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A Constrained and Nonsmooth Hydrothermal Problem

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Abstract: This paper addresses a hydrothermal problem that simultaneously considers non-regular Lagrangian and non-holonomic inequality constraints, obtaining a necessary minimum condition. It is further shown that the discontinuity of the lagrangian does not translate as discontinuity in the derivative of the solution. Finally, a solution algorithm is developed and applied to an example.

Keywords: Optimal Control, Clarke's Gradient, Hydrothermal Optimization

Mathematics Subject Classification: 49J24, 49A52

1 Introduction

This paper deals with the optimization of hydrothermal problems. In a previous paper [1], we considered a hydrothermal system with one hydro-plant and m thermal power plants that had been substituted by their thermal equivalent and addressed the problem of minimizing the cost of fuel $F(P)$ during the optimization interval $[0, T]$

$$F(P) = \int_0^T \Psi(P(t))dt \tag{1.1}$$

$$P(t) + H(t, z(t), z'(t)) = P_d(t),\ \forall t \in [0, T] \tag{1.2}$$

$$z(0) = 0, z(T) = b \tag{1.3}$$

where Ψ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by said plant. The following must be also be verified: the equilibrium equation of active power (1.2), and the boundary conditions (1.3), where $P_d(t)$ is the power demand, $H(t, z(t), z'(t))$ is the power contributed to the system at the instant t by the hydro-plant, $z(t)$ being the volume that is discharged up to the instant t by the plant, $z'(t)$ the rate of water discharge of the plant at the instant t, and b the volume of water that must be discharged during the entire optimization interval. In said paper, we likewise considered constraints for the admissible generated power ($P(t) \geq 0$ and $H(t, z(t), z'(t)) \geq 0$). The mathematical problem (P_1) was stated in the following terms:

$$\min_{z\in\Theta_1} F(z) = \min_{z\in\Theta_1} \int_0^T \Psi\left[P_d(t) - H(t, z(t), z'(t))\right] dt = \min_{z\in\Theta_1} \int_0^T L(t, z(t), z'(t))dt$$

$$\Theta_1 = \{z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b, 0 \leq H(t, z(t), z'(t)) \leq P_d(t), \forall t \in [0, T]\}$$

where ($\widehat{C}^1$) is the set of piecewise C^1 functions. The problem (P_1) was formulated within the framework of optimal control [2] and

$$\mathbb{Y}_q(t) := -L_{z'}(t, q(t), q'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))} ds\right] \tag{1.4}$$

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was called *the coordination function of* $q \in \Theta_1$, obtaining the following result:

Theorem 1. *If q is a solution of (P_1), then $\exists K \in \mathbb{R}^+$ such that:*

$$\mathbb{Y}_q(t) \quad is \quad \begin{cases} \leq K & if \quad H(t, q(t), q'(t)) = 0 \\ = K & if \quad 0 < H(t, q(t), q'(t)) < P_d(t) \\ \geq K & if \quad H(t, q(t), q'(t)) = P_d(t) \end{cases}$$

In another previous paper [3], a problem of hydrothermal optimization with pumped-storage plants was addressed, though without considering constraints for the admissible generated power. In this kind of problem, the derivative of H with respect to z' ($H_{z'}$) presents discontinuity at $z' = 0$, which is the border between the power generation zone (positive values of z') and the pumping zone (negative values of z'). The mathematical problem (P_2) was stated in the following terms:

$$\min_{z \in \Theta_2} F(z) = \min_{z \in \Theta_2} \int_0^T \Psi \left[P_d(t) - H(t, z(t), z'(t)) \right] dt = \min_{z \in \Theta_2} \int_0^T L(t, z(t), z'(t)) dt$$
$$\Theta_2 = \{ z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b \}$$

where $L(\cdot, \cdot, \cdot)$ and $L_z(\cdot, \cdot, \cdot)$ are the class C^0 and $L_{z'}(t, z, \cdot)$ is piecewise continuous ($L_{z'}(t, z, \cdot)$ is discontinuous in $z' = 0$). Denoting by $¥_q(t), q \in \Theta_2$ the function:

$$¥_q(t) := -L_{z'}(t, q(t), q'(t)) + \int_0^t L_z(s, q(s), q'(s)) ds \tag{1.5}$$

and by $¥_q^+(t)$ and $¥_q^-(t)$ the expressions obtained when considering the lateral derivatives with respect to z'. The problem (P_2) was formulated within the framework of nonsmooth analysis [4], using the generalized (or Clarke's) gradient, the following result being proven:

Theorem 2. *If q is a solution of (P_2), then $\exists K \in \mathbb{R}^+$ such that:*

$$\begin{cases} ¥_q^+(t) = ¥_q^-(t) = K & if \quad q'(t) \neq 0 \\ ¥_q^+(t) \geq K \geq ¥_q^-(t) & if \quad q'(t) = 0 \end{cases}$$

This paper merges the two previous studies, simultaneaously considering non-regular Lagrangian and non-holonomic inequality constraints (differential inclusions), obtaining a necessary minimum condition. Furthermore, under certain convexity conditions, we shall establish the result (*smooth transition*) that the derivative of the minimum presents a constancy interval, the constant being the value for which $L_{z'}(t, z, \cdot)$ presents discontinuity. Finally, we shall present a solution algorithm and shall apply it to an example.

2 Mathematical Statement and Resolution of the Problem

In this paper, we consider a hydrothermal system with one thermal plant (the thermal equivalent) and one pumped-hydro plant, which will have certain constraints in both genearation and pumping for H. We shall take $H_{\min}$ (maximum pumping capacity) as the lower boundary and $H_s(t) = min\{H_{\max}, P_d(t)\}$ ($H_{\max}$ being maximum generation) as the upper boundary.

The mathematical problem (P_3) may be stated in the following terms:

$$\min_{z \in \Theta} F(z) = \min_{z \in \Theta} \int_0^T \Psi \left[P_d(t) - H(t, z(t), z'(t)) \right] dt = \min_{z \in \Theta} \int_0^T L(t, z(t), z'(t)) dt$$
$$\Theta = \{ z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b, H_{\min} \leq H(t, z(t), z'(t)) \leq H_s(t), \forall t \in [0, T] \}$$

where $L(\cdot,\cdot,\cdot)$ and $L_z(\cdot,\cdot,\cdot)$ are the class C^0 and $L_{z'}(t,z,\cdot)$ is piecewise continuous. We shall assume that Ψ is strictly increasing and strictly convex, that H verifies $H_{z'}>0$, and $H_z(t,z(t),0)=0$, and the strictly increasing nature of $L_{z'}(t,z,\cdot)$. We shall establish the necessary minimum condition for this problem with non-regular Langrangian and constraints on the admissible functions, employing to this end the coordination function, $\mathbb{Y}_q(t)$.

We shall denote by $\mathbb{Y}_q^+(t)$ and $\mathbb{Y}_q^-(t)$ the expressions obtained when considering in (1.4) the lateral derivatives with respect to z'. We shall prove that these functions also verify Theorem 2 in the same way as $¥_q^+(t)$ and $¥_q^-(t)$, and that for the stated problem, except in those values of z' for which $L_{z'}(t,z,\cdot)$ is not continuous, Theorem 1 will continue to be valid. We thus obtain the following result:

Theorem 3. *If q is a solution of (P_3), then $\exists K\in\mathbb{R}^+$ such that:*

$$\begin{array}{lll} \textit{i) If } L_{z'}(t,q(t),\cdot) \textit{ is discontinuous at } q'(t) & \Longrightarrow & \mathbb{Y}_q^+(t)\le K\le \mathbb{Y}_q^-(t) \\ \textit{ii) If } L_{z'}(t,q(t),\cdot) \textit{ is continuous at } q'(t) & \Longrightarrow & \end{array}$$

$$\mathbb{Y}_q(t)\ \ \textit{is}\ \ \begin{cases} \le K & \textit{if}\quad H(t,q(t),q'(t))=H_{\min} \\ =K & \textit{if}\quad H_{\min}<H(t,q(t),q'(t))<H_s(t) \\ \ge K & \textit{if}\quad H(t,q(t),q'(t))=H_s(t) \end{cases}$$

3 Smooth Transition

In this section, we present a qualitative aspect of the solution of (P_3). We prove that, under certain conditions, the discontinuity of the derivative of the Langrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verifed that the derivative of the extremal where the minimum is reached presents an interval of constancy, the constant being the value for which $L_{z'}(t,z,\cdot)$ presents discontinuity. The character C^1 of the solution is thus guaranteed.

Theorem 4. Let $L(\cdot,\cdot,\cdot)$ *be the Lagrangian of the functional F in the conditons stated above, and let us assume that the function $L_{z'}(t_0,z(t_0),\cdot)$ is strictly increasing (decreasing) and discontinuous in* 0*. If q is minimum (maximum) for F, then:*

(i) t_0 is not an isolated point of a change in the sign of q'.

(ii) $q'\equiv 0$ in some interval that contains t_0.

(iii) q' is continuous in t_0.

Note that this result has a very clear interpretation in terms of pumping plants: under optimum operating conditions, pumping plants never switch brusquely from generating power to pumping water or vice versa, but rather carry out a smooth transition, remaining inactive during a certain period of time.

4 Optimization Algorithm

From the computational point of view, the construction of q_K can be performed with the use of a discretized version of Theorem 3. The problem will consist in finding for each K the function q_K that satisfies conditions i) and ii) of Theorem 3, and from among these functions, an admisible function $q_K\in\Theta$. In general, the construction of q_K cannot be carried out all at once over the entire interval $[0,T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs, until completing the interval $[0,T]$, where:

· $H_{\min}<H(t,q(t),q'(t))<H_s(t)$ (free extremal arcs), or

· $q'(t)=0$ (the hydro-plant is on shut-down), or

· $H(t,q(t),q'(t))=H_s(t)$ (the hydro-plant generates all the demanded power or its technical maximum), or

· $H(t,q(t),q'(t))=H_{\min}$ (the hydro-plant is functioning at its maximum pumping power)

If the values obtained for q and q' do not obey the constraints, we force the solution q_K to belong to the boundary until the moment when the conditions of leaving the domain (established in Theorem 3) are fulfilled.

5 Application to a Hydrothermal Problem

A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to one example of hydrothermal system made up of the thermal equivalent plant and a hydraulic pumped-storage plant.

We use the quadratic model: $\Psi(x) := c_1 + c_2 x + c_3 x^2$, for the fuel cost model of the equivalent thermal plant. The power production H of the hydroplant (variable head) is a function of $z(t)$ and $z'(t)$ and its power consumption during pumping is a lineal function of the amount of water pumped $(M \cdot z'(t))$. Hence the function H is defined piecewise as

$$H(t, z(t), z'(t)) := \begin{cases} A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t) & \text{if} \quad z'(t) > 0 \\ M \cdot z'(t) & \text{if} \quad z'(t) \leq 0 \end{cases}$$

where $A(t) = \frac{B_y}{G}(S_0 + t \cdot i)$, $B = \frac{B_y}{G}$. The parameters are: $G(m^4/h.Mw)$ the efficiency, $i(m^3/h)$ the natural inflow, $S_0(m^3)$ the initial volume, and $B_y(m^{-2})$ a parameter that depends on the geometry of the tanks. $M(h.Mw/m^3)$ is the factor of water-conversion of the pumped-storage plant and we consider two cases: (1) $M_1 = (1,04)A(0)$ and (2) $M_2 = (1,10)A(0)$.

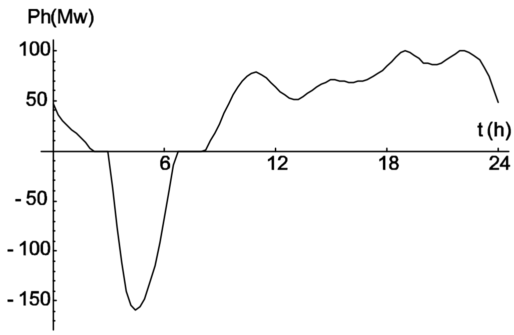

Figure 1. Optimal solution with M_1.

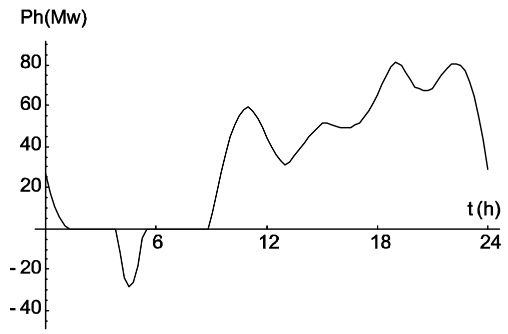

Figure 2. Optimal solution with M_2.

Figures 1 and 2 presents the optimum solution. It can be seen how the interval of smooth transtion varies when considering two different values of M.

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