Corrigendum to "Cullen numbers with the Lehmer Property"

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Abstract

In this note, we correct an oversight from the paper [2] mentioned in the title.

There is an error on Page 131 of the paper [2] mentioned in the title in justifying that the expression A is nonzero. After the sentence "Also, since m_p divides n_1, it follows that $u \le w$" on Page 131 in [2], the argument continues in the following way. The case when $\rho = 1$ implies $n_1 = 1$ and leads to the conclusion that all prime factors of C_n are Fermat primes, and this instance has been dealt with on Page 131 in [2]. Thus, we may assume that $\rho \ge 3$. The relation

$$(2^\alpha \rho^w + \alpha)u = wn_p$$

shows that $u \mid n_p$. Thus,

$$p = m_p 2^{n_p} + 1 = \rho^u 2^{n_p} + 1 = X^u + 1,$$

where $X = \rho 2^{n_p/u}$ is an integer. If $u > 1$, the above expression has $X + 1$ as a proper divisor > 1 (because u is odd), which is impossible since p is prime. Thus, $u = 1$. If $w = 1$, we first get that $m_p = n_1 = \rho$, and then that $n_p = \alpha + 2^\alpha\rho = \alpha + n$, so $p = C_n$, which is not allowed. Otherwise, $w \geq 3$, $n_1 = \rho^w$ and $p = \rho 2^{(\alpha+n)/w} + 1 = (n2^n)^{1/w} + 1$. We now show that there is at most one prime p with the above property. Indeed, assume that there are two of them p_1 and p_2, corresponding to $w_1 < w_2$. Thus, $n_1 = \rho_1^{w_1} = \rho_2^{w_2}$ and both w_1 and w_2 divide $n + \alpha$. Let $W = \mathrm{lcm}[w_1, w_2]$. Then $n_1 = \rho_0^W$ for some positive integer ρ_0. Furthermore, writing $W = w_1\lambda$, we have that $\lambda > 1$, and $\rho_0^\lambda = \rho_1$. Hence,

$$p_1 = \rho_1 2^{(\alpha+n)/w_1} + 1 = Y^\lambda + 1,$$

where $Y = \rho_0 2^{(\alpha+n)/W}$ is an integer. This is false since $\lambda > 1$ is odd, therefore the above expression $Y^\lambda + 1$ has $Y + 1$ as a proper divisor > 1, contradicting the fact that p_1 is prime. Hence, if A is zero for some p, then p is unique. Further, in this case $n_1 = \rho^w$ and $p = (n2^n)^{1/w} + 1 \leq (n2^n)^{1/3} + 1$.

The remaining of the argument from the paper [2] shows that the expression A is nonzero for all other primes q of C_n, so all prime factors q of C_n satisfy inequality (5) in the paper [2] with at most one exception, say p, which satisfies the inequality $p \leq (n2^n)^{1/3} + 1$. Hence, instead of the inequality from Line 2 of Page 132 in [2], we get that

$$C_n < ((n2^n)^{1/3} + 1)2^{6(k-1)(n \log n)^{1/2}},$$

giving

$$2^{6(k-1)(n \log n)^{1/2}} > \frac{n2^n}{(n2^n)^{1/3} + 1} > 2^{2n/3},$$

where the right–most inequality above holds for all $n \geq 3$. This leads to a slightly worse inequality than the inequality (6) in the paper [2], namely

$$k > 1 + \frac{n^{1/2}}{9(\log n)^{1/2}}. \tag{1}$$

Note that inequality (6) from the paper [2] still holds whenever $A \neq 0$ for all primes p dividing n, and in particular for all n except maybe when $n_1 = \rho^w$ for some $\rho \geq 3$ and $w \geq 3$. So, from now on, we shall treat only the case when $n_1 = \rho^w$. Comparing estimate (3) in the paper [2] with (1) leads to

$$\frac{n^{1/2}}{9(\log n)^{1/2}} < 2.4 \log n, \tag{2}$$

which implies that $n < 1.4 \times 10^6$. We now lower the bound in a way similar to the calculation on Page 132 in [2]. Namely, first if $2^{2^\gamma} + 1$ is a

Fermat prime factor of C_n, then $\gamma \leq 20$, so $\gamma \in \{0,1,2,3,4\}$. Furthermore, $\log n/\log 3 \leq 12.9$, therefore $k \leq 5+12=17$. Now inequality (1) shows that

$$\frac{n^{1/2}}{9(\log n)^{1/2}} < 16,$$

giving $n < 260,000$. But then $\log n/\log 3 \leq 11.4$, giving $k \leq 16$. Also, if n is not a multiple of 3, then the number of prime factors p of C_n with $m_p > 1$ is at most $\log 260,000/\log 5 < 7.8$. Thus, C_n can have at most $5+7=12$ distinct prime factors, contradicting the result of Cohen and Hagis [1]. Hence, $3 \mid n$ showing that 3 does not divide C_n. Thus, $k \leq 15$, so

$$\frac{n^{1/2}}{9(\log n)^{1/2}} < 14,$$

giving $n < 200,000$. Also, n cannot be divisible by a prime $q \geq 5$, for otherwise, since $n_1 = \rho^w$ for some $w \geq 3$, we would get that the number of prime factors p of C_n with $m_p > 1$ is at most $3+\log(200,000/q^3)/\log 3 < 9.8$, so $k \leq 9+4 = 13$, contradicting again the result of Cohen and Harris. Hence, $n = 2^\alpha \cdot 3^\beta$ and the proof finishes as in the paper [2] after formula (7).

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References

[1] G. L. Cohen and P. Hagis, 'On the number of prime factors of n if $\phi(n) \mid n-1$', *Nieuw Arch. Wisk.* **28** (1980), 177–185.

[2] J. M. Grau Ribas and F. Luca, "Cullen numbers with the Lehmer property", *Proc. Amer. Math. Soc.* **140** (2012), 129–134.