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A quasi-linear algorithm for calculating the infimal convolution of convex quadratic functions

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Abstract

In this paper we present an algorithm of quasi-linear complexity for exactly calculating the infimal convolution of convex quadratic functions. The algorithm exactly and simultaneously solves a separable uniparametric family of quadratic programming problems resulting from varying the equality constraint.

Key words: Algorithm Complexity, Infimal Convolution, Quadratic Programming

MSC 2000: 90C20, 90C25, 90C60

1 Introduction

The infimal convolution operator is well known within the context of convex analysis. For a survey of the properties of this operation, see [1].

Definition 1. *Let $F, G : \mathbb{R} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ be two functions. We denote as the Infimal Convolution of F and G the operation defined as follows:*

$$(F \bigodot G)(x) := \inf_{y \in \mathbb{R}} \{F(x) + G(y - x)\}$$

Furthermore, if $A = \{1, ..., N\}$, we have that

$$(\bigodot_{i \in A} F_i)(\xi) = \inf_{\sum\limits_{i \in A} x_i = \xi} \sum_{i \in A} F_i(x_i)$$

When the functions are considered to be constrained a certain domain, $Dom(F_i) = [m_i, M_i]$, the above definition continues to be valid by redefining $F_i(x) = +\infty$ if $x \notin Dom(F_i)$. In this case, the equivalent definition may be expressed as follows:

$$\Psi^A(\xi) := (\bigodot_{i \in A} F_i)(\xi) = \min_{\substack{\sum\limits_{i \in A} x_i = \xi \\ m_i \le x_i \le M_i}} \sum_{i \in A} F_i(x_i)$$

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This operator has a microeconomic interpretation that is quite precise: if Ψ^A is the infimal convolution of several production cost functions, $\Psi^A(\xi)$ represents the joint cost for a production level ξ when the latter is shared out among the different units in the most efficient way possible.

In this paper we present an algorithm that leads to the determination of the analytic optimal solution of a particular quadratic programming (QP) problem: Let $\{F_i\}_{i\in A}$ be a family of strictly convex quadratic functions:

$$F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2$$

We denote by $\left\{\Pr^A(\xi)\right\}_{\xi\in\mathbb{R}}$ the family of separable convex QP Problems:

$$\begin{array}{ll} \text{minimize:} & \displaystyle\sum_{i\in A} F_i(x_i) \\ \text{subject to:} & \displaystyle\sum_{i\in A} x_i = \xi; \quad m_i \le x_i \le M_i, \forall i \in A \end{array}$$

QP problems have long been a subject of interest in the scientific community. Thousands of papers [2] have been published that deal with applying QP algorithms to diverse problems. Within this extremely wide-ranging field of research, some authors have sought the analytic solution for certain particular cases of QP problems with additional simplifications. For example, [3] presents an algorithm of linear complexity for the case of a single equality constraint (fixed ξ), only including constraints of the type $x_i \ge 0$. The present paper generalizes prior studies, presenting an algorithm of quasi-linear complexity, $O(N\,\log(N))$, for the family of problems $\left\{\Pr^A(\xi)\right\}_{\xi\in\mathbb{R}}$. This supposes a substantial improvement to a previous paper by the authors [4] in which an algorithm was presented that, as we shall show in this paper, is one of quadratic computational complexity, $O(N^2)$.

2 Algorithm

In this section, we first present the necessary definitions to build our algorithm.

Definition 2. *Let us consider in the set $A \times \{m, M\}$ the binary relation $\preccurlyeq$ defined as follows:*

$$\begin{aligned} (i,m) \preccurlyeq (j,m) &\iff F_i'(m_i) < F_j'(m_j) \text{ or } (F_i'(m_i) = F_j'(m_j) \text{ and } i \le j) \\ (i,m) \preccurlyeq (j,M) &\iff F_i'(m_i) < F_j'(M_j) \text{ or } (F_i'(m_i) = F_j'(M_j) \text{ and } i \le j) \\ (i,M) \preccurlyeq (j,m) &\iff F_i'(M_i) < F_j'(m_j) \text{ or } (F_i'(m_i) = F_j'(M_j) \text{ and } i \le j) \\ (i,M) \preccurlyeq (j,M) &\iff F_i'(M_i) < F_j'(M_j) \text{ or } (F_i'(m_i) = F_j'(M_j) \text{ and } i \le j) \end{aligned}$$

Definition 3. *We denote by g the isomorphism*

$$g(n) := (g_1(n), g_2(n)), \quad g : (\{1, 2, \cdots, 2N\}, \le) \longrightarrow (A \times \{m, M\}, \preccurlyeq)$$

which at each natural number $n \in \{1, 2, \cdots, 2N\}$ corresponds to the n-th element of $A \times \{m, M\}$ following the order established by $\preccurlyeq$.

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We now present the optimization algorithm that leads to the determination of the optimal solution. The algorithm generates all the feasible states of activity/inactivity of the constraints on the solution of the problem. We build a sequence $(\Omega_n, \Theta_n, \Xi_n)$ starting with the triad $(A, \varnothing, \varnothing)$, which represents the fact that all the constraints on minimum are active and ending with the triad $(\varnothing, \varnothing, A)$, which represents the fact that all the constraints on maximum are active. Each step of the process consists in decreasing the number of active constraints on minimum by one unit or increasing the number of active constraints on maximum by one unit, following the order established by the relation $\preccurlyeq$. Let us consider the following recurrent sequence, $X_n := (\Omega_n, \Theta_n, \Xi_n)$, $n = 0, \ldots, 2N$:

$$
\begin{array}{lll}
\Omega_0 = A & \Theta_0 = \varnothing & \Xi_0 = \varnothing \\
\text{If } g_2(n) = M : \Omega_n = \Omega_{n-1} & \Theta_n = \Theta_{n-1} - \{g_1(n)\} & \Xi_n = \Xi_{n-1} \cup \{g_1(n)\} \\
\text{If } g_2(n) = m : \Omega_n = \Omega_{n-1} - \{g_1(n)\} & \Theta_n = \Theta_{n-1} \cup \{g_1(n)\} & \Xi_n = \Xi_{n-1}
\end{array}
$$

We prove the following proposition.

Proposition 1. *The function Ψ^A (infimal convolution) is piecewise quadratic, continuous and, if $\Theta_n \neq \varnothing$, $\forall n, 0 < n < 2N$, then it also belongs to class C^1. Specifically, if $\phi_n \leq \xi \leq \phi_{n+1}$, with*

$$
\phi_{n+1} = \phi_n + \frac{1}{2}\left[s_{n+1} - s_n\right]\frac{1}{\widehat{\gamma}_n}; \quad s_n = \begin{cases} F'_{g_1(n)}(m_{g_1(n)}) & \text{if} \quad g_2(n) = m \\ F'_{g_1(n)}(M_{g_1(n)}) & \text{if} \quad g_2(n) = M \end{cases}
$$

we have

$$
\Psi^A(\xi) = \widehat{\alpha}_n + \widehat{\beta}_n(\xi - \mu_n) + \widehat{\gamma}_n(\xi - \mu_n)^2
$$

where

$$
\mu_n = \begin{cases} \mu_{n-1} - m_{g_1(n)} & \text{if} \quad g_2(n) = m \\ \mu_{n-1} + M_{g_1(n)} & \text{if} \quad g_2(n) = M \end{cases}
$$

$$
\widehat{\alpha}_n = \begin{cases} \widehat{\alpha}_{n-1} + \alpha_{g_1(n)} - \frac{\left(\widehat{\beta}_{n-1} + \beta_{g_1(n)}\right)^2}{4\left(\widehat{\gamma}_{n-1} + \gamma_{g_1(n)}\right)} - F_{g_1(n)}\left(m_{g_1(n)}\right) & \text{if} \quad g_2(n) = m \\ \widehat{\alpha}_{n-1} - \alpha_{g_1(n)} - \frac{\left(\widehat{\beta}_{n-1} - \beta_{g_1(n)}\right)^2}{4\left(\widehat{\gamma}_{n-1} - \gamma_{g_1(n)}\right)} - F_{g_1(n)}\left(M_{g_1(n)}\right) & \text{if} \quad g_2(n) = M \end{cases}
$$

$$
\widehat{\beta}_n = \begin{cases} \frac{1}{\widehat{\gamma}_{n-1} + \gamma_{g_1(n)}}\left[\widehat{\beta}_{n-1} \cdot \gamma_{g_1(n)} + \beta_{g_1(n)} \cdot \widehat{\gamma}_n\right] & \text{if} \quad g_2(n) = m \\ \frac{1}{\widehat{\gamma}_{n-1} - \gamma_{g_1(n)}}\left[-\widehat{\beta}_{n-1} \cdot \gamma_{g_1(n)} + \beta_{g_1(n)} \cdot \widehat{\gamma}_n\right] & \text{if} \quad g_2(n) = M \end{cases}
$$

$$
\widehat{\gamma}_n = \begin{cases} \frac{\widehat{\gamma}_{n-1} \cdot \gamma_{g_1(n)}}{\widehat{\gamma}_{n-1} + \gamma_{g_1(n)}} & \text{if} \quad g_2(n) = m \\ -\frac{\widehat{\gamma}_{n-1} \cdot \gamma_{g_1(n)}}{\widehat{\gamma}_{n-1} - \gamma_{g_1(n)}} & \text{if} \quad g_2(n) = M \end{cases}
$$

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3 Computational Complexity of the Algorithm

In this section we analyze the complexity of the previous algorithm and compare it to the one presented in [4]. Given the family of strictly convex quadratic functions $F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2$ with $i = 1.....N$ and $Dom(F_i) = [m_i, M_i]$, each one of these shall be represented by the list $\{m_i, M_i, \alpha_i, \beta_i, \gamma_i\}$. The union of all these functions constitutes the input for the algorithm:

$$\{\{m_1, M_1, \alpha_1, \beta_1, \gamma_1\}, \{m_2, M_2, \alpha_2, \beta_2, \gamma_2\}, \cdots, \{m_N, M_N, \alpha_N, \beta_N, \gamma_N\}\}$$

The output shall represents the infimal convolution, which we symbolize as:

$$\{\{\phi_1, \phi_2, \widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1\}, \cdots, \{\phi_n, \phi_{n+1}, \widehat{\alpha}_n, \widehat{\beta}_n, \widehat{\gamma}_n\}, \cdots, \{\phi_{2N-1}, \phi_{2N}, \widehat{\alpha}_{2N}, \widehat{\beta}_{2N}, \widehat{\gamma}_{2N}\}\}$$

The algorithm presents the following phases:

- A) Construction of the set $A \times \{m, M\}$.
- B) Ordering of the set $A \times \{m, M\}$ following the ordering relation $\preccurlyeq$.
- C) Construction of the recurrent sequence $X_n := (\Omega_n, \Theta_n, \Xi_n)$, $n = 0, \ldots, 2N$.
- D) Construction of the sequence $s_n,\ n = 0, \ldots, 2N$.
- E) Construction of the sequences $\widehat{\alpha}_n, \widehat{\beta}_n, \widehat{\gamma}_n,\ n = 1, \ldots, 2N-1$.
- F) Construction of the sequences $\phi_n,\ n = 1, \ldots, 2N$.

We prove that:

Proposition 2. *The complexity of the aforementioned algorithm is quasi-linear:* $O(N \log(N))$, *and the complexity of the algorithm* [4] *is quadratic:* $O(N^2)$.

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