Mathematical
Inequalities
& Applications
[0590] Volume 0, Number 0 (0000), 000–000

THE FIRST WEIERSTRASS–ERDMANN CONDITION IN VARIATIONAL PROBLEMS INVOLVING DIFFERENTIAL INCLUSIONS

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(*communicated by R. Agarwal*)

Abstract. In this paper, the authors continue a previous study about the broken extremals in variational problems with differential inclusions. In said paper, we presented a necessary condition for extremals with corner points that is valid for shapeable sets. This condition has been obtained by adapting a novel proof of the first Weiertrass-Erdmann condition.

In the present paper we extend the class of shapeable sets and demonstrate that the set

$$\Omega := \{z \in KC^1[a,b] \mid G_1(t,z(t)) \leqslant z'(t) \leqslant G_2(t,z(t)), \forall t \in [a,b] \text{ a.e.}\}$$

with $G_1, G_2 \in C^1$, is shapeable for every t.

Finally, we present two examples, the second being a classic engineering problem: the optimization of hydrothermal systems.

1. Introduction

The extremal values of the functional

$$F(z) = \int_a^b L(t, z(t), z'(t))dt$$

on

$$D = \{z \in KC^1[a,b] \mid z(a) = \alpha,\ z(b) = \beta\}$$

may be achieved in functions with corner points. For $KC^1[a,b]$, we denote the set of continuous with piecewise continuous derivative functions. In all the paper, when reference is made to properties of the derivative of a function, these shall be understood to be fulfilled for the two lateral derivatives.

The Weiertrass-Erdmann conditions (W-E conditions) show that the discontinuities of q' that are permitted at corner points of a local extremal q are limited to those which preserve the continuity of both

$$\begin{cases} \text{(i) } L_{z'}(t, q(t), q'(t)) \\ \text{(ii) } L(t, q(t), q'(t)) - q'(t)L_{z'}(t, q(t), q'(t)) \end{cases}$$

The W-E conditions are of crucial importance in determining broken extremals and sometimes allow one to prove that such extremals do not exist. Although these two

Mathematics subject classification (2000): 49K24, 49K30.

Key words and phrases: Calculus of Variations, optimization, Weierstrass-Erdmann, broken extremals, inequality constraints.

conditions of continuity have been known since the end of the 19th century [12], they have been expounded on diverse occasions with insufficient care. Both are correct when dealing with strong extremals, but only the first is true for weak extremals. An incorrect formulation of the second of these conditions was expounded by the authors of [24] and [26], who assumed that the condition was true for the weak minima. The counterexamples presented in [4-5] show that this assumption was incorrect.

The so-called first W-E condition is not always satisfied in variational problems where the admissible functions are subject to certain constraints. For example, this condition is not fulfilled in the problems of reflection of the extremals; one can easily discover other examples where the condition is also violated.

We shall analyze the first condition and study the possibility of its extension to variational problems with constraints on the admissible functions.

The classic proofs of the W-E conditions are based on the fact that either the Gâteaux differential of the functional vanishes at the extremal [15], or [26] employ the equation

$$L_{z'}(t,q(t),q'(t)) = Const. + \int_a^t L_z(x,q(x),q'(x))dx \tag{1}$$

These techniques do not work if the constraints are taken into account, because the functional need not admit bilateral variations at the extremum, or Equation (1) is simply not satisfied.

Variational problems in which the derivatives of the admissible functions must be subject to certain inequality constraints (differential inclusion $z' \in E(t,z)$) have traditionally been dealt with by recurring to a large number of diverse techniques. The first studies in this field were conducted by Flodin [13] for simpler constraints of the type $A \leqslant z'(t) \leqslant B$ and by Follinger, who in [14] deals in a very complex way with a more general constraint of the type $H(t,z(t)) \leqslant z'(t) \leqslant G(t,z(t))$ (for present-day existence theorems see [6-16]). In [7], Clarke deals with necessary conditions for problems in the calculus of variations that incorporate inequality constraints of the form $f(z,z') \leqslant 0$. In [8], the author determines necessary conditions, in terms of generalized gradients, for the existence of an extremal arc for calculus of variations and optimal control problems with differential multi-inclusion $z' \in E(t,z)$.

In [9], Clarke and Loewen consider an optimal control problem on a fixed time interval $[0,T]$, and a variety of necessary conditions are derived for the original optimal control problem. The same authors, in [10], develop an existence theory for solutions to the original problem with $|z'(t)| < R$. In [17-18], Loewen and Rockafellar consider the classical Bolza problem in the calculus of variations, incorporating endpoint and velocity constraints through infinite penalties. The integrand L are allowed to be nondifferentiable. In [19], the authors have recurred to techniques of optimal control and formulate a sufficient optimality condition for broken extremals in terms of the solution of the Hamilton-Jacobi-Bellman equation.

In [25], the simplest problem of the calculus of variations is investigated, along with the corresponding Euler equation. Some new results on the Euler equation are obtained and a minimizing sequence whose derivatives form a family of equicontinuous functions at a point is studied. Examples of the problem with singular extremals that are local minima are given.

In [21] the simplest problem of the calculus of variations is considered. The authors aim to prove that the classical second-order conditions formulated in terms of a conjugate point and Riccati equation can also be generalized to the case of a broken extremal. However, the type of minimum considered is weaker than a strong minimum and stronger than a weak minimum. It is called the Θ-weak minimum. Osmolovskii [22] distinguishes five basic types of minimum: weak minimum, Θ-weak minimum, Pontryagin minimum, bounded-strong minimum, and strong minimum. The method of the strengthening of the quadratic conditions is used throughout the paper, and quadratic conditions are formulated for broken extremals. In [23], the authors obtain sufficient conditions for positive definiteness of the quadratic form in terms of the Riccati equation and hence sufficient optimality conditions for broken extremals.

Noble and Schättler [20] develop sufficient conditions for a relative minimum for broken extremals in an optimal control problem based on the method of characteristics.

In the present paper, the authors continue a previous study [1] into broken extremals in variational problems with differential inclusions. In said paper, we presented a novel proof of the first W-E condition. This proof is based on the analysis of the Gâteaux variations in certain directions which we will call $h_\varepsilon^{t_0}$.

DEFINITION 1. Let us take $t_0 \in (a,b)$ and $\varepsilon > 0$. We consider the auxiliary function $h_\varepsilon^{t_0}$

$$h_\varepsilon^{t_0}(t) := \begin{cases} 0 & \text{if } \; t \in [a, t_0-\varepsilon] \cup [t_0+\varepsilon, b] \\ (t-t_0+\varepsilon) & \text{if } \; t \in [t_0-\varepsilon, t_0] \\ -(t-t_0-\varepsilon) & \text{if } \; t \in [t_0, t_0+\varepsilon] \end{cases}$$

THEOREM 1. *If $L(t,z,z') \in C^1([a,b]\times\mathbb{R}^2)$ and $q \in KC^1[a,b]$ provides a (weak) local extremal value for $F(z) = \int_a^b L(t,z(t),z'(t))dt$ on $D = \{z \in KC^1[a,b] \mid \; z(a) = \alpha \; , \; z(b) = \beta\}$, then $\forall t \in [a,b]$ the first condition W-E holds: $L_{z'}(t,q(t),q'(t_-)) = L_{z'}(t,q(t),q'(t_+))$.*

The method proposed for the proof can be adapted to study the extremum of the functional restricted to the sets where

$$DF(q;h_\varepsilon^{t_0}) := \lim_{x\to 0^+} \frac{F(q+xh_\varepsilon^{t_0}) - F(q)}{x}$$

exists. We call these constraints shapeable sets.

2. Shapeable sets

Let us establish the concept of a shapeable set of functions. This will allow us to introduce a class of constraints on the admissible functions under which the necessary condition for broken extremals that we present is satisfied.

DEFINITION 2. We will say that ω is W-admisible at q if $\exists\theta > 0$ such that $q + x\omega \in W$, $\forall x \in [0,\theta]$.

DEFINITION 3. We will say that a set of functions $\Omega \subset KC^1[a,b]$ is shapeable in $t_0 \in (a,b)$ if $\forall q \in \Omega$

i) $q'(t_{0-}) < q'(t_{0+}) \Longrightarrow \exists \varepsilon > 0$ such that $h_\varepsilon^{t_0}$ is $\Omega -$ *admisible* at q.

ii) $q'(t_{0-}) > q'(t_{0+}) \Longrightarrow \exists \varepsilon > 0$ such that $-h_\varepsilon^{t_0}$ is $\Omega -$ *admisible* at q.

Let us see a necessary condition for broken extremals.

THEOREM 2. (THE FIRST W-E GENERALIZED CONDITION). *If $L(t,z,z') \in C^1([a,b]\times \mathbb{R}^2)$, Ω is shapeable in t_0 and q provides a (weak) local minimum value for $F(z) = \int_a^b L(t,z(t),z'(t))dt$ on $D = \Omega \cap \{z \in KC^1[a,b] \mid z(a) = \alpha \ \wedge \ z(b) = \beta\}$ then it holds that:*

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) \leqslant 0$$

We show how, by imposing a certain property on $L_{z'}$, the necessary condition (Theorem 2) becomes the classic first W-E condition.

THEOREM 3. *If $L(t,z,z') \in C^1([a,b]\times \mathbb{R}^2)$, $\psi(x) = L_{z'}(t_0, q(t_0), x)$ is nondecreasing, Ω is shapeable at t_0, and q provides a (weak) local minimum value for $F(z) = \int_a^b L(t,z(t),z'(t))dt$ on $D = \Omega \cap \{z \in KC^1[a,b] \mid z(a) = \alpha \ \wedge \ z(b) = \beta\}$, then the first W-E condition holds:*

$$L_{z'}(t_0, q(t_0), q'(t_{0-})) = L_{z'}(t_0, q(t_0), q'(t_{0+})).$$

And it is now obvious that the property that $L_{z'}$ is strictly increasing with respect to z' allows the existence of extremals with corner points to be rejected.

THEOREM 4. *If $L(t,z,z') \in C^1([a,b]\times \mathbb{R}^2)$, and $\psi(x) = L_{z'}(t,z,x)$ is strictly increasing $\forall (t,z) \in (a,b)\times \mathbb{R}$, Ω is shapeable for every $t \in [a,b]$, and q provides a (weak) local minimum value for $F(z) = \int_a^b L(t,z(t),z'(t))dt$ on $D = \Omega \cap \{z \in KC^1[a,b] \mid z(a) = \alpha \ \wedge \ z(b) = \beta\}$, then q is C^1.*

In [24] we see, with examples, that the concept of the shapeable set embraces the constraints considered in the classic obstacle problem and in problems with velocity constraints.

PROPOSITION 1. *If $g_1, g_2 \in C^1[a,b]$, then the set $\{z \in KC^1[a,b] \mid g_1(t) \leqslant z(t) \leqslant g_2(t), \forall t \in [a,b]\}$ is shapeable $\forall t_0 \in (a,b)$.*

PROPOSITION 2. *If $g_1, g_2 \in C[a,b]$, then the set $\{z \in KC^1[a,b] \mid g_1(t) \leqslant z'(t) \leqslant g_2(t), \forall t \in [a,b]\}$ is shapeable $\forall t_0 \in (a,b)$.*

In the present paper, the class of shapeable constraints is extended and it is demonstrated that, given two functions $G_1, G_2 \in C^1$, the set

$$\{z \in KC^1[a,b] \mid G_1(t,z(t)) \leqslant z'(t) \leqslant G_2(t,z(t)),\ \forall t \in [a,b] \text{ a.e.}\}$$

is shapeable for each t.

As a consequence of this, and of Theroems 3 and 4, it is once more established that under adequate conditions of convexity, the broken solutions of certain variational problems with differential inclusion constraints satisfy the first classic Weierstrass-Erdmann condition. It is likewise concluded that, in the case of the Langrangian being strictly convex with respect to z', the minimum value of the functional is necessarily achieved in functions of class C^1.

Finally we present two examples. The first is of a geometrical type and the second is a classic engineering problem: the optimization of hydrothermal systems.

3. Shapeable sets and differential inclusions

Employing the following theorem, we shall demonstrate that the set associated with certain differential inclusion constraints is also shapeable for every $t \in (a, b)$.

THEOREM 5. *If* $G \in C^1([a, b] \times \mathbb{R})$ *, the set*

$$\Omega := \{z \in KC^1[a, b] \mid z'(t) \leqslant G(t, z(t)), \ \forall t \in [a, b] \ a.e.\}$$

is shapeable for every $t_0 \in (a, b)$.

Proof. Let us assume firstly that

$$q'(t_{0-}) < q'(t_{0+}) \leqslant G(t_0, z(t_0))$$

Let

$$m := \min_{t \in [a,b]} G'_z(t, q(t))$$

It is evident that there exists sufficiently small ε and θ so as to verify, for every

$$(t, x) \in [t_0 - \varepsilon, t_0) \times [0, \theta]$$

that

$$q'(t) + x(h_\varepsilon^{t_0})'(t) = q'(t) + x < G(t, q(t) + xh_\varepsilon^{t_0}(t)) \tag{1}$$

and so that, for every $t \in (t_0, t_0 + \varepsilon]$

$$-1 < (-t + t_0 + \varepsilon) \cdot m$$

Employing the Theorem of Lagrange, we shall also have that, for every $(t, x) \in (t_0, t_0 + \varepsilon] \times [0, \theta]$

$$\begin{aligned} G(t, q(t) + xh_\varepsilon^{t_0}(t)) &= G(t, q(t) + x(-t + t_0 + \varepsilon)) \\ &= G(t, q(t)) + x(-t + t_0 + \varepsilon)G'_z(t, c_t) \end{aligned}$$

where $c_t \in [z(t), z(t) + x(-t + t_0 + \varepsilon)]$.

Hence, for every $(t, x) \in (t_0, t_0 + \varepsilon] \times [0, \theta]$

$$\begin{aligned} q'(t) + x(h_\varepsilon^{t_0})'(t) = q'(t) - x &\leqslant G(t, q(t)) + x(-t + t_0 + \varepsilon) \cdot G'_z(t, c_t) \\ &= G(t, q(t) + xh_\varepsilon^{t_0}(t)) \end{aligned} \tag{2}$$

In short, (1) and (2) guarantee that $h_\varepsilon^{t_0}$ is $\Omega - admissible$ at t_0.

In the case of

$$G(t_0, z(t_0)) \geqslant q'(t_{0-}) > q'(t_{0+})$$

by analogous reasoning, we reach the conclusion that $-h_\varepsilon^{t_0}$ is $\Omega - admissible$ at t_0. □

THEOREM 6. *If $G \in C^1$, the set*

$$\{z \in KC^1[a,b] \mid G(t,z(t)) \leqslant z'(t),\ \forall t \in [a,b]\ a.e.\}$$

is shapeable for every $t_0 \in (a,b)$.

THEOREM 7. *If $G_1, G_2 \in C^1([a,b]\times\mathbb{R})$, the set*

$$\{z \in KC^1[a,b] \mid G_1(t,z(t)) \leqslant z'(t) \leqslant G_2(t,z(t)),\ \forall t \in [a,b]\ a.e.\}$$

is shapeable for every $t_0 \in (a,b)$.

Therefore, applying Theorem 3, under adequate conditions of convexity, the broken solutions to certain variational problems with differential inclusion constraints satisfy the first Weierstrass-Erdmann condition. It is likewise concluded, applying Theorem 4, that in the case of the Langrangian being strictly convex with respect to z', the minimum value of the functional is necessarily achieved in functions of the class C^1.

4. A numerical example

Let us take $L \in C^1[\mathbb{R}]$ with the strictly increasing L'. Let us consider the problem of minimizing

$$F(z) = \int_0^1 L(z'(t))dt$$

on

$$D = \Omega \cap \{z \in KC^1[0,1] \mid z(0) = 0\,,\, z(1) = b\}$$

where

$$\Omega := \{z \in KC^1[0,1] \mid z(t) + t \leqslant z'(t) \leqslant z(t) + t + 2, \forall t \in [0,1]\}$$

We shall denote as:

f_s : the solution of the differential equation $z'(t) = z(t) + t + 2$ with the intial condition $z(0) = 0$.

f_i : the solution of the differential equation $z'(t) = z(t) + t$ with the final condition $z(1) = b$.

It is necessary for the solution q to account for the arcs of the extremal $(C_1 + C_2 t)$ and the boundary arcs ($-1 - t + C_3 e^t$ o $-3 - t + C_4 e^t$). Hence, since Ω is shapeable at every point, and by virtue of Theorem 4, its derivative must be continuous and can only be of the form

$$q'(t) = \begin{cases} -1 - t + e^t & \text{if} \quad t \in [0,\alpha] \\ -1 - \alpha + e^\alpha & \text{if} \quad t \in [\alpha, \alpha+\beta] \\ -3 - t + C_4 e^t & \text{if} \quad t \in [\alpha+\beta, 1] \end{cases}$$

for a certain α, with $\beta = \dfrac{-4 + (-2+\alpha)\alpha}{2(-\alpha + e^\alpha)}$ and $C_4 = \dfrac{t - \alpha - \beta}{e^{\alpha+\beta} - e^t}$.

We denote by k_{ex} the slope of the extremal; $k_{fs} = f_s'(0)$; $k_{fi} = f_i'(1)$.

If $k_{fs} > b > k_{fi}$ is fulfilled, the solution is the free extremal (Fig. 1-a): $q(t) = bt$.

Let us analyse the cases with boundary arcs in more detail.

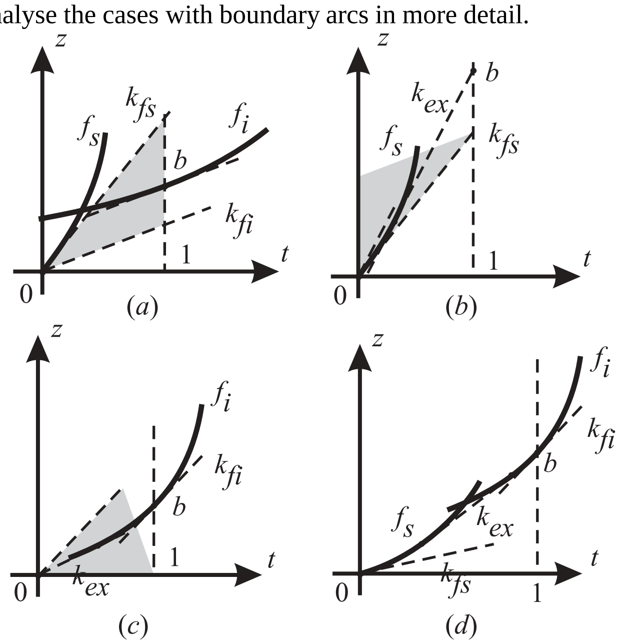

Fig. 1. Solution q.

Case a) If $k_{fs} \leqslant b$; $k_{fi} \leqslant b$, the solution is formed by a boundary arc $f_s(t)$ and an extremal arc tangential to $f_s(t)$ (Fig. 1-b). That is, $\alpha + \beta = 1$.

Case b) If $k_{fs} \geqslant b$; $k_{fi} \geqslant b$, the solution is formed by an extremal arc followed by a boundary arc $f_i(t)$ (Fig. 1-c). Tha is, $\alpha = 0$.

Case c) If $k_{fs} \leqslant b \leqslant k_{fi}$, the solution consists in both boundary arcs and an extremal arc between these (Fig. 1-d).

This example shows how the assertion of Theorem 4 can exclude the presence of the corner points and therefore the unique solution is obtained in a much simpler way than by means of any traditional method (for example, optimal control or an equivalent Caratheodory formulation).

5. A hydrothermal problem

A hydrothermal system is made up of hydraulic and thermal power plants which during a definite time interval must jointly satisfy a certain demand in electric power. Thermal plants generate power at the expense of fuel consumption (which is the object of minimization), while hydraulic plants obtain power from the energy liberated by water that moves a turbine; a limited amount of water being available during the optimization period.

In prior studies [11-2], it has been proven that the problem of optimization of the fuel costs of a hydrothermal system with m thermal power plants may be reduced to the study of a hydrothermal system made up of one single thermal power plant, called the thermal equivalent. In the present paper, we consider a hydrothermal system with one hydraulic power plant and m thermal power plants that have been substituted by their thermal equivalent. With these conditions, we present the problem from the Electrical Engineering perspective to then go on to resolve the mathematical problem thus formulated.

5.1. Hydrothermal statement of the problem

The problem consists in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0,T]$. Said cost may be represented by the functional

$$F(P(t)) = \int_0^T \Psi(P(t))dt$$

where Ψ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by said plant. Moreover, the following equilibrium equation of active power will have to be fulfilled

$$P(t) + H(t,z(t),z'(t)) = P_d(t),\ \forall t \in [0,T]$$

$P_d(t)$ being the power demand and $H(t,z(t),z'(t))$ the power contributed to the system at the instant t by the hydraulic plant, where: $z(t)$ is the volume that is discharged up to the instant t (in what follows, simply volume) by the plant, and $z'(t)$ the rate of water discharge at the instant t of the plant.

Taking into account the equilibrium equation, the problem reduces to calculating the minimum of the functional

$$F(z) = \int_0^T \Psi\left(P_d(t) - H\left(t,z(t),z'(t)\right)\right)dt$$

If we assume that b is the volume of water that must be discharged during the entire optimization interval, the following boundary conditions will have to be fulfilled

$$z(0) = 0,\ z(T) = b$$

For the sake of convenience, we assume throughout the paper that they are sufficiently smooth and are subject to the following additional assumptions.

Let us assume that the cost function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies $\Psi'(x) > 0$, $\forall x \in \mathbb{R}^+$ and thus is strictly increasing. This restriction is totally natural: it reads more cost to more generated power. Let us assume as well that $\Psi''(x) > 0,\ \forall x \in \mathbb{R}^+$ and is therefore strictly convex. The models traditionally employed meet this restriction.

Let us assume that the hydraulic generation $H(t,z,z') : \Omega_H = [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is strictly increasing with respect to the rate of water discharge z', with $H_{z'} > 0$. Let us also assume that $H(t,z,z')$ is concave with respect to z', i.e. $H_{z'z'} \leqslant 0$. The real models meet these two restrictions, and the former means more power to a higher rate of water discharge.

We see that we only admit non-negative thermal power $(P(t))$ and we will solely admit non-negative volumes $(z(t))$ and rates of water discharge $(z'(t))$, therefore we may present the mathematical problem in the following terms.

6. Variational statement of the problem

We will call Π_b the problem of minimization of the functional

$$F(z(t)) = \int_0^T L(t, z(t), z'(t))dt$$

with L of the form

$$L(t, z(t), z'(t)) = \Psi(P_d(t) - H(t, z(t), z'(t)))$$

over the set

$$D = \Omega \cap \{z \in KC^1[0, T] \mid z(0) = 0 \, , z(T) = b\}$$

where

$$\Omega := \{z \in KC^1[0, T] \mid 0 \leqslant H(t, z(t), z'(t)) \leqslant P_d(t)\}$$

So the problem involves inequality non-holonomic constraints in the derivative $z'(t)$. Let us take $\Psi \in C^1[\mathbb{R}]$, (fuel cost) strictly convex, $P_d \in C^1([0, T])$ and $H \in C^1([0, T] \times \mathbb{R}^2)$ (strictly increasing with respect to its second component).

It is easy to see that Ω is shapeable if we bear in mind that H is strictly increasing with respect to z' and that it may be expressed as

$$\Omega := \{z \in KC^1[0, T] \mid G_1(t, z(t)) \leqslant z'(t) \leqslant G_2(t, z(t)), \; \forall t \in [0, T]\}$$

where $H(t, z(t), G_1(t, z(t))) = 0$ and $H(t, z(t), G_2(t, z(t))) = P_d(t)$.

It is necessary for the solution q to account for the arcs of the extremal and the boundary arcs ($q'(t) = 0$ or $H(t, q'(t), q(t)) \leqslant P_d(t)$). Hence, since Ω is shapeable at every point, and by virtue of Theorem 4, its derivative must be continuous.

If z satisfies Euler's equation for the functional

$$F(z) = \int_0^T L(t, z(t), z'(t))dt$$

for every $t \in [\alpha, \alpha + \beta]$, where $L(t, z(t), z'(t)) = \Psi\left(P_d(t) - H(t, z(t), z'(t))\right)$, we have that

$$L_z(t, z(t), z'(t)) - \frac{d}{dt}\left(L_{z'}(t, z(t), z'(t))\right) = 0$$

If we divide Euler's equation by $L_{z'}(t, z(t), z'(t)) < 0$, $\forall t$, we have that

$$\frac{L_z(t, z(t), z'(t))}{L_{z'}(t, z(t), z'(t))} - \frac{\frac{d}{dt}\left[L_{z'}(t, z(t), z'(t))\right]}{L_{z'}(t, z(t), z'(t))} = 0$$

and, integrating, we have

$$-L_{z'}(t, z(t), z'(t)) \cdot \exp\left[-\int_\alpha^t \frac{H_z(s, z(s), z'(s))}{H_{z'}(s, z(s), z'(s))} ds\right] = -L_{z'}(\alpha, z(\alpha), z'(\alpha)) = K \in \mathbb{R}^+$$

We shall call the preceding relation *the coordination equation* for $z(t)$, and the positive constant K will be termed *the coordination constant* of the extremal.

In real problems, H is decreasing with respect to z $(H_z < 0)$ and $H_{z'}$ is increasing with respect to time $(H_{z't} > 0)$. This allows us to assert in a simple way that the optimal thermal power $P(t) = P_d(t) - H(t, z(t), z'(t))$ is decreasing in the intervals corresponding to the extremal arcs. Effectively, with the above relations, the exponential

$$\exp\left[-\int_0^t \frac{H_z(s, z(s), z'(s))}{H_{z'}(s, z(s), z'(s))} ds\right]$$

is increasing with respect to time, and so for the coordination constant to be maintained, the expression

$$-L_{z'}(t, z(t), z'(t)) = \Psi'\left(P_d(t) - H(t, z(t), z'(t))\right) \cdot H_{z'}(t, z(t), z'(t))$$

(with $H_{z'} > 0$) leads us to the conclusion that the optimal thermal power is decreasing.

This circumstance allows us to assert that there exists α, $\beta \in [0, T]$ such that the solution q satisfies

$$\begin{cases} H(t, q'(t), q(t)) = 0 & \text{if} \quad t \in [0, \alpha] \\ \text{free extremal} & \text{if} \quad t \in [\alpha, \alpha + \beta] \\ H(t, q'(t), q(t)) = P_d(t) & \text{if} \quad t \in [\alpha + \beta, T] \end{cases}$$

where β may be calculated from α.

In these conditions, the aim is to consider for very $\alpha \in [0, T]$ the function $q_\alpha \in C^1[0, T]$ that fulfills $q_\alpha(0) = 0$ and the following conditions

$$\begin{cases} H(t, q'_\alpha(t), q_\alpha(t)) = 0 & \text{if} \quad t \in [0, \alpha] \\ \text{free extremal} & \text{if} \quad t \in [\alpha, \alpha + \beta_\alpha] \\ H(t, q'_\alpha(t), q_\alpha(t)) = P_d(t) & \text{if} \quad t \in [\alpha + \beta_\alpha, T] \end{cases}$$

and to determine the value of α for which the final volume condition $q_\alpha(T) = b$ is satisfied. All this may be done, at least in an approximate way, using simple numerical techniques.

A program was elaborated using the Mathematica package which resolves the optimization problem and was then applied to a hydrothermal system made up of the thermal equivalent and a hydraulic plant.

For the fuel cost model of the equivalent thermal plant, we use the quadratic model

$$\Psi(P(t)) = \alpha + \beta P(t) + \gamma P(t)^2$$

The units for the coefficients are: α in $(\$/h)$; β in $(\$/h.Mw)$; γ in $(\$/h.MW^2)$. The hydro-plant's active power generation is given by

$$P_h(t) = -A(t)z'(t) - Bz'(t)z(t) - Cz'(t)^2$$

where the coefficients A, B and C are

$$A(t) = \frac{-1}{G} B_y(S_0 + t \cdot i), \quad B = \frac{B_y}{G}, \quad C = \frac{B_T}{G}$$

We consider that the transmission losses for the hydro-plant are expressed by Kirchmayer's model, with the following loss equation: $b_l \cdot (P_h(t))^2$. So

$$H(t) = P_h(t) - b_l \cdot (P_h(t))^2$$

The units for the coefficients of the hydro-plant are: the efficiency G in $(m^4/h.Mw)$, the restriction on the volume b in (m^3), the loss coefficient b_l in $(1/Mw)$, the natural inflow i in (m^3/h), the initial volume S_0 in (m^3), the coefficients B_T in $(m^{-2}.h)$ and the coefficients B_y in (m^{-2}) (parameters that depend on the geometry of the tanks).

The data for the thermal and hydraulic plants are summarized in Table I.

α	β	γ	G	i
9127.31	19.8841	0.0012718	$570.834 \cdot 10^3$	$301.952 \cdot 10^6$

S_0	B_T	B_y	b_l
$407.808 \cdot 10^8$	$219.597 \cdot 10^{-8}$	$149.5 \cdot 10^{-11}$	0

Table I.- Coefficients

The values of the power demand (in Mw) were adjusted to the following curve

$$P_d(t) = 350 + 5t(24 - t)$$

An optimization interval of 24 h. was considered, and a final volume $b = 90.120 \cdot 10^6$ m^3.

Fig. 2 presents the plots of power demand (P_d) and thermal power (P_t).

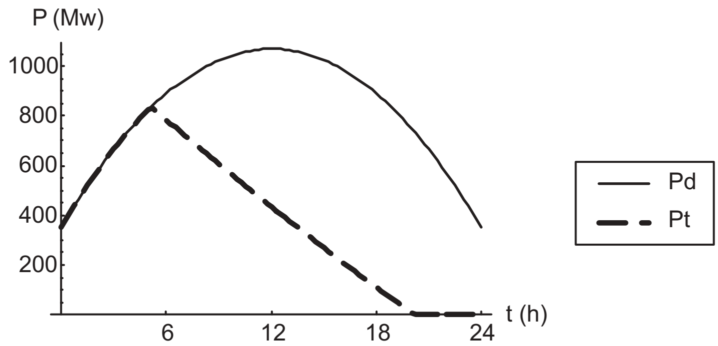

Figure 2. A hydrothermal example.

The method of resolution may be consulted in more detail in [3].

7. Conclusions

This paper continues a previous study by the authors into broken extremals in variational problems with differential inclusions.

In particular, it is demonstrated that, given certain functions G_1, G_2 of the class C^1, the set

$$\{z \in KC^1[a,b] \mid G_1(t,z(t)) \leqslant z'(t) \leqslant G_2(t,z(t)), \forall t \in [a,b] \text{ a.e.}\}$$

is shapeable for every t.

Finally, we present two examples. The first is of the geometrical type and the second is a classic engineering problem: the optimization of hydrothermal systems. These examples show how the theory developed can exclude the presence of the corner points, thus obtaining the unique solution in a very simple way.

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(Received April 2, 2003)

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