

The Analytic Solution of the Firm's Cost-Minimization Problem with Box Constraints and the Cobb-Douglas model

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Abstract. One of the most well-known problems in the field of Microeconomics is the Firm's Cost-Minimization Problem. In this paper we establish the analytical expression for the cost function using the Cobb-Douglas model and considering maximum constraints for the inputs. Moreover we prove that it belongs to the class C^1 .

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INTRODUCTION

One of the most well-known problems in the field of Microeconomics is the Firm's Cost-Minimization Problem [1]. This problem can be expressed as follows: Produce a given output y , and choose inputs to minimize its cost:

$$\begin{aligned} c(\mathbf{w}, y) &= \min_{\mathbf{x} \geq \mathbf{0}} \mathbf{w}\mathbf{x} \\ \text{s.t. } f(\mathbf{x}) &= y \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^m$ are the inputs, and $\mathbf{w} \in \mathbb{R}^m$ are the factor prices. There are several different ways to mathematically express how inputs are transformed into outputs. The production function $f(\mathbf{x})$ maps a vector of input quantities into the maximal quantity of the output good that can be produced with these inputs. Popular production functions are (for the sake of simplicity, let us assume that we have only two inputs):

$$\begin{aligned} f(x_1, x_2) &= \min(ax_1, bx_2) && \text{(Leontief production function)} \\ f(x_1, x_2) &= x_1^\alpha x_2^\beta && \text{(Cobb-Douglas)} \\ f(x_1, x_2) &= ax_1 + bx_2 && \text{(Linear production function)} \end{aligned}$$

In this paper we study the classic Cobb-Douglas production function. The formulas for the production cost function $c(\mathbf{w}, y)$ are well known when the production function follows the Cobb-Douglas model. These formulas, which can be obtained by simply using the Lagrange multipliers method, present the inconvenience that they are not applicable when upper limit constraints are considered for the different inputs.

In this paper we establish the analytical expression for the cost function $c(\mathbf{w}, y)$ using the Cobb-Douglas model, considering maximum constraints for the inputs. Moreover, we prove, under certain assumptions, the existence and uniqueness of the equivalent minimizer and that it belongs to the class C^1 . Finally, the method is applied to a test system.

STATEMENT OF THE PROBLEM

Our problem (P1) will be:

$$\begin{aligned} c(\mathbf{w}, y) &= \min \sum_{i=1}^m w_i x_i \\ \text{s.t. } y &= A \prod_{i=1}^m x_i^{\alpha_i} \\ 0 &\leq x_i \leq M_i \end{aligned}$$

We shall usually measure units so that the efficiency parameter $A = 1$. The sum of α_i determines the returns to scale.

Problems of this kind, with box constraints, become complicated in the presence of Boundary solutions. We shall address this problem in an exact way in this paper, transforming it into a Non-Linear, Separable, Programming Problem [2], which we state as a constrained Infimal Convolution problem. The problem (P1) is equivalent to a new problem (P2):

$$\begin{aligned} \tilde{c}(\mathbf{w}, \xi) &= \min \sum_{i=1}^m w_i e^{\frac{1}{\alpha_i} p_i} \\ \text{s.t. } \sum_{i=1}^m p_i &= \xi \\ -\infty < p_i &\leq \alpha_i \ln M_i = P_i^{\max} \end{aligned}$$

in which only the following changes in the variables need to be taken into account:

$$\begin{aligned} \ln y &= \xi \\ \alpha_i \ln x_i &= p_i, \quad i = 1, \dots, m \end{aligned}$$

The function $\tilde{c}(\mathbf{w}, \cdot)$ is in fact the infimal convolution of the exponential functions

$$F_i(p_i) := w_i e^{\frac{1}{\alpha_i} p_i}$$

The case of quadratic F_i functions is well known and has been studied by the authors in [3] within the framework of hydrothermal optimization. However, the same kind of study is unknown for exponential functions. In this paper we develop the necessary mathematical tools to justify the proposed method for solving the stated problem.

EQUIVALENT MINIMIZER FOR (P2)

Let us calculate the equivalent minimizer for the functions $F_i(p_i)$. Let the function $F : (-\infty, P_1^{\max}] \times \dots \times (-\infty, P_m^{\max}] \rightarrow \mathbb{R}$ given by:

$$F(p_1, \dots, p_m) := \sum_{i=1}^m F_i(p_i)$$

Let C_ξ be the set:

$$C_\xi := \{(p_1, \dots, p_m) \in (-\infty, P_1^{\max}] \times \dots \times (-\infty, P_m^{\max}] \mid \sum_{i=1}^m p_i = \xi\}$$

Let us now see the definitions of the elements which are present in our problem.

Definition 1. Let us call *the i -th distribution function* the function $\Psi_i : (-\infty, \sum_{j=1}^m P_j^{\max}] \rightarrow (-\infty, P_i^{\max}]$ defined by

$$\Psi_i(\xi) = p_i, \quad \forall i = 1, \dots, m$$

where (p_1, \dots, p_m) is the unique minimum of $F(p_1, \dots, p_m)$ over the set C_ξ .

Definition 2. We shall call the *equivalent minimizer* of $\{F_i\}_1^m$, the function $\Psi : (-\infty, \sum_{j=1}^m P_j^{\max}] \rightarrow \mathbb{R}$ defined by

$$\Psi(\xi) = \min_{\sum_{i=1}^m p_i = \xi} \left[\sum_{i=1}^m F_i(p_i) \right]$$

Observation 1. It follows that

$$\sum_{i=1}^m \Psi_i(\xi) = \xi \quad \text{and} \quad \sum_{i=1}^m F_i(\Psi_i(\xi)) = \Psi(\xi)$$

Moreover, let us impose the following hypotheses: Let $\{F_i\}_{i=1}^m \subset C(-\infty, P_i^{\max}]$ be a set of functions such that:

- (H1) F_i' is strictly increasing, $\forall i = 1, \dots, m$
- (H2) ordered such that: $F_i'(P_i^{\max}) \leq F_{i+1}'(P_{i+1}^{\max})$, $\forall i = 1, \dots, m$

Note 1. If $w_i > 0$ and $\alpha_i > 0$, then (H1) is verified. This assumption is systematically accepted in the studies on this topic.

The following lemma guarantees that if p_i reaches its maximum value, all those p_k for which the derivative of F_k at its maximum value is less than or equal to the derivative corresponding to F_i will likewise have already reached their maximum.

Lemma 1. Under the above hypotheses, if the function $F(p_1, \dots, p_m)$ reaches at (a_1, \dots, a_m) the minimum over the set C_ξ , then, if for a certain $i \in \{1, \dots, m\}$, $a_i = P_i^{\max}$, then,

$$\forall k \in \{1, \dots, m\} / F'_k(P_k^{\max}) \leq F'_i(P_i^{\max}) \implies a_k = P_k^{\max}$$

The following lemma establishes the order of the points where the variables reach their maximum value.

Lemma 2. Under the above hypotheses, the parameters

$$\theta_k = \sum_{i=k}^m \frac{\alpha_i}{\alpha_k} P_k^{\max} + \sum_{i=k}^m \ln \left(\frac{\alpha_i w_k}{\alpha_k w_i} \right)^{\alpha_i} + \sum_{i=1}^{k-1} P_i^{\max}$$

satisfy

$$\theta_1 \leq \theta_2 \leq \dots \leq \theta_m = \sum_{i=1}^m P_i^{\max}$$

The following theorem establishes a Necessary and Sufficient Condition to obtain the Interior Solution.

Theorem 1. Under the above hypotheses, the function $F(p_1, \dots, p_m)$ reaches the minimum over the set C_ξ at the point $(a_1, \dots, a_m) \in \overset{\circ}{C}_\xi$ iff

$$\xi < \sum_{i=1}^m \frac{\alpha_i}{\alpha_1} P_1^{\max} + \sum_{i=1}^m \ln \left(\frac{\alpha_i w_1}{\alpha_1 w_i} \right)^{\alpha_i} = \theta_1$$

Having proven the above results, we are now in a position to obtain the Distribution Functions:

Theorem 2. For every $k = 1, \dots, m$ the k -th distribution function is

$$\Psi_k(\xi) = \begin{cases} \frac{\alpha_k}{\sum_{i=j+1}^m \alpha_i} \left[\xi + \sum_{i=j+1}^m \ln \left(\frac{\alpha_k w_i}{\alpha_i w_k} \right)^{\alpha_i} - \sum_{i=1}^j P_i^{\max} \right] & \text{if } \theta_j \leq \xi < \theta_{j+1} \leq \theta_k \\ P_k^{\max} & \text{if } \xi \geq \theta_k \end{cases}$$

with the coefficients:

$$\theta_k = \sum_{i=k}^m \frac{\alpha_i}{\alpha_k} P_k^{\max} + \sum_{i=k}^m \ln \left(\frac{\alpha_i w_k}{\alpha_k w_i} \right)^{\alpha_i} + \sum_{i=1}^{k-1} P_i^{\max}$$

We may also now express the Equivalent Minimizer:

Theorem 3. The equivalent minimizer is an exponential piece-wise function:

$$\Psi(\xi) = \begin{cases} \tilde{w}_1 e^{\frac{\xi}{\tilde{\alpha}_1}} & \text{if } \xi < \theta_1 \\ \tilde{\mu}_k + \tilde{w}_k e^{\frac{\xi}{\tilde{\alpha}_k}} & \text{if } \theta_{k-1} \leq \xi < \theta_k \end{cases}$$

with the coefficients:

$$\tilde{\mu}_k = \sum_{i=1}^{k-1} w_i e^{\frac{P_i^{\max}}{\tilde{\alpha}_i}} ; \quad \tilde{\alpha}_k = \sum_{i=k}^m \alpha_i ; \quad \tilde{w}_k = \exp \left[\left(- \sum_{i=1}^{k-1} P_i^{\max} \right) / \tilde{\alpha}_k \right] \left[\sum_{i=k}^m w_i \prod_{j=k}^m \left(\frac{\alpha_i w_j}{\alpha_j w_i} \right)^{\frac{\alpha_j}{\tilde{\alpha}_k}} \right]$$

Moreover, it belongs to the class C^1 .

SOLUTION OF THE PROBLEM (P1)

Considering the fact that $c(\mathbf{w}, y) = \tilde{c}(\mathbf{w}, \ln y)$, the following theorem is verified:

Theorem 4. *The demand of the k-th input is:*

$$x_k = \begin{cases} \exp \left[\left(- \sum_{i=1}^j P_i^{\max} \right) / \tilde{\alpha}_{j+1} \right] \cdot \prod_{i=j+1}^m \left(\frac{\alpha_k w_i}{\alpha_i w_k} \right)^{\frac{\alpha_i}{\tilde{\alpha}_{j+1}}} \cdot y^{\frac{1}{\tilde{\alpha}_{j+1}}} & \text{if } e^{\theta_j} \leq y < e^{\theta_{j+1}} \leq e^{\theta_k} \\ \frac{P_k^{\max}}{e^{\alpha_k}} & \text{if } y \geq e^{\theta_k} \end{cases}$$

and the cost function is:

$$c(\mathbf{w}, y) = \begin{cases} \tilde{w}_1 y^{\frac{1}{\tilde{\alpha}_1}} & \text{if } y < e^{\theta_1} \\ \tilde{\mu}_k + \tilde{w}_k y^{\frac{1}{\tilde{\alpha}_k}} & \text{if } e^{\theta_{k-1}} \leq y < e^{\theta_k} \end{cases}$$

being $\tilde{\mu}_k$, $\tilde{\alpha}_k$ and \tilde{w}_k the coefficients defined in Theorem 3.

CONCLUSIONS

In this paper we have established the analytical solution for the cost function in the classic firm's cost minimization problem in the general case with m inputs. We have used the Cobb-Douglas model for the production function and, for the first time, we have considered maximum constraints for the inputs. Our study has a number of advantages with respect to other methods: the exact boundary solution is obtained and the method is not affected by the dimension of the problem. At the same time, it is easy to generalize to other studies such as the classic profit maximization problem, including maximum constraints for the inputs.

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