Cyclic coordinate descent in a class of bang-singular-bang problems

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November 28, 2014

1 Introduction

A continuous production process on a time interval, $[0, T]$, whose output has variable price $p(t)$ and whose production function is linear with respect to the consumption rate over time of the inputs, of limited availability, is posed as the maximization of a functional:

$$\int_0^T p(t)(f_1(t)z_1'(t) + ... + f_n(t)z_n'(t))dt \tag{1}$$

with $z_i \in \left\{\widehat{C}^1[0, T] \;|z_i(0) = 0, z_i(T) = b_i\right\}$, with technical constraints for the consumption rate over time: $m_i \leq \dot{z}_i(t) \leq M_i$, $\forall i = 1, \dots, n$.

We abstract from this situation to tackle a general problem which studies a production process whose efficiency of production with respect to each input depends on time, on the stock of the various inputs, and on the consumption rate of other inputs. The main limitation will be the linearity of the functional with respect to consumption rates and stocks, which will imply that it is a bang-singular-bang optimal control problem. We propose an efficient method for finding the bang-singular-bang solution using a cyclic coordinate descent strategy combined with a suitable adaptation of the shooting method

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([2], and [1]). The corresponding boundary value problem derived from Pontryagin's Maximum Principle is solved without any initial guess regarding the structure of the solution.

The coordinate descent gives a convergent iterative method because of our assumptions, which are more general ([5]) than the usual ones ([3], [4]).

2 Statement of the variational problem

We consider the problem of maximizing the multidimensional functional

$$J(\mathbf{z}) = \int_0^T L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))dt$$

with the integrand being

$$L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \mathbf{z}^t A(t)\mathbf{z} + \mathbf{z}^t B(t)\dot{\mathbf{z}} + \dot{\mathbf{z}}^t C(t)\dot{\mathbf{z}} + \mathbf{s}(t)^t \cdot \dot{\mathbf{z}} + \mathbf{z}^t P\dot{\mathbf{z}} + \mathbf{r}(t)^t \cdot \mathbf{z}$$

on a suitable product $\mathbb{D} = \prod \mathbb{D}_i$ for

$$\mathbb{D}_i = \left\{ z_i \in \widehat{C}^1[0,T] | z_i(0) = 0, z_i(T) = b_i \, \text{and} \, m_i \leq \dot{z}_i(t) \leq M_i \right\}$$

and where $A(t)$, $B(t)$, $C(t)$, $P(t)$, $r(t)$ and $s(t)$ are suitable matrices of continuous functions (not totally general).

3 Dimension one: the shooting method

The general case iterates over the one-dimensional one. To solve the latter, assume that all the components of $\mathbf{z}$ and $\dot{\mathbf{z}}$ are fixed but the $i-$th one. Let $\mathbf{q} = (q_1, \cdots, q_m) \in \mathbb{D}$ and write $L_q^i(t, z_i, \dot{z}_i)$ as

$$L_q^i := L(t, q_1(t), \cdots, q_{i-1}(t), z_i, q_{i+1}(t), \cdots, q_m(t), \dot{q}_1(t), \cdots, \dot{z}_i, \cdots, \dot{q}_m(t)).$$

We wish to solve the problem of maximizing the functional $J_{\mathbf{q}}^i : \mathbb{D}_i \longrightarrow \mathbb{R}$

$$J_{\mathbf{q}}^i(z_i) := J(q_1, \cdots, q_{i-1}, z_i, q_{i+1}, \cdots, q_m) = \int_0^T L_{\mathbf{q}}^i(t, z_i(t), \dot{z}_i(t))dt \qquad (1)$$

on $\mathbb{D}_i$ where one can write, for suitable functions:

$$L_{\mathbf{q}}^i(t, z_i, \dot{z}_i) = F_{\mathbf{q}}^i(t) + G_{\mathbf{q}}^i(t)z_i(t) + \left(H_{\mathbf{q}}^i(t) + P_{\mathbf{q}}^i(z_i)\right)\dot{z}_i(t).$$

Definition 1 *The i-th* efficiency function *associated to* $\mathbf{q} \in \mathbb{D}$ *is:*

$$\mathbb{Y}^i_{\mathbf{q}}(t) := \int_0^t G^i_{\mathbf{q}}(s)ds - H^i_{\mathbf{q}}(t)$$

Solutions of the optimization problem are characterized by

Theorem 1 *Given* $\mathbf{q} \in \mathbb{D}$*, its* $i-$*th component* $q_i(t)$ *solves the optimization problem* (1) *for* $J^i_{\mathbf{q}}$ *if and only if there exists* $k_i \in \mathbb{R}$ *satisfying:*

$$\mathbb{Y}^i_{\mathbf{q}}(t) \text{ is } \begin{cases} \leq k_i & \text{if } \dot{q}_i(t) = m_i \\ = k_i & \text{if } m_i < \dot{q}_i(t) < M_i \\ \geq k_i & \text{if } \dot{q}_i(t) = M_i \end{cases}$$

Hence, the one-dimensional problem is reduced to finding the $i-$th critical efficieny level. We prove that this can be done using an adaptation of the shooting method because the efficieny level is the zero of a continuous function, so that Bolzano's Theorem applies.

4 General case: cyclic coordinate descent

The result for the general $m-$dimensional case is

Theorem 2 *An admissible element* $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{D}$*, is a solution of problem* (1)*, if and only if there exist* $\{k_i\}_{i=1}^m \subset \mathbb{R}$ *satisfying:*

$$\mathbb{Y}^i_{\mathbf{q}}(t) \text{ is } \begin{cases} \leq k_i & \text{if } \dot{q}_i(t) = m_i \\ = k_i & \text{if } m_i < \dot{q}_i(t) < M_i \\ \geq k_i & \text{if } \dot{q}_i(t) = M_i \end{cases}$$

This theorem allows us to solve for each coordinate and then iterate (thus doing a "coordinate descent"). The compactness of the sets on which we work guarantees convergence.

5 Conclusions

We first solve, using techiques similar to the shooting method, the one-dimensional optimal control problem with a Lagrangian of the form:

$$L(t, z, \dot{z}) = F(t) + G(t)z(t) + (H(t) + P(z))\,\dot{z}(t)$$

The shooting method in combination with Pontryagin's maximum principle and the theory of singular control provides the theoretical basis that has enabled us to construct an algorithm for solving the problem approximately and, in some cases, even analytically, despite the possible apparition of singular arcs of infinite order.

After tackling that one-dimensional problem, we show how we can also solve multidimensional problems whose Lagrangian is of the form (for specific types of matrices A, B, C and P):

$$L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \mathbf{z}^t A(t)\mathbf{z} + \mathbf{z}^t B(t)\dot{\mathbf{z}} + \dot{\mathbf{z}}^t C(t)\dot{\mathbf{z}} + \mathbf{s}(t)^t \cdot \dot{\mathbf{z}} + \mathbf{z}^t P\dot{\mathbf{z}} + \mathbf{r}(t)^t \cdot \mathbf{z}$$

using the cyclical coordinate descent method.

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