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The valve points of the thermal cost function: A Hydrothermal Problem with non-regular Lagrangian

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Abstract: This paper deals with the optimization of a hydrothermal problem that considers non-regular Lagrangian $L(t,z,z')$. We consider a general case where the functions $L_{z'}(t,z,\cdot)$ and $L_z(t,z,\cdot)$ are discontinuous in $z'=\phi(t,z)$, which is the borderline point between two power generation zones. This situation arises in problems of optimization of hydrothermal systems where the thermal plant input-output curve considers the shape of the cost curve in the neighborhood of the valve points. The problem shall be formulated in the framework of nonsmooth analysis, using the generalized (or Clarke's) gradient. We shall obtain a new necessary minimum condition and we shall generalize the known result (smooth transition) that the derivative of the minimum presents a constancy interval. Finally, we shall present an example.

Keywords: Optimal Control, Clarke's Gradient, Hydrothermal Optimization

Mathematics Subject Classification: 49J24, 49A52

1 Introduction

In a previous paper [1], a problem of hydrothermal optimization with pumped-storage plants was considered. The problem consisted in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0,T]$. The mathematical problem was stated in the following terms:

$$\min_{z\in\Theta} F(z)=\min_{z\in\Theta}\int_0^T \Psi\left[P_d(t)-H(t,z(t),z'(t))\right]dt=\min_{z\in\Theta}\int_0^T L(t,z(t),z'(t))dt \tag{1.1}$$

$$\Theta=\{z\in\widehat{C}^1[0,T]\mid z(0)=0, z(T)=b\}$$

By ($\widehat{C}^1$) we denote the set of piecewise C^1 functions from $[0,T]$ to $\mathbb{R}$, P_d is the power demand, H the function of effective hydraulic generation, $z(t)$ the volume that is discharged up to the instant t by the hydroplant, $z'(t)$ the rate of water discharge at the instant t by the hydraulic plant, b the volume of water that must be discharged during the entire optimization interval and Ψ is the cost function of the thermal plant. In this kind of problem, the derivative of H with respect to z' ($H_{z'}$) presents discontinuity at $z'=0$, which is the border between the power generation zone (positive values of z') and the pumping zone (negative values of z').

Thus, the Lagrangian $L(\cdot,\cdot,\cdot):[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $L_z(\cdot,\cdot,\cdot)$ belong to class C^0 and the function $L_{z'}(t,z,\cdot)$ is piecewise continuous ($L_{z'}(t,z,\cdot)$ is discontinuous in $z'=0$). Denoting by $\yen_q(t), q\in\Theta$ the function:

$$\yen_q(t):=-L_{z'}(t,q(t),q'(t))+\int_0^t L_z(s,q(s),q'(s))ds \tag{1.2}$$

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and by $¥_q^+(t)$ and $¥_q^-(t)$ the expressions obtained when considering the lateral derivatives with respect to z'. The problem was formulated within the framework of nonsmooth analysis [2], using the generalized (or Clarke's) gradient, the following result being proven:

Theorem 1. *If q is a solution of* (1.1)*, then $\exists K \in \mathbb{R}^+$ such that:*

$$\begin{cases} ¥_q^+(t) = ¥_q^-(t) = K & \text{if} \quad q'(t) \neq 0 \\ ¥_q^+(t) \leq K \leq ¥_q^-(t) & \text{if} \quad q'(t) = 0 \end{cases} \tag{1.3}$$

In another previous paper [3], we presented a qualitative aspect of the solution: the *smooth transition*. The following result was proven: under certain convexity conditions, the discontinuity of the derivative of the Langrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verifed that the derivative of the extremal where the minimum is reached presents an interval of constancy, the constant being the value for which $L_{z'}(t, z, \cdot)$ presents discontinuity. The character C^1 of the solution is thus guaranteed.

This paper generalizes the two previous studies, considering a more general and non-regular Lagrangian: $L(\cdot,\cdot,\cdot)$ belongs to class C^0, but $L_{z'}(t, z, \cdot)$ and $L_z(t, z, \cdot)$ are piecewise continuous, i.e. both are discontinuous in

$$z' = \phi(t, z) \tag{1.4}$$

where ϕ belongs to class C^1. This situation arises in problems of optimization of hydrothermal systems where the thermal plant input-output curve considers the shape of the cost curve in the neighborhood of the valve points. Let us consider a thermal plant defined by several quadratic cost function such that Ψ is continuous but Ψ' is discontinuous at the valve points.

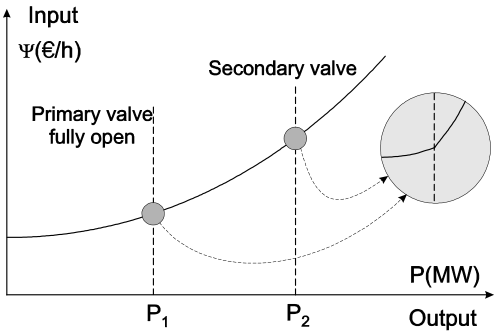

Fig. 1. Thermal plant input-output curve.

In Fig. 1 we see that Ψ' is discontinuous at P_1 and P_2. At P_1, for example, we have that

$$P_1 = P_d(t) - H(t, z(t), z'(t)) \Rightarrow z' = \phi(t, z) \tag{1.5}$$

so $L_{z'}(t, z, \cdot)$ and $L_z(t, z, \cdot)$ are discontinuous in $z' = \phi(t, z)$. We shall obtain a necessary minimum condition using the generalized (or Clarke's) gradient. Furthermore, we shall generalize the smooth transition and shall prove that the derivative of the minimum presents a interval where (1.4) is verified. Finally, we shall present a solution algorithm and shall apply it to an example.

2 A Necessary Condition

We now consider the mathematical problem

$$\min_{z\in\Theta} F(z) = \min_{z\in\Theta} \int_0^T \Psi\left[P_d(t) - H(t, z(t), z'(t))\right] dt = \min_{z\in\Theta} \int_0^T L(t, z(t), z'(t)) dt \tag{2.1}$$

$$\Theta = \{z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b\}$$

where $L(\cdot,\cdot,\cdot)$ belongs to class C^0, and $L_{z'}(t,z,\cdot)$ and $L_z(t,z,\cdot)$ are piecewise continuous (both are discontinuous in $z'=\phi(t,z)$).

Nonsmooth analysis [2] works with locally Lipschitz functions that are differentiable almost everywhere (the set of points at which f fails to be differentiable is denoted Ω_f). Let $f(x):\mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz near x, and let us assume that S is any set of Lebesgue measure 0 in $\mathbb{R}^n$. The generalized (or Clarke's) gradient ∂f can be calculated as a convex hull of (almost) all converging sequences of the gradients

$$\partial f(x) = \text{co}\left\{\lim \nabla f(x_i) : x_i \longrightarrow x,\ x_i \notin S,\ x_i \notin \Omega_f\right\} \tag{2.2}$$

We now extend this study to integral functionals, which will be taken over the σ-finite positive measure space $(\mathbb{T},\Im,\mu)=[0,T]$ with Lebesgue measure. $L^\infty(\mathbb{T},Y)$ denotes the space of measurable essentially bounded functions mapping $\mathbb{T}$ to Y, equipped with the usual supremum norm, with Y being the separable Banach space $Y=\mathbb{R}\times\mathbb{R}$. We are also given a closed subspace X of $L^\infty(\mathbb{T},Y)$

$$X=\left\{(s,v)\in L^\infty(\mathbb{T},Y)\ \text{ for some }\ c\in\mathbb{R},\ s(t)=c+\int_0^t v(\tau)d\tau\right\} \tag{2.3}$$

and a family of functions $f_t : Y \longrightarrow \mathbb{R}$ $(t\in\mathbb{T})$ with $f_t(s,v)=L(t,s,v)$. We define a function f

$$f(s,v)=\int_0^T L(t,s(t),v(t))dt$$

Under the above hypotheses, f is Lipschitz in a neighborhood of $(\widehat{s},\widehat{v})\in X$ and the following holds:

$$\partial f(\widehat{s},\widehat{v}) \subset \int_0^T \partial L(t,\widehat{s}(t),\widehat{v}(t))dt \tag{2.4}$$

Hence, if $\xi\in\partial f(\widehat{s},\widehat{v})$, we deduce the existence of a measurable function $\xi_t=(r(t),p(t))$ such that

$$(r(t),p(t))\in\partial L(t,\widehat{s}(t),\widehat{v}(t))\ a.e. \tag{2.5}$$

(∂L denotes the generalized gradient with respect to (s,v)) and where, for any $(s,v)\in X$

$$<\xi,(s,v)>=\int_0^T <\xi_t,(s,v)> dt = \int_0^T [r(t)s(t)+p(t)v(t)]\,dt \tag{2.6}$$

If $\xi=0$ (as when F attains a local minimum at $\widehat{s}$), then $0\in\partial f(\widehat{s},\widehat{v})$, it hence follows easily (Dubois-Reymond lemma) that $p(\cdot)$ is absolutely continuous and that $r=p'$ *a.e.* Thus, in this case we have a nonsmooth version (generalized subgradient version) of the Euler-Lagrange equation

$$(p'(t),p(t))\in\partial L(t,\widehat{s}(t),\widehat{s}'(t))\ a.e. \tag{2.7}$$

For our problem, we assume the following notations throughout the paper:

$$\begin{aligned} L^+_{z'}(t,z,z') &:= L_{z'}(t,z,z'_+); L^-_{z'}(t,z,z') := L_{z'}(t,z,z'_-) \\ L^+_{z}(t,z,z') &:= L_{z}(t,z,z'_+); L^-_{z}(t,z,z') := L_{z}(t,z,z'_-) \\ \yen^+_z(t) &= -L^+_{z'}(t,z(t),z'(t))+\int_0^t L^-_z(\tau,z(\tau),z'(\tau))d\tau \\ \yen^-_z(t) &= -L^-_{z'}(t,z(t),z'(t))+\int_0^t L^+_z(\tau,z(\tau),z'(\tau))d\tau \end{aligned} \tag{2.8}$$

With the above definitions, we can prove the following result (necessary condition for minimum).

Theorem 2. *If q is a solution of* (2.1)*, then $\exists K\in\mathbb{R}^+$ such that:*

$$\begin{cases} \yen^+_q(t)=\yen^-_q(t)=K & if \quad q'(t)\neq\phi(t,q(t)) \\ \yen^+_q(t)\leq K\leq\yen^-_q(t) & if \quad q'(t)=\phi(t,q(t)) \end{cases} \tag{2.9}$$

3 Smooth Transition

In this section, we present a qualitative aspect of the solution of (2.1). We prove that, under certain conditions, the discontinuity of the derivative of the Langrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verifed that the derivative of the extremal where the minimum is reached presents an interval where (1.4) is verified. The character C^1 of the solution is thus guaranteed.

Theorem 3. Let $L(\cdot,\cdot,\cdot)$ *be the Lagrangian of the functional* F *in the conditions stated above, and let us assume that the function* $L_{z'}(t_0, z(t_0), \cdot)$ *is strictly increasing and discontinuous in* $\phi(t_0, q(t_0))$. *If* q *is minimum for* F, *then* $q'(t) = \phi(t, q(t))$ *in some interval that contains* t_0 *and* q' *is continuous in* t_0.

This result has a very clear interpretation: under optimum operating conditions, thermal plants never switch brusquely from one generating power zone to other, but rather carry out a smooth transition, remaining above the boundary $q'(t) \equiv \phi(t, q(t))$ a certain interval.

4 Application to a Hydrothermal Problem

A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to an example of hydrothermal system made up of one thermal plant and one hydro plant. The Optimization Algorithm is very similar to the algorithm that we present in [3]. Let us consider a thermal plant with

$$\Psi(P) = \begin{cases} \alpha_1 + \beta_1 P + \gamma_1 P^2 & \text{if} \quad P_{\min} \le P < P_1 \\ \alpha_2 + \beta_2 P + \gamma_2 P^2 & \text{if} \quad P_1 \le P < P_2 \\ \alpha_3 + \beta_3 P + \gamma_3 P^2 & \text{if} \quad P_2 \le P < P_{\max} \end{cases} \tag{4.1}$$

where Ψ' is discontinuous at P_1 and P_2 (as we can see in Fig. 1). This model in the cost curves is due to sharp increases in throttle losses due to wire drawing effects occurring at valve points. These are loading (output) levels at which a new steam admission valve is being opened. The shape of the cost curve in the neighborhood of the valve points is difficult to determine by actual testing. Most utility systems find it satisfactory to represent the input-output characteristic by a smooth curve that can be defined by a polynomial or, even better, by means of a piecewise C^1 quadratic function. We accept this more approximate model.

For the power production H of the hydroplant (variable head), we consider a function of $z(t)$ and $z'(t)$ defined as

$$H(t, z(t), z'(t)) := A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t) \text{ with } A(t) = \frac{B_y}{G}(S_0 + t \cdot i), B = \frac{B_y}{G} \tag{4.2}$$

The parameters are: $G(m^4/h.Mw)$ representing efficiency, $i(m^3/h)$ the natural inflow, $S_0(m^3)$ the initial volume, and $B_y(m^{-2})$ a parameter that depends on the geometry of the tanks. We shall present the optimal solution.

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