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The Profit Maximization Problem in Economies of Scale

L. Bayón[1], J.A. Otero[1], M.M. Ruiz[1], P.M. Suárez[1] and C. Tasis[1]

[1] *Department of Mathematics, University of Oviedo. Spain*

emails: `bayon@uniovi.es`, `jaurelio@uniovi.es`, `mruiz@uniovi.es`, `pedrosr@uniovi.es`, `ctasis@uniovi.es`

Abstract

In this paper we present a generalization of the classic Firm's Profit Maximization Problem, using the linear model for the production function, considering a decreasing price $w_i(x_i) = b_i - c_i x_i$ and maximum constraints for the inputs or, equivalently, considering inputs that are in turn outputs in a economies of scale with quadratic concave cost functions. We formulate the problem by previously calculating the analytical minimum cost function in the quadratic concave case. This minimum cost function will be calculated for each production level via the infimal convolution of quadratic concave functions whose result is a piecewise quadratic concave function.

Key words: Concave Programming, Economies of Scale, Infimal Convolution, Piecewise Concave Functions, Algorithm Complexity
MSC 2000: 90C20, 90C26, 90C60

1 Introduction

Problems involving economies of scale (in production and sales) can often be formulated as concave quadratic programming problems [1], [2]. Consider a case in which n products are being produced, with x_i being the number of units of product i and w_i being the unit production cost of product i. As the number of units produced increases, the unit cost usually decreases. This can often be correlated by a linear functional

$$w_i(x_i) = b_i - c_i x_i \tag{1}$$

where $c_i > 0$. Thus, given constraints on production demands and the availability of each product and using the classic linear production function model [3], [4], the Firm's Cost Minimization (FCM) problem [5], [6] can be written as:

$$\begin{aligned} & C(y) = \min_{\mathbf{x}} \sum_{i=1}^{n} x_i w_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_i x_i = y;\ a_i \neq 0,\ i = 1, ..., n \\ & 0 \leq x_i \leq U_i;\ i = 1, ..., n \end{aligned} \tag{2}$$

where y is the output. This is a concave minimization problem. As well as representing a situation in which the inputs are acquired with a discount proportional to the amount, the affine function model for the prices (1) can also be interpreted as dealing with inputs that are in turn outputs of a prior production process of economies of scale with a quadratic cost: $x_i b_i - c_i x_i^2$. On the other hand, the linear production function is presented in a natural way when the output is the result of the sum of the inputs ($a_i = 1$) or, in general, a specific fraction of each of these.

Similarly, when the Firm's Profit Maximization (FPM) Problem is considered:

$$\begin{aligned} &\pi(p, \mathbf{w}) = \max_{\mathbf{x},y}(py - \sum_{i=1}^{n} x_i w_i(x_i)) \\ \text{s.t.} \quad &\sum_{i=1}^{n} a_i x_i = y;\ a_i \neq 0,\ i = 1, ..., n \\ &0 \leq x_i \leq U_i;\ i = 1, ..., n \end{aligned} \tag{3}$$

the economies of scale dictate that the profit per unit rises linearly with the number of units produced. In this case, therefore, the problem becomes one of maximization of a convex functional.

To solve the FPM problem, we formulate the problem by previously calculating the analytical minimum cost function $C(y)$ and then maximizing over the output quantity:

$$\pi(p, \mathbf{w}) = \max_{y}(py - c(\mathbf{w}, y)) = \max_{y}(py - C(y))$$

Concave programming [7], [8] constitutes one of the most fundamental and most widely studied problem classes in deterministic nonconvex optimization. Concave programming has a remarkably broad range of direct and indirect applications. Many of the mathematical properties of concave programming are even identical to the properties of linear programming. The goal in concave programming, or the concave minimization problem (CPM):

$$\begin{aligned} &glob \min \quad f(x) \\ &\text{s.t.} \qquad\quad x \in D \end{aligned}$$

is to find the global minimum value that f achieves over D, where D is a nonempty, closed convex set in $\mathbb{R}^n$ and f is a real-valued, concave function defined on some open convex set A in $\mathbb{R}^n$ that contains D. The application of standard algorithms designed for solving constrained convex programming problems will generally fail to solve CMP. Accordingly, in this paper we shall present an algorithm specifically designed for the problem we are going to solve that, as we shall see, presents very advantageous features.

To develop the algorithm which determines the optimal production level, we shall make use of the infimal convolution operator. This operator is well known within the context of convex analysis [9], [10] and [11]. However, convexity is only one desirable property so as to be able to resort to differential techniques to tackle its calculation and its use should definitely not be restricted to this context alone.

Definition 1. *Let $F, G : \mathbb{R} \longrightarrow \bar{\mathbb{R}}$ be two functions of $\mathbb{R}$ in $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$. We denote as the Infimal Convolution of F and G the operation defined below:*

$$(F \bigodot G)(x) := \inf_{y \in \mathbb{R}} \{F(x) + G(y - x)\}$$

It is well known that $(\mathcal{F}(\mathbb{R},\bar{\mathbb{R}}),\bigodot)$ is a commutative semigroup. Furthermore, for every finite set $E\subset\mathbb{N}$, it is verified that

$$(\bigodot_{i\in E}F_i)(K)=\inf_{\sum_{i\in E} x_i=K}\sum_{i\in E}F_i(x_i)$$

When the functions are considered constrained to a certain domain, $Dom(F_i)=[m_i,M_i]$, the above definition continues to be perfectly valid redefining $F_i(x)=+\infty$ if $x\notin Dom(F_i)$. In this case, the definition may be expressed as follows:

$$(F_1\bigodot F_2)(\xi):=\min_{\substack{x_1+x_2=\xi\\ m_i\le x_i\le M_i}}(F_1(x_1)+F_2(x_2))=\min_{\substack{m_1\le x\le M_1\\ m_2\le \xi-x\le M_2}}((F_1(x)+F_2(\xi-x))$$

2 Statement of the Generalized Problem

We first consider the FCM problem (2). Using (1) and making these changes in the variables

$$\begin{aligned} a_ix_i &= z_i;\quad a_iU_i=M_i\\ \frac{b_i}{a_i} &= \beta_i;\quad \frac{c_i}{a_i^2}=\gamma_i \end{aligned}$$

the FCM problem may be re-written as follows:

$$\begin{aligned} & C(y)=\min_{\mathbf{z}}\sum_{i=1}^{n}\beta_iz_i-\gamma_iz_i^2\\ \text{s.t.}\quad & \sum_{i=1}^{n}z_i=y\\ & 0\le z_i\le M_i;\ i=1,...,n \end{aligned}\tag{4}$$

which makes $C(y)$ the infimal convolution of the quadratic functions:

$$F_i(z_i):=\beta_iz_i-\gamma_iz_i^2$$

respectively constrained to the domains $[0,M_i]$; i.e.

$$C=F_1\bigodot F_2\bigodot\ldots\bigodot F_n$$

In this paper we shall demonstrate that $C(y)$ is piecewise concave such that the solution to the FPM problem:

$$\begin{aligned} & \max_y(py-C(y))\\ \text{s.t.}\quad & \sum_{i=1}^{n}z_i=y\\ & 0\le z_i\le M_i;\ i=1,...,n \end{aligned}\tag{5}$$

cannot be tackled by means of marginalistic techniques (coinciding of marginal cost and price). In fact, the maximum profit will be obtained at a production level y^* where C is not differentiable, or at boundary values

$$y^*=0\ \text{ or }\ y^*=\sum_{i=1}^{n}M_i$$

3 The infimal convolution in the concave case

In this section we shall study the infimal convolution of two concave functions, which is crucial as the basis for the optimization algorithm.

Lemma 1. *Let F_1 and F_2 be concave functions with domains $[m_1, M_1]$ and $[m_2, M_2]$, respectively. We shall consider the following four functions:*

$$\begin{array}{lll} \Psi_1^-(x) := F_1(x-m_2)+F_2(m_2) & \text{with domain} & [m_1+m_2, M_1+m_2] \\ \Psi_1^+(x) := F_1(x-M_2)+F_2(M_2) & \text{with domain} & [m_1+M_2, M_1+M_2] \\ \Psi_2^-(x) := F_2(x-m_1)+F_1(m_1) & \text{with domain} & [m_1+m_2, m_1+M_2] \\ \Psi_2^+(x) := F_2(x-M_1)+F_1(M_1) & \text{with domain} & [M_1+m_2, M_1+M_2] \end{array}$$

then

$$(F_1 \bigodot F_2)(x) = \min\{\Psi_1^-(x), \Psi_1^+(x), \Psi_2^-(x), \Psi_2^+(x)\}$$

Proof. Due to the concavity of the functions involved, the minimum value of $F_1(x_1) + F(x_2)$ constrained to $x_1 + x_2 = \xi$ can only be achieved in those pairs (x_1, x_2) in which only one of the components can be inside the corresponding domain of F_i. In other words, the aforementioned minimum value can only be achieved in pairs of the following form

$$(m_1, \xi - m_2), (m_1, \xi - M_2), (m_2, \xi - m_1) \text{ and } (m_2, \xi - M_1)$$

Thus, for each value of ξ, we have that

$$(F_1 \bigodot F_2)(\xi) = \min\{F_1(\xi - m_2) + F_2(m_2), F_1(x - M_2) + \\ + F_2(M_2), F_2(\xi - m_1) + F_1(m_1), F_2(\xi - M_1) + F_1(M_1)\}$$

□

Unfortunately, the operator of the infimal convolution does not preserve the concave nature of the functions. In general, the result is a piecewise concave function. This means that the infimal convolution of more than two functions cannot be obtained by means of a simple reiteration of the aforestated lemma, but requires resorting to calculating the infimal convolution of several piecewise concave functions. To carry out this calculation, we shall interpret a piecewise concave function as the minimum function of several concave functions, preceding as shown in the following obvious lemma.

Lemma 2. *Let the function*

$$F(x) = \begin{cases} F_1(x) & \text{if} \quad x \in [m_1, M_1] \\ \dots & \dots \quad \dots \\ F_k(x) & \text{if} \quad x \in [m_k, M_k] \end{cases}$$

be piecewise concave (concave in each interval $[m_k, M_k]$). Thus,

$$F(x) = \min_{i \in \{1,\dots k\}} F_i(x)$$

where, we have redefined each function $F_i(x)$ as

$$F_i(x) := \begin{cases} F_i(x) & \text{if} \quad x \in [m_i, M_i] \\ \infty & \text{if} \quad x \notin [m_i, M_i] \end{cases}, \; i = 1, \dots k$$

Once redefined in this way, the calculation of the infimal convolution of two piecewise concave functions requires a combinatorial exploration that is reflected in the following theorem.

Theorem 1. *Let* $F(x) := \min_{i\in A}(F_i(x))$ *and* $G(x) := \min_{i\in B}(G_i(x))$*, then:*

$$(F\bigodot G)(t) = \min_{(i,j)\in A\times B}(F_i\bigodot G_j)(t)$$

Proof.

$$\begin{aligned}(F\bigodot G)(t) &= \min_x(F(t-x)+G(x)) = \min_x(\min_{i\in A}(F_i(t-x)) + \min_{j\in B}(G_j(x)))\\ &= \min_x(\min_{(i,j)\in A\times B}(F_i(t-x)+G_j(x))) = \\ &= \min_{(i,j)\in A\times B}(\min_x(F_i(t-x)+G_j(x))) = \min_{(i,j)\in A\times B}(F_i\bigodot G_j)(t)\end{aligned}$$

□

This theorem justifies the construction of the infimal convolution of the two functions defined piecewise as the minimum function of all the possible infimal convolutions of "*pairs of pieces*".

Now, bearing in mind the associative nature of the infimal convolution operation, the infimal convolution may be calculated by means of a recursive process, carrying out n operations of infimal convolution considering the following recurrence:

$$H_1\bigodot H_2\bigodot\cdots\bigodot H_n = (H_1\bigodot H_2\bigodot\cdots\bigodot H_{n-1})\bigodot H_n$$

4 Algorithm and complexity

In this section we analyze the computational complexity of the previously proposed recursive algorithm for calculating the analytical solution for the piecewise concave quadratic functions. We first analyze the calculation of the minimum of a set of piecewise quadratic functions.

4.1 Algorithm

Let G be a quadratic function and let F be a piecewise quadratic function:

$$F(x) = \begin{cases} F_1(x) & \text{if} \quad x\in[m_1,M_1] \\ ... & ... \quad\quad ... \\ F_k(x) & \text{if} \quad x\in[m_k,M_k]\end{cases}$$

considering $F_j(x) := \infty$ if $x \notin [m_j, M_j]$ and $G(x) := \infty$ if $x \notin [\tilde{m}, \tilde{M}]$. Hence,

$$F(x) = \min_{i\in A=\{1,\dots k\}} F_i(x)$$

The calculation of the infimal convolution

$$(F\bigodot G)(x) = \min_i((F_i\bigodot G)(x))$$

is carried out in two phases:

PHASE (1) Calculation of $F_i \bigodot G$ for each i
PHASE (2) Calculate $\min_i (F_i \bigodot G)(x)$

4.2 Computational Complexity

The nature of the underlying problem in the calculation of the infimal convolution of piecewise concave functions suggests that the computational complexity of the algorithm is exponential seeing as it entails exploring all the combinations of intervals of concavity of the functions involved. In certain cases, this is effectively so; however, we shall see that the complexity is polynomial in some other cases.

Theorem 2. *Let* $\{F_i\}_{i=1}^n$, where $F_i(x) := \beta_i x - \gamma_i x^2$, *with* $\gamma_i > 0$, *with the same domain* $[0, M]$. *If* $F_i(x) \neq F_j(x)$ *for all* $0 \neq x \in [0, M]$, *then the computational complexity of the algorithm is cubic in order; i.e.* $T \in O(n^3)$.

5 Example

A program that solves the FPM problem was written using the Mathematica package and was then applied to one example using the previously developed model for the cost function

$$C(y) = \min_{\mathbf{z}} \sum_{i=1}^{n} \beta_i z_i - \gamma_i z_i^2$$

and maximum constraints for the $n = 4$ inputs.

$$\begin{array}{ll} & \max_{\mathbf{x},y} (py - \sum_{i=1}^{n} \left(\beta_i z_i - \gamma_i z_i^2\right)) \\ \text{s.t.} & \sum_{i=1}^{n} z_i = y \\ & 0 \leq z_i \leq M_i;\ \ i = 1, ..., n \end{array}$$

The data on the inputs is summarized in Table 1.

Table 1. Example data.

i	1	2	3	4
β_i	1	2	3	4
γ_i	-0.01	-0.03	-0.03	-0.01
M_i	10	15	4	2

Applying the aforementioned algorithm, we have that the infimal convolution

$$C = (F_1 \bigodot F_2 \bigodot F_3 \bigodot F_4)$$

is a piecewise quadratic function:

$$C(y) = \begin{cases} y - 0.01y^2 & \text{if} \quad 0 \leq y \leq 10 \\ -14 + 2.6y - 0.03y^2 & \text{if} \quad 10 \leq y \leq 25 \\ -61.5 + 4.5y - 0.03y^2 & \text{if} \quad 25 \leq y \leq 29 \\ -80.64 + 4.58y - 0.01y^2 & \text{if} \quad 29 \leq y \leq 31 \end{cases}$$

Finally, considering different values of the price p, we calculate the solution to the FPM problem

$$\max_y (py - C(y))$$

The results are summarized in Table 2.

Table 2. Solution y^*.

p	2	1	$\frac{1}{2}$	5
y^*	25	10	0	31

As already mentioned, despite having the analytical cost expression, $C(y)$, the optimum level of output cannot be obtained via marginalistic techniques; i.e. $\partial C(y)/\partial y$ coincides with the price p. The maximum profit is always obtained with a level of output y^* in which either C is not differentiable or y^* is one of the extreme values of the interval $[0, \sum_{i=1}^{n} M_i]$.

In fact, for $p = 2 \rightarrow y^* = 25$ and for $p = 1 \rightarrow y^* = 10$, the solution is obtained from angle points of $C(y)$, whereas, as we have already seen, for $p = 1/2 \rightarrow y^* = 0$, i.e. production is not profitable, and for $p = 5 \rightarrow y^* = 31$, the maximum is produced at the technical maximum.

6 Conclusions

Concave quadratic problems often arise involving economies of scale. In this paper we present an algorithm for calculating the analytical solution for the classic firm's cost minimization problem in the case of economies of scale, with n inputs, maximum constraints for the inputs and a general output y (i.e. a family of monoparametric problems). The algorithm uses the infimal convolution of piecewise concave functions. For the firm's profit maximization problem, the solution cannot be obtained using derivatives and our method calculates the exact solution, without any kind of simplification, searching non-differentiable points of the analytical formulae of the cost or extreme values of the output.

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