives are concerned, it would be very interesting to apply this method when the system is made up of variable-head hydroplants of the type: $P(t) = f(Q(t)) \cdot q(t)$ or $P(t) = f(t, Q(t)) \cdot q(t)$.

References

[1] R.M. Lewis, *Defintions of Order and Junction Conditions in Singular Optimal Control Problems*, SIAM Journal Control and Optimization **18** (1980) 21-32

[2] H.T. Jongen, K. Meer and E. Triesch, *Optimization Theory*, Springer, 2004.

[3] H. Maurer, *Numerical Solution of Singular Control Problems Using Multiple Shooting Techniques*, JOTA **18** (1976) 235-257.

[4] G. Fraser-Andrews, *Numerical Methods for Singular Optimal Control*, JCTA **61** (1989) 377-401.

[5] D.J. Bell and D.H. Jacobson, *Singular Optimal Control Problems*, Mathematics in Science and Engineering **117**, Academic Press, London 1975.

[6] R. Luus, *On the Application of Iterative Dynamic Programming to Singular Optimal Control Problems*, IEEE Trans. on Automatic control **37** (1992) 1802-1806.

[7] H. Maurer, C. Buskens, J.H.R. Kim and C. Y. Kaya, *Optimization methods for the verification of second order sufficient conditions for bang–bang controls*, Optim. Control Appl. Meth. **26** (2005) 129-156.

[8] H.J. Kelley, R.E. Kopp and H.G. Moyer, *Singular Extremals, in Topics in Optimization*, Academic Press, 1967, 63-101.

[9] M.E. El-Hawary and G.S. Christensen, *Optimal Economic Operation of Electric Power Systems*, Academic Press, New York, 1979.

[10] D.P. Kothari and J. S. Dhillon, *Power System Optimization*, PHI Learning Pvt. Ltd., 2004.

[11] L. Bayon, J. M. Grau, M. M. Ruiz and P. M. Suarez, *An environmentally constrained economic dispatch: CFBC boilers in the day-ahead market*, International Journal of Computer Mathematics **85(3)** (2008) 345-358.

[12] L. Bayon, J. M. Grau, M. M. Ruiz and P. M. Suarez, *An optimization problem in deregulated electricity markets solved with the nonsmooth maximum principle*, International Journal of Computer Mathematics **86(2)** (2009) 237-249.

[13] Compañía Operadora del Mercado Español de Electricidad, S.A. http://www.omel.es.