Turing instability for a nonlinear reaction-diffusion system with cross-diffusion

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**Bifurcation:** qualitative change in behavior of the equilibrium of a system.

Due to variation of a bifurcation parameter. Leads to a new steady state.

Stability of uniform steady states related to the sign of $\lambda_k$ of the linearized system. If the steady state is initially stable, $\text{Re}(\lambda_k) < 0$.

At bifurcation, at least one eigenvalue crosses the imaginary axis:

- Turing bifurcation, in which one eigenvalue crosses the origin,
- Hopf bifurcation, where a pair of imaginary eigenvalues crosses the real axis and results in a limit cycle with oscillations.

We analyze the occurrence of **Turing bifurcation**.
The contents is extracted from:

Outline

1. Linear self-diffusion problem
   - Conditions for linear instability
   - Linear stability of the competitive Lotka-Volterra system

2. Cross-diffusion problem
   - Conditions for linear instability
     - Election of the bifurcation parameter
     - Critical value of the bifurcation parameter
   - Amplitude equations and weakly nonlinear analysis
   - The supercritical case
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Problem:

\[
\begin{align*}
\partial_t u_1 - d_1 \Delta u_1 &= \gamma f_1(u_1, u_2) & \text{in } Q_T = (0, T) \times \Omega, \\
\partial_t u_2 - d_2 \Delta u_2 &= \gamma f_2(u_1, u_2) & \text{in } Q_T, \\
\nabla u_1 \cdot n &= \nabla u_2 \cdot n = 0 & \text{on } \Gamma_T = \partial(0, T) \times \Omega, \\
\nabla u_1 \cdot n &= \nabla u_2 \cdot n = 0 & \text{in } \Omega.
\end{align*}
\]

We take \( \Omega = [0, 2\pi] \), and \((f_1, f_2)\) nonlinear, e.g. competitive Lotka-Volterra.

In vector form:

\[
\partial_t u - d \Delta u = \gamma f(u), \quad d = \text{diag}(d_1, d_2).
\]

Non-trivial uniform steady state \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2) \) are constant positive solutions of

\[
f(\tilde{u}) = 0.
\]
Linear self-diffusion problem

Linearization around $\tilde{u}$ gives, for $w = u - \tilde{u}$,

$$\partial_t w - d\Delta w = \gamma Df(\tilde{u})w,$$

with

$$Df(u) = \begin{pmatrix}
\partial_1 f_1(u) & \partial_2 f_1(u) \\
\partial_1 f_2(u) & \partial_2 f_2(u)
\end{pmatrix}.$$

We look for a particular solution of the form

$$w = \exp(\lambda_k t + ikx)u_k,$$

where

- $u_k$ is a constant eigenvector,
- $\lambda_k$ is an eigenvalue, representing the linear growth rate,
- $k$ is the wavenumber of the perturbation.
Upon substitution, one gets the eigenvalue problem

\[ A_k w = \lambda_k w, \quad \text{with} \quad A_k = \gamma Df(\bar{u}) - k^2 d. \]

For each \( k \), we obtain the particular solution

\[ (c_{1k} u_{1k} e^{\lambda_{1k} t} + c_{2k} u_{2k} e^{\lambda_{2k} t}) e^{ikx}, \]

where \( c_{jk} \) depend on the initial data.

The general solution can be expressed as

\[ w(t, x) = \sum_k \left( c_{1k} u_{1k} e^{\lambda_{1k} t} + c_{2k} u_{2k} e^{\lambda_{2k} t} \right) e^{ikx}. \]
The characteristic equation is, for $A_k = \gamma Df(\tilde{u}) - k^2 d$

$$\lambda_k^2 - \text{tr}(A_k)\lambda_k + \det(A_k) = 0,$$

where

$$\text{tr}(A_k) = \gamma(\partial_1 f_1(\tilde{u}) + \partial_2 f_2(\tilde{u})) - k^2(d_1 + d_2),$$
$$\det(A_k) = d_1 d_2 k^4 - \gamma(d_2 \partial_1 f_1(\tilde{u}) + d_1 \partial_2 f_2(\tilde{u}))k^2 + \gamma^2 \det(Df(\tilde{u})), $$

having the roots

$$\lambda_k = \frac{1}{2} \left( \text{tr}(A_k) \pm \sqrt{\text{tr}(A_k) - 4 \det(A_k)} \right).$$
We look for diffusion-driven instability: if no spatial variations ($k = 0$) then $\text{Re}(\lambda_{j0}) < 0$. This implies

$$\text{tr}(A_0) = \text{tr}(Df(\tilde{u})) < 0, \quad \text{det}(A_0) = \text{det}(Df(\tilde{u})) > 0.$$ 

Returning to spatial-dependent problem: look for changes of sign of $\text{Re}(\lambda_k)$ when varying diffusion coefficients.

We have $\text{tr}(A_k) < 0$. The only way for $\text{Re}(\lambda_k) > 0$ is $\text{det}(A_k) < 0$, with

$$\text{det}(A_k) = d_1 d_2 k^4 - \gamma(d_2 \partial_1 f_1(\tilde{u}) + d_1 \partial_2 f_2(\tilde{u})) k^2 + \gamma^2 \text{det}(Df(\tilde{u})).$$
The point of minimum is

\[ k_c^2 = \gamma \frac{d_2 \partial_1 f_1(\tilde{u}) + d_1 \partial_2 f_2(\tilde{u})}{2d_1 d_2}, \]

and the minimum value is

\[ h(k_c^2) = \gamma^2 \left[ \det(Df(\tilde{u})) - \frac{(d_2 \partial_1 f_1(\tilde{u}) + d_1 \partial_2 f_2(\tilde{u}))^2}{4d_1 d_2} \right]. \]

We have \( h(k_c^2) = 0 \) (bifurcation) if \( d_c \) is a positive root of

\[ (\partial_1 f_1(\tilde{u}))^2 d_c^2 + 2(2\partial_2 f_1(\tilde{u})\partial_1 f_2(\tilde{u}) - \partial_1 f_1(\tilde{u})\partial_2 f_2(\tilde{u})) d_c + (\partial_2 f_2(\tilde{u}))^2 = 0, \]

where \( d_c \) is the critical diffusion ratio.

If \( d_c \) exists, then critical wavenumber is obtained from \( k_c^2 \), with \( d_2^c/d_1^c = d_c \).
For $d^* > d_c$, exists a range of unstable wavenumbers in $[k_1^2, k_2^2]$, where $\det(A_{k_1}) = \det(A_{k_2}) = 0$.

The wavenumbers are discrete and a finite number in $[k_1^2, k_2^2]$.

Within this range, $\text{Re}(\lambda_k)$ is positive and assumes its maximum value for $k_c^2$.

For large $t$,

$$w(t, x) \approx \sum_{k=k_1}^{k_2} u_k e^{\lambda_k t} e^{ikx}.$$
Linear stability of the competitive Lotka-Volterra system

Consider

$$f_i(u) = \alpha_i u_i - (\beta_{i1} u_1 + \beta_{i2} u_2) u_i,$$

with $\alpha_i, \beta_{ij} \geq 0$, for $i, j = 1, 2$. The co-existence equilibrium is

$$\tilde{u} = \left( \frac{\beta_{22} \alpha_1 - \beta_{12} \alpha_2}{\beta_{11} \beta_{22} - \beta_{12} \beta_{21}}, \frac{\beta_{11} \alpha_2 - \beta_{21} \alpha_1}{\beta_{11} \beta_{22} - \beta_{12} \beta_{21}} \right),$$

with $\tilde{u}_i > 0$, for which

$$Df(u) = \begin{pmatrix} -\beta_{11} \tilde{u}_1 & -\beta_{12} \tilde{u}_1 \\ -\beta_{21} \tilde{u}_2 & -\beta_{22} \tilde{u}_2 \end{pmatrix}.$$
\( \tilde{u} \) is stable for the dynamical system if the eigenvalues of \( Df(u) \) are negative,
\[
\mu^2 - \text{tr}(Df(u)) + \det(Df(u)) = 0.
\]
Thus, the conditions are
\[
\text{tr}(Df(u)) < 0, \quad \text{and} \quad \det(Df(u)) > 0 \quad \text{(for negative real part)},
\]
\[
\text{tr}(Df(u))^2 - 4\det(Df(u)) \geq 0 \quad \text{(for null imaginary part)}.
\]
The second condition is equivalent to
\[
(\beta_{11}\tilde{u}_1 - \beta_{22}\tilde{u}_2)^2 + 4\beta_{12}\beta_{21}\tilde{u}_1\tilde{u}_2 \geq 0.
\]
Both conditions are satisfied if \( \beta_{ij} \geq 0 \), and
\[
\text{tr}(B) > 0, \quad \text{and} \quad \det(B) > 0, \quad \text{with} \quad B = (\beta_{ij}).
\]
Returning to the spatial-dependent problem and writing

\[ (\partial_1 f_1(\tilde{u}))^2 d_c^2 + 2(2\partial_2 f_1(\tilde{u})\partial_1 f_2(\tilde{u}) - \partial_1 f_1(\tilde{u})\partial_2 f_2(\tilde{u})) d_c + (\partial_2 f_2(\tilde{u}))^2 = 0, \]

as \(ad_c^2 + bd_c + c = 0\), the solutions are \(d_c = \frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})\). For real and positive solutions

\[ b^2 - 4ac > 0 \quad \text{and} \quad b < 0. \]

After some computations,

\[ b^2 - 4ac > 0 \iff \beta_{12}\beta_{21} > \beta_{11}\beta_{22}; \]

contradicts the stability assumption \(\det(B) > 0\) for the dynamical system.
Thus \(\tilde{u}\) is linearly stable for any choice of the diffusion coefficients.
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May the Lotka-Volterra system be instable in more complex situations?

We study the SKT cross-diffusion case:

\[
\begin{align*}
\partial_t u_1 - \text{div} J_1(u) &= \gamma f_1(u_1, u_2) \quad \text{in } Q_T, \\
\partial_t u_2 - \text{div} J_2(u) &= \gamma f_2(u_1, u_2) \quad \text{in } Q_T, \\
J_1(u) \cdot n &= J_2(u) \cdot n = 0 \quad \text{on } \Gamma_T, \\
u_1(\cdot, 0) &= u_{10}, \quad u_2(\cdot, 0) = u_{20} \quad \text{in } \Omega,
\end{align*}
\]

with flows and reaction terms

\[
\begin{align*}
J_i(u) &= \nabla \left( u_i(d_1 + a_{i1}\nabla u_1 + a_{i2}\nabla u_2) \right), \\
f_i(u) &= \alpha_i u_i - (\beta_{i1} u_1 + \beta_{i2} u_2) u_i,
\end{align*}
\]

with the coefficients \( B = (\beta_{ij}) \) satisfying the kinetic stability conditions

\[
\text{tr}(B) > 0, \quad \text{and} \quad \det(B) > 0.
\]
We study the co-existence homogeneous stationary state

\[ \tilde{u} = \left( \frac{\beta_{22}\alpha_1 - \beta_{12}\alpha_2}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}, \frac{\beta_{11}\alpha_2 - \beta_{21}\alpha_1}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} \right), \]

with \( \tilde{u}_i > 0 \), for which,

\[ K := Df(\tilde{u}) = \begin{pmatrix} -\beta_{11}\tilde{u}_1 & -\beta_{12}\tilde{u}_1 \\ -\beta_{21}\tilde{u}_2 & -\beta_{22}\tilde{u}_2 \end{pmatrix}. \]

Linearization around \( \tilde{u} \) gives the following system for \( w = u - \tilde{u} \)

\[ \partial_t w - D\Delta w = \gamma K w, \]

with

\[ D = \begin{pmatrix} d_1 + 2a_{11}\tilde{u}_1 + a_{12}\tilde{u}_2 & a_{12}\tilde{u}_1 \\ a_{21}\tilde{u}_2 & d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2 \end{pmatrix}. \]
The corresponding eigenvalue problem leads to

\[ \lambda_k^2 - \text{tr}(A_k)\lambda_k + h(k^2) = 0, \]

with \( A_k = \gamma K - k^2 D \), and

\[ h(k^2) = \det(A_k) = \det(D)k^4 + \gamma qk^2 + \gamma^2 \det(K), \]

being

\[ q = \beta_{11} \tilde{u}_1 \left( 2a_{22} \tilde{u}_2 + d_2 \right) + \beta_{22} \tilde{u}_2 \left( 2a_{11} \tilde{u}_1 + d_1 \right) + a_{12} \tilde{u}_2 \left( \beta_{22} \tilde{u}_2 - \beta_{21} \tilde{u}_1 \right) + a_{21} \tilde{u}_1 \left( \beta_{11} \tilde{u}_1 - \beta_{12} \tilde{u}_2 \right). \]
Conditions for linear instability

Spatial patterns arise for \( \text{Re}(\lambda_k) > 0 \).

Since \( \tilde{u} \) is stable for the kinetics then \( \text{tr}(A_k) < 0 \).

Therefore, the only way to have \( \text{Re}(\lambda_k) > 0 \) is \( h(k^2) < 0 \).

Condition for marginal stability

\[
\min(h(k_c^2)) = 0.
\]

The minimum of \( h \) is attained for

\[
k_c^2 = -\frac{\gamma q}{2 \det(D)},
\]

which requires \( q < 0 \).

The only potential destabilizing mechanism in \( q \) is the cross-diffusion.
Election of the bifurcation parameter

\[ q = \beta_{11} \tilde{u}_1 (2a_{22} \tilde{u}_2 + d_2) + \beta_{22} \tilde{u}_2 (2a_{11} \tilde{u}_1 + d_1) + a_{12} \tilde{u}_2 (\beta_{22} \tilde{u}_2 - \beta_{21} \tilde{u}_1) + a_{21} \tilde{u}_1 (\beta_{11} \tilde{u}_1 - \beta_{12} \tilde{u}_2). \]

Conditions on the positiveness and stability of \( \tilde{u} \) imply that

\[ \beta_{22} \tilde{u}_2 - \beta_{21} \tilde{u}_1 < 0 \quad \text{OR} \quad \beta_{11} \tilde{u}_1 - \beta_{12} \tilde{u}_2 < 0. \]

When \( a_{12} \) destabilizes then \( a_{21} \) stabilizes and vice versa.

We choose \( \beta_{22} \tilde{u}_2 - \beta_{21} \tilde{u}_1 < 0 \) and

\[ b := a_{12} \quad \text{as the bifurcation parameter.} \]
Critical value of the bifurcation parameter

Since $h(k^2)$ depends on $b$, one gets the bifurcation value from

$$\min(h(k_c^2)) = 0.$$ 

Consider

$$m_1 = \tilde{u}_2(\beta_{21} \tilde{u}_1 - \beta_{22} \tilde{u}_2) \geq 0,$$

$$m_2 = \beta_{11} \tilde{u}_1(2a_{22} \tilde{u}_2 + d_2) + \beta_{22} \tilde{u}_2(2a_{11} \tilde{u}_1 + d_1) + a_{21} \tilde{u}_1(\beta_{11} \tilde{u}_1 - \beta_{12} \tilde{u}_2) \geq 0,$$

so $q = -m_1 b + m_2$. The minimum value of $h(k^2)$ is

$$\min(h(k_c^2)) = \gamma^2 \left( \det(K) - \frac{(-m_1 b + m_2)^2}{4 \det(D)} \right).$$

Let $\xi \in \mathbb{R}$, to be determined, and set $b = m_2/m_1 + \xi$. We get the marginal stability condition

$$\frac{m_2^2}{4 \det(K)} \xi^2 - \det(D) = 0.$$
Replacing $a_{12} \equiv b = m_2 / m_1 + \xi$ in $D$, we get

$$\det(D) = \bar{u}_2(d_2 + 2a_{22}\bar{u}_2)\xi + \left(\frac{m_2}{m_1}\bar{u}_2(d_2 + 2a_{22}\bar{u}_2)\right)$$

$$+ (d_1 + 2a_{11}\bar{u}_1)(d_2 + a_{21}\bar{u}_1 + 2a_{22}\bar{u}_2).$$

Therefore, $\frac{m_1^2}{4\det(K)}\xi^2 - \det(D) = 0$ has a positive root, denoted by $\xi^+$. 

The critical value for bifurcation is

$$b^c = \frac{m_2}{m_1} + \xi^+.$$

Observe that $q := -m_1 b + m_2 < 0$ is guaranteed.
For $b > b^c$ the system has a finite $k$ pattern-forming stationary instability.

Unstable wavenumbers are between the roots of $h(k^2)$, denoted by $k_1^2$ and $k_2^2$.

It is straightforward to check that these roots are proportional to $\gamma$.

For pattern formation, $\gamma$ must be big enough so that at least one of the modes allowed by the boundary conditions is in $[k_1^2, k_2^2]$. 
Linear stability theory is useful for understanding pattern formation:

- Diffusion is the key mechanism.
- Determine conditions on system parameters.
- Gives length scale of pattern formation, $1/k_c$.

But,

- the exponentially growing solutions are physically meaningless.

**To predict the amplitude and the form, nonlinear terms must be included.**

We perform a weakly nonlinear analysis based on multiple scales.
Nonlinear expansion

In Turing bifurcation,

- Vlose to the bifurcation, \( \text{Re}(\lambda_k) < 0 \).
- The linear instability must be preceded by \( \text{Re}(\lambda_k) = 0 \).

Therefore, the pattern evolves on a slow temporal scale, like \( e^{\lambda_k t} \), with \( \lambda_k \approx 0 \).

- New, scaled, magnitudes are considered, and treated as separate variables.
- We fix a control parameter

\[
\varepsilon^2 = \frac{b - b_c}{b_c},
\]

and write the solution of the original system as a expansion in terms of \( \varepsilon^2 \).
Considering a random perturbation, \( w \), around the steady state, we recast the original nonlinear system as

\[
\partial_t w = \mathcal{L}^b w + \mathcal{N}^b w,
\]

where \( \mathcal{L}^b = \gamma K + D^b \Delta \) is the linear part, and \( \mathcal{N}^b \) contains 2nd order terms,

\[
\mathcal{N}^b = \frac{1}{2} Q_K(w, w) + \frac{1}{2} \Delta Q_D^b(w, w),
\]

with the bilinear forms

\[
Q_K(x, y) = \gamma \begin{pmatrix}
-2\beta_{11} x^1 y^1 - \beta_{12} (x^1 y^2 + x^2 y^1) \\
-2\beta_{22} x^2 y^2 - \beta_{21} (x^1 y^2 + x^2 y^1)
\end{pmatrix},
\]

\[
Q_D^b(x, y) = \begin{pmatrix}
2a_{11} x^1 y^1 + b(x^1 y^2 + x^2 y^1) \\
2a_{22} x^2 y^2 + a_{21} (x^1 y^2 + x^2 y^1)
\end{pmatrix}.
\]
The idea is:
Expand $w$ in $\varepsilon$, so the leading term is $A(t)e^{ik_0x}$, with $A$ slowly varying.

We consider

$$b = b^c + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 b_3 + O(\varepsilon^4),$$

$$w = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + O(\varepsilon^4),$$

$$\partial_t = \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2} + \varepsilon^3 \partial_{T_3} + O(\varepsilon^4).$$

Then,

$$D^b = \begin{pmatrix}
    d_1 + 2a_{11} \tilde{u}_1 + b\tilde{u}_2 & b\tilde{u}_1 \\
    a_{21} \tilde{u}_2 & d_2 + a_{21} \tilde{u}_1 + 2a_{22} \tilde{u}_2
\end{pmatrix}$$

$$= D^{bc} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix}
    b_j \tilde{u}_2 & b_j \tilde{u}_1 \\
    0 & 0
\end{pmatrix} + O(\varepsilon^4).$$
\( \mathcal{L}^b = \gamma K + D^b \Delta \) takes the form

\[
\mathcal{L}^b = \mathcal{L}^{bc} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix} b_j \tilde{u}_2 & b_j \tilde{u}_1 \\ 0 & 0 \end{pmatrix} \Delta + O(\varepsilon^4), \quad \text{with } \mathcal{L}^{bc} = \gamma K + D^{bc} \Delta.
\]

For the quadratic terms

\[
Q_K(\mathbf{w}, \mathbf{w}) = \varepsilon^2 Q_K(\mathbf{w}_1, \mathbf{w}_1) + 2\varepsilon^3 Q_K(\mathbf{w}_1, \mathbf{w}_2) + O(\varepsilon^4),
\]

\[
Q^b_D(\mathbf{w}, \mathbf{w}) = \varepsilon^2 Q^b_D(\mathbf{w}_1, \mathbf{w}_1) + 2\varepsilon^3 \left( Q^b_D(\mathbf{w}_1, \mathbf{w}_2) + (b_1 w_1^1 w_1^2, 0)^t \right) + O(\varepsilon^4).
\]

For the time derivative expansion,

\[
\partial_t \mathbf{w} = \varepsilon^2 \partial_{T_1} \mathbf{w}_1 + \varepsilon^3 \left( \partial_{T_1} \mathbf{w}_2 + \partial_{T_2} \mathbf{w}_1 \right) + O(\varepsilon^4).
\]
Introducing these expansions in $\partial_t w = \mathcal{L}^b w + \mathcal{N}^b w$, leads to

$$O(\varepsilon) : \quad \mathcal{L}^{bc} w_1 = 0,$$

$$O(\varepsilon^2) : \quad \mathcal{L}^{bc} w_2 = \partial_{T_1} w_1 - \frac{1}{2} (Q_K(w_1, w_1) + \Delta Q_D^b(w_1, w_1))$$

$$- b_1 \begin{pmatrix} \tilde{u}_2 \\ 0 \\ \tilde{u}_1 \end{pmatrix} \Delta w_1 =: F,$$

$$O(\varepsilon^3) : \quad \mathcal{L}^{bc} w_3 = \partial_{T_1} w_2 + \partial_{T_2} w_1 - Q_K(w_1, w_2) - \Delta Q_D^b(w_1, w_2) - b_1 \Delta \begin{pmatrix} w_1^1 & w_1^2 \\ 0 \end{pmatrix}$$

$$- \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} (b_1 \Delta w_2 + b_2 \Delta w_1) =: G,$$

with

$$\mathcal{L}^{bc} = \gamma K + D^{bc} \Delta.$$
Studying the orders of the expansion

We solve in \( x \in (0, 2\pi/k_c) \), and later adapt to \( \Omega = (0, 2\pi) \).

**Order \( \varepsilon \):** The solution of the linear problem \( L^{bc} w_1 = 0 \) in \( (0, 2\pi/k_c) \) with Neumann boundary conditions is

\[
 w_1 = A(T_1, T_2) \rho \cos(k_c x), \quad \text{with} \quad \rho \in \ker(\gamma K - k_c^2 D^{bc}),
\]

where \( A \) is still arbitrary.

\( \rho \) is defined up to a multiplicative constant, we normalize

\[
 \rho = (1, M)^t, \quad \text{with} \quad M = \frac{-\gamma K_{21} + D^{bc}_{21} k_c^2}{\gamma K_{22} - D^{bc}_{22} k_c^2},
\]

where \( K_{ij}, D^{bc}_{ij} \) are the \( i, j \)-entries of the matrices \( K \) and \( D^{bc} \).
Order $\varepsilon^2$: $L^{bc} w_2 = F$.

Observing that

$$Q_K(w_1, w_1) = A(T_1, T_2)^2 \cos^2(k_c x) Q_K(\rho, \rho),$$

$$Q^{bc}_D(w_1, w_1) = A(T_1, T_2)^2 \cos^2(k_c x) Q^{bc}_D(\rho, \rho),$$

and using standard trigonometric identities, we find

$$\frac{1}{2} \left( Q_K(w_1, w_1) + \Delta Q^{bc}_D(w_1, w_1) \right)$$

$$= \frac{1}{4} A(T_1, T_2)^2 \left( Q_K(\rho, \rho) + (Q_K(\rho, \rho) - 4k_c^2 Q^{bc}_D(\rho, \rho)) \cos(2k_c x) \right)$$

$$= \frac{1}{4} A(T_1, T_2)^2 \sum_{j=0,2} M_j(\rho, \rho) \cos(jk_c x),$$

with

$$M_j = Q_K - j^2 k_c^2 Q^{bc}_D.$$
Therefore,

\[ F = -\frac{1}{4}A(T_1, T_2)^2 \sum_{j=0,2} M_j(\rho, \rho) \cos(jk_c x) \]
\[ + \left( \partial_{T_1} A_1 \rho + b_1 k_c^2 A_1 (\bar{u}_2 + \bar{u}_1 M, 0)^t \right) \cos(k_c x). \]

Fredholm alternative: since \( \ker \mathcal{L}^{b^c} = \text{span}(\rho) \), a solution if and only if

\[ \langle F, \psi \rangle = 0 \quad \text{for} \quad \psi \in \ker((\mathcal{L}^{b^c})^*), \]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( L^2(0, 2\pi/k_c) \), and

\[ \psi = (1, M^*)^t \cos(k_c x), \quad \text{with} \quad M^* = \frac{-\gamma K_{12} + D_{12}^{b^c} k_c^2}{\gamma K_{22} - D_{22}^{b^c} k_c^2}. \]
0 = \langle \mathbf{F}, \psi \rangle = - \frac{1}{4} A^2 \sum_{j=0,2} \mathcal{M}_j(\rho, \rho)(1, M^*)^t \int_0^{2\pi/k_c} \cos(jk_cx) \cos(k_cx) \, dx \\
+ \left( \partial_{T_1} A \rho + b_1 k_c^2 A(\tilde{u}_2 + \tilde{u}_1 M, 0)^t \right)(1, M^*)^t \int_0^{2\pi/k_c} \cos^2(k_cx) \, dx.

The first integrand at the right hand side vanishes, so we obtain

$$
\partial_{T_1} A(T_1, T_2) = \chi A(T_1, T_2), \quad \text{with} \quad \chi = -\frac{b_1 k_c(\tilde{u}_2 + \tilde{u}_1 M)}{1 + MM^*}.
$$

No indication on the asymptotic behavior, $T_i \to \infty$, of the amplitude!

Suppress secular terms in $\mathbf{F}$ setting $T_1 = b_1 = 0$.

Then the compatibility condition is satisfied, and

$$
\mathbf{F} = - \frac{1}{4} A(T_2)^2 \sum_{j=0,2} \mathcal{M}_j(\rho, \rho) \cos(jk_cx).
$$
The solution of $\mathcal{L}^{bc}w_2 = F$ is explicitly computed: Introducing

$$w_2 = A(T_2)^2 \sum_{j=0,2} w_{2j} \cos(jk_c x),$$

we get

$$\mathcal{L}^{bc}w_2 = (\gamma K + D^{bc} \Delta)w_2 = A(T_2)^2 \sum_{j=0,2} \left( \gamma K - (jk_c)^2 D^{bc} \right) w_{2j} \cos(jk_c x).$$

$w_{2j}$ must satisfy the linear systems

$$L_j w_{2j} = -\frac{1}{4} \mathcal{M}_j(\rho, \rho), \quad \text{for } j = 0, 2,$$

with $L_j = \gamma K - j^2 k_c^2 D^{bc}$. 
Order $\varepsilon^3$: $L^{bc}w_3 = G$.

Since $T_1 = b_1 = 0$,

$$
G = \partial_{T_2}w_1 - Q_K(w_1, w_2) - \Delta Q_D^{bc}(w_1, w_2) - \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \end{pmatrix} b_2 \Delta w_1,
$$

where, for $\rho = (\tilde{u}_2 + \tilde{u}_1 M, 0)^t$,

$$
w_1 = A(T_2)\rho \cos(k_c x), \quad w_2 = A(T_2)^2(w_{20} + w_{22} \cos(2k_c x)).
$$

Then

$$
\partial_{T_2}w_1 = \rho \cos(k_c x) \partial_{T_2}A(T_2),
$$

$$
- \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \end{pmatrix} b_2 \Delta w_1 = A(T_2)k_c^2 \cos(k_c x)b_2(\tilde{u}_2 + \tilde{u}_1 M, 0)^t.
$$
Using that $Q_K$ and $Q_D^{bc}$ are bilinear, and

$$2 \cos(x) \cos(y) = \cos(x + y) + \cos(x - y),$$

and recalling

$$w_1 = A(T_2) \rho \cos(k_c x), \quad w_2 = A(T_2)^2 (w_{20} + w_{22} \cos(2k_c x)).$$

we get

$$Q_K(w_1, w_2) = A(T_2)^2 Q_K(w_1, w_{20}) + A(T_2)^2 \cos(2k_c x) Q_K(w_1, w_{22})$$

$$= A(T_2)^3 \cos(k_c x) Q_K(\rho, w_{20})$$

$$+ A(T_2)^3 \cos(2k_c x) \cos(k_c x) Q_K(\rho, w_{22})$$

$$= A(T_2)^3 \left( \cos(k_c x) (Q_K(\rho, w_{20})

+ \frac{1}{2} Q_K(\rho, w_{22})) + \frac{1}{2} \cos(3k_c x) Q_K(\rho, w_{22}) \right).$$
Similarly

$$
\Delta Q_D^{bc}(w_1, w_2) = A(T_2)^3 \left( -k_c^2 \cos(k_c x) \left( Q_D^{bc}(\rho, w_{20}) + \frac{1}{2} Q_D^{bc}(\rho, w_{22}) \right)
- \frac{9}{2} k_c^2 \cos(3k_c x) Q_D^{bc}(\rho, w_{22}) \right).
$$

Recalling the definition $M_j = Q_K - j^2 k_c^2 Q_D^{bc}$,

$$
Q_K(w_1, w_2) + \Delta Q_D^{bc}(w_1, w_2) = A(T_2)^3 \left( \cos(k_c x) \left( M_1(\rho, w_{20}) + \frac{1}{2} M_1(\rho, w_{22}) \right)
+ \frac{1}{2} \cos(3k_c x) M_3(\rho, w_{22}) \right).
$$

Thus,

$$
G = \left( \rho \partial_{T_2} A + G_1^{(1)} A + G_1^{(3)} A^3 \right) \cos(k_c x) + G_3 A^3 \cos(3k_c x),
$$

$$
G_1^{(1)} = (\tilde{u}_2 + \tilde{u}_1 M) k_c^2 b_2 (1, 0)^t,
$$

$$
G_1^{(3)} = - \left( M_1(\rho, w_{20}) + \frac{1}{2} M_1(\rho, w_{22}) \right),
$$

$$
G_3 = - \frac{1}{2} M_3(\rho, w_{22}).
$$
The solvability condition is $\langle \mathbf{G}, \psi \rangle = 0$, leading to

$$
\langle \rho \cos(k_c x), \psi \rangle \partial T_2 A + \langle \mathbf{G}_1^{(1)} \cos(k_c x), \psi \rangle A + \langle \mathbf{G}_1^{(3)} \cos(k_c x), \psi \rangle A^3 = 0.
$$

Thus, recalling the definition of $\psi = (1, M^*)^t \cos(k_c x)$, and defining

$$
\sigma = \frac{\mathbf{G}_1^{(1)} \cdot \eta}{\rho \cdot \eta}, \quad L = \frac{\mathbf{G}_3^{(1)} \cdot \eta}{\rho \cdot \eta},
$$

for $\eta = (1, M^*)^t$, the resulting Stuart-Landau equation is

$$
\partial T_2 A = \sigma A - LA^3.
$$

We may check that $\sigma > 0$. Two qualitatively cases depending on the sign of $L$:

- the supercritical case, for $L > 0$,
- the subcritical case, for $L < 0$. 
The supercritical case

If $\sigma, L > 0$ in $\partial T_2 A = \sigma A - LA^3$ then $A_\infty = \sqrt{\sigma/L}$.

The amplitude and the form of the asymptotic pattern is

$$\tilde{w} = \varepsilon \rho \sqrt{\frac{\sigma}{L}} \cos(k_c x) + \varepsilon^2 \frac{\sigma}{L} (w_{20} + w_{22} \cos(2k_c x)) + O(\varepsilon^3).$$

In general, this solution is not compatible with the Neumann boundary conditions in $\Omega = [0, 2\pi]$, that require $k_c$ to be integer or semi-integer.

We define $\bar{k}_c$ as the first integer or semi-integer to become unstable when $b > b_c$, and take

$$\tilde{w} = \varepsilon \rho \sqrt{\frac{\sigma}{L}} \cos(\bar{k}_c x) + \varepsilon^2 \frac{\sigma}{L} (w_{20} + w_{22} \cos(2\bar{k}_c x)) + O(\varepsilon^3).$$