

## A HYDROTHERMAL PROBLEM WITH NON-SMOOTH LAGRANGIAN

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(Communicated by Helmut Maurer)

**ABSTRACT.** This paper deals with the optimization of a hydrothermal problem that considers a non-smooth Lagrangian  $L(t, z, z')$ . We consider a general case where the functions  $L_{z'}(t, \cdot, \cdot)$  and  $L_z(t, \cdot, \cdot)$  are discontinuous in  $\{(t, z, z')/z' = \phi(t, z)\}$ , which is the borderline point between two power generation zones. This situation arises in problems of optimization of hydrothermal systems where the thermal plant input-output curve considers the shape of the cost curve in the neighborhood of the valve points. The problem shall be formulated in the framework of nonsmooth analysis, using the generalized (or Clarke's) gradient. We shall obtain a necessary minimum condition and we shall generalize the known result (smooth transition) that the derivative of the minimum presents a constancy interval. Finally, we shall present an example.

**1. Introduction.** The economic dispatch (ED) problem ([18], [5]) is one of the important optimization problems in a hydrothermal power system. In two previous papers ([2], [3]) a problem of hydrothermal optimization with pumped-storage plants was considered. The problem consisted in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval  $[0, T]$ . The mathematical problem was stated in the following terms:

$$\min_{z \in \Theta} F(z) = \min_{z \in \Theta} \int_0^T \Psi [P_d(t) - H(t, z(t), z'(t))] dt = \min_{z \in \Theta} \int_0^T L(t, z(t), z'(t)) dt, \quad (1.1)$$

$$\Theta = \{z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b\}.$$

By  $(\widehat{C}^1)$ , we denote the set of piecewise  $C^1$  functions from  $[0, T]$  to  $\mathbb{R}$ ,  $b$  is the volume of water that must be discharged during the entire optimization interval,  $P_d : [0, T] \rightarrow \mathbb{R}$  is a continuous function that represents the power demand at each instant of the optimization interval,  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  is the cost function of the thermal plant,  $H : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  the function of effective hydraulic generation,  $z(t)$  the volume that is discharged up to the instant  $t$  by the hydroplant, and  $z'(t)$  the rate of water discharge at the instant  $t$  by the hydroplant. We shall assume that

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2010 *Mathematics Subject Classification.* Primary: 65K10, 49J52.

*Key words and phrases.* Optimization, nonsmooth analysis, variational methods, hydrothermal problem, valve-point.

This work was supported by the Oviedo University, project: UNOV-12-MA-01.

$\Psi$  is strictly increasing and strictly convex and that  $H$  is strictly increasing and concave with respect to  $z'$ . In this kind of problem, the derivative of  $H$  with respect to  $z'$  ( $H_{z'}$ ) presents discontinuity at  $z' = 0$ , which is the border between the power generation zone (positive values of  $z'$ ) and the pumping zone (negative values of  $z'$ ). Thus, the Lagrangian  $L(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $L_z(\cdot, \cdot, \cdot)$  belong to class  $C^0$  and the function  $L_{z'}(t, z, \cdot)$  is piecewise continuous ( $L_{z'}(t, z, \cdot)$  is discontinuous in  $z' = 0$ ).

Denoting by  $\Upsilon_z(t), z \in \Theta$  the function:

$$\Upsilon_z(t) := -L_{z'}(t, z(t), z'(t)) + \int_0^t L_z(s, z(s), z'(s)) ds \quad (1.2)$$

and by  $\Upsilon_z^+(t)$  and  $\Upsilon_z^-(t)$  the expressions obtained when considering the lateral derivatives with respect to  $z'$ . The problem was formulated within the framework of nonsmooth analysis [7], using the generalized (or Clarke's) gradient, the following result being proven:

**Theorem 1.** *If  $q$  is a solution of (1.1), then  $\exists K \in \mathbb{R}^+$  such that:*

$$\begin{cases} \Upsilon_q^+(t) = \Upsilon_q^-(t) = K & \text{if } q'(t) \neq 0, \\ \Upsilon_q^+(t) \leq K \leq \Upsilon_q^-(t) & \text{if } q'(t) = 0. \end{cases} \quad (1.3)$$

In another previous paper [4], we presented a qualitative aspect of the solution: the *smooth transition*. The following result was proven: under certain convexity conditions, the discontinuity of the derivative of the Lagrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verified that the derivative of the extremal where the minimum is reached presents an interval of constancy, the constant being the value for which  $L_{z'}(t, z, \cdot)$  presents discontinuity. The character  $C^1$  of the solution is thus guaranteed.

This paper generalizes the two previous studies, considering a more general and nonsmooth Lagrangian:  $L(\cdot, \cdot, \cdot)$  belongs to class  $C^0$ , but  $L_{z'}(t, \cdot, \cdot)$  and  $L_z(t, \cdot, \cdot)$  are continuous, except in

$$\Xi = \{(t, z, z')/z' = \phi(t, z)\}, \quad (1.4)$$

where  $\phi$  belongs to class  $C^1$ .

This situation arises in problems of optimization of hydrothermal systems in which the thermal plant input-output curve considers the shape of the cost curve in the neighborhood of the valve-points. Traditionally, the cost function for each thermal plant in the ED problem has been approximately represented by a single quadratic function

$$\Psi(P) = \alpha + \beta P + \gamma P^2 \quad \text{if} \quad P_{\min} \leq P < P_{\max}, \quad (1.5)$$

where  $\Psi(\text{Euro}/h)$  is the cost,  $P(\text{MW})$  is the power generated,  $P_{\min}$  and  $P_{\max}$  are the minimum and maximum power outputs of the unit, valve-point effects being ignored.

The ED problem with valve-point effects is represented as a nonsmooth optimization problem. Some studies of the ED problem, such as the improved genetic algorithm (IGA) [6], [1], a particle swarm optimization (PSO) technique [14], hybrid solution methodology integrating the particle swarm optimization (PSO) algorithm

with the sequential quadratic programming (SQP) method [19], the differential evolution (DE) algorithm combined with the sequential quadratic programming (SQP) technique [8], the Taguchi method [10], a novel Stochastic Search (SS) method [16], and a group search optimizer (GSO) methodology [20] consider valve-point effects.

The cost function is obtained from a data point taken during tests when input and output data are measured, as the thermal plant slowly varies through its operating region. The shape of the cost curve in the neighborhood of the valve-points is difficult to determine by testing. Wire drawing effects, which occur as each steam admission valve in a turbine starts to open, produce a rippling effect on the cost curve. This curve contains higher order nonlinearity and discontinuity due to the valve-point effect and should be refined. Therefore, (1.5) can be modified by a sine function

$$\Psi(P) = \alpha + \beta P + \gamma P^2 + e \sin [f(P_{\min} - P)],$$

where  $e$  and  $f$  are constants of the valve-point effect of generators, or by several piecewise quadratic functions

$$\Psi(P) = \begin{cases} \alpha_1 + \beta_1 P + \gamma_1 P^2 & \text{if } P_{\min} \leq P < P_1 \\ \alpha_2 + \beta_2 P + \gamma_2 P^2 & \text{if } P_1 \leq P < P_2 \\ \dots & \dots \\ \alpha_k + \beta_k P + \gamma_k P^2 & \text{if } P_k \leq P < P_{\max}. \end{cases} \quad (1.6)$$

We accept this more approximate model (1.6). So, let us consider a thermal plant defined by several quadratic cost function such that  $\Psi$  is continuous but  $\Psi'$  is discontinuous at the valve points (a piecewise  $C^1$  quadratic function). In Fig. 1 we see that  $\Psi'$  is discontinuous at  $P_1$  and  $P_2$ . At  $P_1$ , for example, we have that

$$P_1 = P_d(t) - H(t, z(t), z'(t)) \Rightarrow z' = \phi(t, z),$$

so  $L_{z'}(t, \cdot, \cdot)$  and  $L_z(t, \cdot, \cdot)$  are discontinuous in  $z' = \phi(t, z)$ .

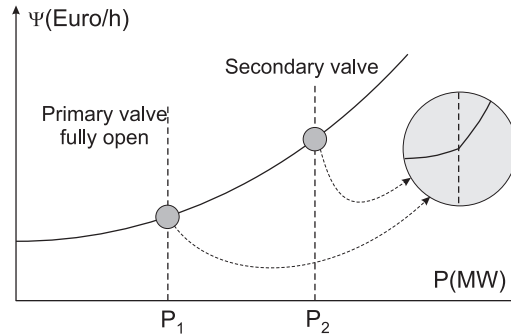


Fig. 1. Thermal plant cost curve.

The paper is organized as follows. In Section 2, we shall obtain a necessary minimum condition using the generalized (or Clarke's) gradient. The study of nonsmooth variational problems has been widely developed in recent years and the development of nonsmooth analysis has allowed researchers to deal with nonsmooth problems. In particular, it has permitted the achievement of nonsmooth versions of the classical Euler–Lagrange condition [11], [9], [17], [12], [13]. In Section 3, we shall generalize the smooth transition and shall prove that the derivative of the minimum presents a interval where  $z' = \phi(t, z)$  is verified. In Section 4, we shall present a solution

algorithm and shall apply it to an example. Finally, Section 5 summarizes the main conclusions of our research.

**2. A necessary condition.** Let us once more consider the mathematical problem (1.1):

$$\min_{z \in \Theta} F(z) = \min_{z \in \Theta} \int_0^T \Psi [P_d(t) - H(t, z(t), z'(t))] dt = \min_{z \in \Theta} \int_0^T L(t, z(t), z'(t)) dt, \quad (2.1)$$

$$\Theta = \{z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b\},$$

though now assuming new conditions: We shall assume a known  $P_d(t)$ , that  $\Psi$  is strictly increasing and strictly convex and that  $\Psi'$  is discontinuous at the valve points represented by the set  $\Xi = \{(t, z, z')/z' = \phi(t, z)\}$ , and that  $H$  verifies  $H_{z'} > 0$  and  $H_z < 0$ . In this problem, the Lagrangian  $L(\cdot, \cdot, \cdot)$  belongs to class  $C^0$ , but  $L_{z'}(t, \cdot, \cdot)$  and  $L_z(t, \cdot, \cdot)$  are discontinuous in  $\Xi$ .

We shall establish the necessary minimum condition for this problem with non-smooth Lagrangian, employing nonsmooth analysis for this purpose.

Nonsmooth analysis [9] works with locally Lipschitz functions that are differentiable almost everywhere (the set of points at which  $f$  fails to be differentiable is denoted by  $\Omega_f$ ). Let  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  be Lipschitz near  $x$ , and let us assume that  $S$  is any set of Lebesgue measure 0 in  $\mathbb{R}^n$ . The generalized (or Clarke's) gradient  $\partial f$  can be calculated as a convex hull of (almost) all converging sequences of the gradients

$$\partial f(x) = \text{co} \{ \lim \nabla f(x_i) : x_i \rightarrow x, x_i \notin S, x_i \notin \Omega_f \}. \quad (2.2)$$

We now extend this study to integral functionals, which will be taken over the  $\sigma$ -finite positive measure space  $(\mathbb{T}, \mathfrak{F}, \mu) = [0, T]$  with Lebesgue measure.  $L^\infty(\mathbb{T}, Y)$  denotes the space of measurable essentially bounded functions mapping  $\mathbb{T}$  to  $Y$ , equipped with the usual supremum norm, with  $Y$  being the separable Banach space  $Y = \mathbb{R} \times \mathbb{R}$ . We are also given a closed subspace  $X$  of  $L^\infty(\mathbb{T}, Y)$

$$X = \left\{ (s, v) \in L^\infty(\mathbb{T}, Y) \text{ for some } c \in \mathbb{R}, s(t) = c + \int_0^t v(\tau) d\tau \right\}$$

and a family of functions  $f_t : Y \rightarrow \mathbb{R}$  ( $t \in \mathbb{T}$ ) with  $f_t(s, v) = L(t, s, v)$ . We define a function  $f$

$$f(s, v) = \int_0^T L(t, s(t), v(t)) dt.$$

Under the above hypotheses,  $f$  is Lipschitz in a neighborhood of  $(\widehat{s}, \widehat{v}) \in X$  and the following [7] holds:

$$\partial f(\widehat{s}, \widehat{v}) \subset \int_0^T \partial L(t, \widehat{s}(t), \widehat{v}(t)) dt. \quad (2.3)$$

Hence, if  $\xi \in \partial f(\widehat{s}, \widehat{v})$ , we deduce the existence of a measurable function  $\xi_t = (r(t), p(t))$  such that

$$(r(t), p(t)) \in \partial L(t, \widehat{s}(t), \widehat{v}(t)) \text{ a.e.}$$

( $\partial L$  denotes the generalized gradient with respect to  $(s, v)$ ) and where, for any  $(s, v) \in X$

$$\langle \xi, (s, v) \rangle = \int_0^T \langle \xi_t, (s, v) \rangle dt = \int_0^T [r(t)s(t) + p(t)v(t)] dt.$$

If  $\xi = 0$  (as when  $F$  attains a local minimum at  $\widehat{s}$ ), then  $0 \in \partial f(\widehat{s}, \widehat{v})$ , it hence follows easily ([15], Dubois-Reymond lemma) that  $p(\cdot)$  is absolutely continuous and that  $r = p'$  a.e. In this case, therefore, we have a nonsmooth version (generalized subgradient version) of the Euler-Lagrange equation

$$(p'(t), p(t)) \in \partial L(t, \widehat{s}(t), \widehat{s}'(t)) \text{ a.e.} \quad (2.4)$$

For our problem, we assume the following notations throughout the paper: denoting by  $z$  the state variable, and by  $q$  the optimal value of said variable. We denote:

$$L(t, z, z') := \begin{cases} L^+(t, z, z') & \text{if } z' \geq \phi(t, z), \\ L^-(t, z, z') & \text{if } z' \leq \phi(t, z). \end{cases} \quad (2.5)$$

We shall use  $L_z^+$  and  $L_{z'}^+$  to represent the derivatives of  $L$  with respect to  $z$  and  $z'$ , respectively, when  $z' \geq \phi(t, z)$ , and, similarly,  $L_z^-$  and  $L_{z'}^-$  when  $z' \leq \phi(t, z)$ .

We define by  $\Upsilon_q^1(t)$  the set of instants prior or equal to  $t$ , where  $\phi(\cdot, q(\cdot)) - q'(\cdot)$  is no longer null and the Lagrangian becomes smooth,

$$\Upsilon_q^1(t) := \left\{ s / 0 \leq s \leq t \wedge \exists \delta > 0 / \begin{cases} \forall \varepsilon \in [s - \delta, s], q'(\varepsilon) = \phi(\varepsilon, q(\varepsilon)) \\ \forall \varepsilon \in [s, s + \delta], q'(\varepsilon) \neq \phi(\varepsilon, q(\varepsilon)) \end{cases} \right\}$$

and we denote by  $N(t)$  its cardinal:

$$N(t) = \text{card}(\Upsilon_q^1(t))$$

and its elements by

$$\Upsilon_q^1(t) = \{t_{2k}/1 \leq k \leq N(t)\}.$$

We define by  $\Gamma_q^1(t)$  the set of instants prior or equal to  $t$ , where  $\phi(\cdot, q(\cdot)) - q'(\cdot)$  becomes null; i.e. the Lagrangian becomes nonsmooth,

$$\Gamma_q^1(t) := \left\{ s / 0 \leq s \leq t \wedge \exists \delta > 0 / \begin{cases} \forall \varepsilon \in [s - \delta, s], q'(\varepsilon) \neq \phi(\varepsilon, q(\varepsilon)) \\ \forall \varepsilon \in [s, s + \delta], q'(\varepsilon) = \phi(\varepsilon, q(\varepsilon)) \end{cases} \right\}$$

and we denote by  $m(t)$  its cardinal:

$$m(t) = \text{card}(\Gamma_q^1(t))$$

and its elements by

$$\Gamma_q^1(t) = \{t_{2k-1}/1 \leq k \leq m(t)\}.$$

We shall assume that the set  $\Xi$  in (1.4) is not “active” at  $t = 0$  and  $t = T$ , the reasoning would be analogous in any other case. Moreover, there are no “transition points” where formally  $t_{2k-1} = t_{2k}$  holds (a fact which will be demonstrated in the following section and which we shall call “smooth transition”). We hence have:

$$0 < t_1 < t_2 < \dots < t_{2k-1} < t_{2k} < \dots < t_{2N(T)-1} < t_{2N(T)} < T.$$

We shall call times  $t_{2k-1}$  “entry-times” and times  $t_{2k}$  “exit-times”. We define by  $\Upsilon_q^2(t)$  the set formed by those intervals of continuity of  $L_{z'}(s, q(s), q'(s))$  and  $L_z(s, q(s), q'(s))$  lower than or equal to  $t$ , denoted by  $t_0 = 0$ ,

$$\Upsilon_q^2(t) := \left\{ \bigcup_{i=0}^{N(t)} [t_{2i}, t_{2i+1}] / t_{2i+1} \leq t \right\}.$$

We denote by  $\delta_q(t)$  the value of the integral of  $L_z$  in the areas of discontinuity it presents prior to  $t$ :

$$\delta_q(t) := \begin{cases} \sum_{i=1}^{N(t)} \int_{t_{2i-1}}^{t_{2i}} L_z^+(\tau, q(\tau), q'(\tau)) d\tau & \text{if } \begin{matrix} \exists \mu > 0 / \\ \forall \varepsilon \in (t_{2i}, t_{2i} + \mu), q'(\varepsilon) > \phi(\varepsilon, q(\varepsilon)) \end{matrix} \\ \sum_{i=1}^{N(t)} \int_{t_{2i-1}}^{t_{2i}} L_z^-(\tau, q(\tau), q'(\tau)) d\tau & \text{if } \begin{matrix} \exists \mu > 0 / \\ \forall \varepsilon \in (t_{2i}, t_{2i} + \mu), q'(\varepsilon) < \phi(\varepsilon, q(\varepsilon)) \end{matrix} \end{cases}$$

and by  $\mathbb{Y}_q^+(t)$  and  $\mathbb{Y}_q^-(t)$  the functions:

$$\begin{aligned} \mathbb{Y}_q^+(t) &:= -L_{z'}^+(t, q(t), q'(t)) + \int_{\Upsilon_q^2(t)} L_z(\tau, q(\tau), q'(\tau)) d\tau + \\ &\quad + \delta_q(t) + \int_{t_{2N(t)}}^t L_z^+(\tau, q(\tau), q'(\tau)) d\tau, \\ \mathbb{Y}_q^-(t) &:= -L_{z'}^-(t, q(t), q'(t)) + \int_{\Upsilon_q^2(t)} L_z(\tau, q(\tau), q'(\tau)) d\tau + \\ &\quad + \delta_q(t) + \int_{t_{2N(t)}}^t L_z^-(\tau, q(\tau), q'(\tau)) d\tau. \end{aligned} \quad (2.6)$$

With the hypotheses imposed on problem (2.1) at Section 2, which are verified in our hydraulic model, is easy to prove that  $L_{z'}^+ > L_{z'}^-$  and  $L_z^+ < L_z^-$  in  $\Xi$ . With the above definitions, we can prove the following result (necessary condition for minimum).

**Theorem 2.** *If  $q$  is a solution of (2.1), then  $\exists K \in \mathbb{R}^+$  such that:*

$$\begin{cases} \mathbb{Y}_q^+(t) = \mathbb{Y}_q^-(t) = K & \text{if } q'(t) \neq \phi(t, q(t)), \\ \mathbb{Y}_q^+(t) \leq K \leq \mathbb{Y}_q^-(t) & \text{if } q'(t) = \phi(t, q(t)). \end{cases} \quad (2.7)$$

*Proof.* It is easy to see that the functional (2.1) satisfies the necessary conditions to verify (2.3). Bearing in mind that the functions  $L_z(t, \cdot, \cdot), L_{z'}(t, \cdot, \cdot)$  are discontinuous in  $\Xi$ , using (2.2), we have that the generalized gradient is

$$\partial L(t, q(t), q'(t)) = \begin{cases} (L_z, L_{z'}) & \text{if } q'(t) \neq \phi(t, q(t)) \\ (L_z^-, L_{z'}^-) + u(t)(L_z^+ - L_z^-, L_{z'}^+ - L_{z'}^-) & \text{if } q'(t) = \phi(t, q(t)) \end{cases}$$

for  $0 \leq u(t) \leq 1$ , where  $(t, q(t), q'(t))$  is the argument of the functions  $L_z^+, L_z^-, L_{z'}^+$  and  $L_{z'}^-$ .

Hence, Equation (2.4) is

$$(p'(t), p(t)) \in \begin{cases} (L_z, L_{z'}) & \text{if } q'(t) \neq \phi(t, q(t)) \\ ((L_z^-, L_{z'}^-) + u(t)(L_z^+ - L_z^-, L_{z'}^+ - L_{z'}^-)) & \text{if } q'(t) = \phi(t, q(t)) \end{cases} \quad a.e. \quad (2.8)$$

We shall first prove that the theorem is verified in an interval  $[t_0, t_2]$ ,  $t_0 = 0$ ,  $t_2 \in \Upsilon_q^1(t)$  where the solution  $q$  of (2.1) reaches points in  $\Xi$ . We shall then generalize for  $t \geq t_2$ , where the solution shall be formed by a concatenation of arcs  $q'(t) \neq \phi(t, q(t))$  and  $q'(t) = \phi(t, q(t))$ , like those already studied in  $[t_0, t_2]$ .

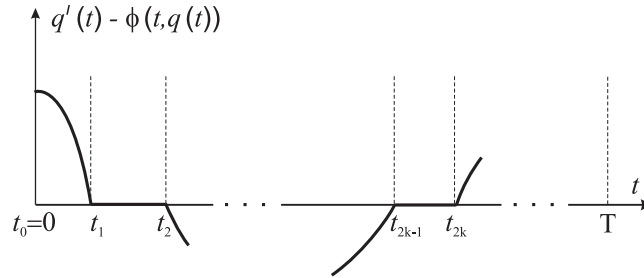


Fig. 2. Concatenating the extremal arcs and the boundary arcs.

Let us assume that  $q'(t) > \phi(t, q(t))$  in  $[0, t_1)$  and  $q'(t) = \phi(t, q(t))$  for  $t \in [t_1, t_2]$ .

We shall reason analogically in the case that  $q'(t) < \phi(t, q(t))$  in  $[0, t_1)$  and  $q'(t) = \phi(t, q(t))$  for  $t \in [t_1, t_2]$ .

Hence, from (2.8), for  $q'(t) = \phi(t, q(t))$ , we have that:

$$\begin{cases} p'(t) = u(t)L_z^+(t, q(t), q'(t)) + (1 - u(t))L_z^-(t, q(t), q'(t)) \\ p(t) = u(t)L_{z'}^+(t, q(t), q'(t)) + (1 - u(t))L_{z'}^-(t, q(t), q'(t)) \end{cases}; 0 \leq u(t) \leq 1. \quad (2.9)$$

An expression that is also verified for  $q'(t) \neq \phi(t, q(t))$  by simply bearing in mind (2.5) and (2.6), since:

$$\begin{aligned} L_{z'}(t, z, z') &= L_z^+(t, z, z') = L_z^-(t, z, z'), \\ L_z(t, z, z') &= L_{z'}^+(t, z, z') = L_{z'}^-(t, z, z'). \end{aligned}$$

From (2.9), integrating  $p'(t)$  in  $[0, t]$ ,  $t \leq t_2$ , we have that

$$\begin{cases} p(t) - p(0) = \int_0^t (u(\tau)L_z^+(\tau, q(\tau), q'(\tau)) + (1 - u(\tau))L_z^-(\tau, q(\tau), q'(\tau))) d\tau, \\ -p(t) = -u(t)L_{z'}^+(t, q(t), q'(t)) - (1 - u(t))L_{z'}^-(t, q(t), q'(t)), \end{cases}$$

with  $0 \leq u(t) \leq 1$ . Summing, and denoting  $-p(0) = K$ , we have that

$$\begin{aligned} K &= -u(t)L_{z'}^+(t, q(t), q'(t)) - (1 - u(t))L_{z'}^-(t, q(t), q'(t)) + \\ &+ \int_0^t (u(\tau)L_z^+(\tau, q(\tau), q'(\tau)) + (1 - u(\tau))L_z^-(\tau, q(\tau), q'(\tau))) d\tau, \end{aligned}$$

with  $0 \leq u(t) \leq 1$  and as  $L_{z'}^+ > L_z^-$  and  $L_z^+ < L_{z'}^-$ , we can conclude that

$$\begin{aligned} -L_{z'}^+(t, q(t), q'(t)) + \int_0^t L_z^+(\tau, q(\tau), q'(\tau)) d\tau &\leq K \leq \\ &\leq -L_{z'}^-(t, q(t), q'(t)) + \int_0^t L_z^-(\tau, q(\tau), q'(\tau)) d\tau. \end{aligned} \quad (2.10)$$

i) If  $0 \leq t \leq t_1$ ,  $N(t) = 0$  and  $q'(t) \neq \phi(t, q(t))$ , then  $L_{z'}^+ \equiv L_{z'}^-, L_z^+ \equiv L_z^-$  and from (2.10)

$$\begin{aligned} -L_{z'}^+(t, q(t), q'(t)) + \int_0^t L_z^+(\tau, q(\tau), q'(\tau)) d\tau &= K \\ K &= -L_{z'}^-(t, q(t), q'(t)) + \int_0^t L_z^-(\tau, q(\tau), q'(\tau)) d\tau, \end{aligned}$$

from which

$$\mathbb{Y}_q^+(t) = \mathbb{Y}_q^-(t) = K.$$

ii) If  $t < t_2$ ,  $N(t) = 0$  and in such a case from (2.10), we have that

$$\mathbb{Y}_q^+(t) \leq K \leq \mathbb{Y}_q^-(t).$$

iii) If  $t = t_2$ ,  $t_2 \in \Upsilon_q^1(t_2)$ ,  $N(t_2) = 1$  and, in (2.10) an equality will be verified, for example

$$K = -L_{z'}^-(t_2, q(t_2), q'(t_2)) + \int_0^{t_2} L_z^-(\tau, q(\tau), q'(\tau)) d\tau.$$

Moreover,

$$\begin{aligned} \int_0^{t_2} L_z^-(\tau, q(\tau), q'(\tau)) d\tau &= \int_0^{t_1} L_z^-(\tau, q(\tau), q'(\tau)) d\tau + \int_{t_1}^{t_2} L_z^-(\tau, q(\tau), q'(\tau)) d\tau \\ &= \int_{\Upsilon_q^2(t_2)} L_z^-(\tau, q(\tau), q'(\tau)) d\tau + \delta_q(t), \end{aligned}$$

from which

$$K = -L_{z'}^-(t_2, q(t_2), q'(t_2)) + \int_{\Upsilon_q^2(t_2)} L_z^-(\tau, q(\tau), q'(\tau)) d\tau + \delta_q(t) = \mathbb{Y}_q^-(t_2). \quad (2.11)$$

iv) If  $t \geq t_2$ ,  $\forall s \in [t_2, t]$ ,  $q'(s) \neq \phi(s, q(s))$ , we shall reiterate the process once more, taking  $t_2$  as the initial point. For the concatenation of arcs, we must bear in mind that:

On the one hand, we are in a zone in which  $q'(t) \neq \phi(t, q(t))$  ( $L_{z'}^+ \equiv L_{z'}^-, L_z^+ \equiv L_z^-$ ) and, hence,  $\exists K^*$  such that

$$K^* = -L_{z'}^-(t, q(t), q'(t)) + \int_{t_2}^t L_z^-(\tau, q(\tau), q'(\tau)) d\tau,$$

with

$$K^* = -L_{z'}^-(t_2, q(t_2), q'(t_2)).$$

On the other hand, from (2.11) it is verified that

$$K = -L_{z'}^-(t_2, q(t_2), q'(t_2)) + \int_{\Upsilon_q^2(t)} L_z^-(\tau, q(\tau), q'(\tau)) d\tau + \delta_q(t),$$

from which

$$-L_{z'}^-(t_2, q(t_2), q'(t_2)) = K - \int_{\Upsilon_q^2(t)} L_z^-(\tau, q(\tau), q'(\tau)) d\tau - \delta_q(t).$$

Therefore,

$$K^* = K - \int_{\Upsilon_q^2(t)} L_z^-(\tau, q(\tau), q'(\tau)) d\tau - \delta_q(t).$$

Thus,

$$K - \int_{\Upsilon_q^2(t)} L_z(\tau, q(\tau), q'(\tau)) d\tau - \delta_q(t) = -L_{z'}(t, q(t), q'(t)) + \int_{t_2}^t L_z(\tau, q(\tau), q'(\tau)) d\tau,$$

from which

$$K = -L_{z'}(t, q(t), q'(t)) + \int_{\Upsilon_q^2(t)} L_z(\tau, q(\tau), q'(\tau)) d\tau + \delta_q(t) + \int_{t_2}^t L_z(\tau, q(\tau), q'(\tau)) d\tau,$$

and in such a case, once more

$$K = \mathbb{Y}_q^+(t) = \mathbb{Y}_q^-(t).$$

Once the arcs of the extremal have been concatenated in  $t_2$ , the theorem is demonstrated, simply be bearing in mind that only situations like those already analyzed above may arise.  $\square$

**3. Smooth transition.** In this section, we present a qualitative aspect of the solution of (2.1). We prove that, under certain conditions, the discontinuity of the derivative of the Lagrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verified that the derivative of the extremal where the minimum is reached presents an interval where  $z' = \phi(t, z)$  is verified. The character  $C^1$  of the solution is thus guaranteed.

**Theorem 3.** *Let  $L(\cdot, \cdot, \cdot)$  be the Lagrangian of the functional  $F$  in the conditions stated above, and let us assume that the function  $L_{z'}(t_0, z(t_0), \cdot)$  is strictly increasing and discontinuous in  $\phi(t_0, z(t_0))$ . If  $q$  is minimum for  $F$ , and  $q'(t_0) = \phi(t_0, q(t_0))$ , then  $q'(t) = \phi(t, q(t))$  in some interval that contains  $t_0$  and  $q'$  is continuous in  $t_0$ .*

*Proof.* Let us see, first, that  $q'(t) = \phi(t, q(t))$  in some interval that contains  $t_0$ . We shall proceed by contradiction.

Let  $q \in \Theta$  be a minimum of  $F$ , and let us first assume that for  $t_0 \in (0, T)$  there exist  $\varepsilon > 0$  such that:

$$\begin{aligned} q'(t) &< \phi(t, q(t)), \quad \forall t \in (t_0 - \varepsilon, t_0), \\ q'(t) &> \phi(t, q(t)), \quad \forall t \in (t_0, t_0 + \varepsilon). \end{aligned}$$

The strict growth of  $L_{z'}$ , implies that:

$$\begin{aligned} L_{z'}(t, q(t), q'(t)) &< L_{z'}^-(t_0, q(t_0), \phi(t_0, q(t_0))), \quad \forall t \in (t_0 - \varepsilon, t_0) \\ L_{z'}^+(t_0, q(t_0), \phi(t_0, q(t_0))) &< L_{z'}(\tilde{t}, q(\tilde{t}), q'(\tilde{t})), \quad \forall \tilde{t} \in (t_0, t_0 + \varepsilon) \end{aligned}$$

and, together with the discontinuity of  $L_{z'}$ , we have:

$$\begin{aligned} L_{z'}(t, q(t), q'(t)) &< L_{z'}^-(t_0, q(t_0), \phi(t_0, q(t_0))) < \\ &< L_{z'}^+(t_0, q(t_0), \phi(t_0, q(t_0))) < L_{z'}(\tilde{t}, q(\tilde{t}), q'(\tilde{t})), \quad \forall t \in (t_0 - \varepsilon, t_0), \forall \tilde{t} \in (t_0, t_0 + \varepsilon). \end{aligned} \quad (3.1)$$

We consider the auxiliary function  $h_\varepsilon^{t_0}$  defined on  $[0, T]$ :

$$h_\varepsilon^{t_0}(t) := \begin{cases} 0 & \text{if } t \in [0, t_0 - \varepsilon] \cup [t_0 + \varepsilon, T], \\ (t - t_0 + \varepsilon) & \text{if } t \in [t_0 - \varepsilon, t_0], \\ -(t - t_0 - \varepsilon) & \text{if } t \in [t_0, t_0 + \varepsilon]. \end{cases}$$

Notice that  $h_\varepsilon^{t_0} \in \widehat{C}^1[0, T]$ ,  $0 \leq h_\varepsilon^{t_0}(t) \leq \varepsilon$ ,  $\forall t \in [0, T]$ , and

$$(h_\varepsilon^{t_0})'(t) = \begin{cases} 0 & \text{if } t \in [0, t_0 - \varepsilon] \cup (t_0 + \varepsilon, T], \\ 1 & \text{if } t \in (t_0 - \varepsilon, t_0), \\ -1 & \text{if } t \in (t_0, t_0 + \varepsilon). \end{cases}$$

It is evident of (3.1) that we may choose the previous  $\varepsilon$  sufficiently small for the following inequality to be verified:

$$\begin{aligned} \sup_{t \in (t_0 - \varepsilon, t_0)} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] < \\ < \inf_{t \in (t_0, t_0 + \varepsilon)} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))], \end{aligned}$$

from which the following inequalities are deduced:

$$\begin{aligned} I_1 &= \int_{t_0 - \varepsilon}^{t_0} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] dt \leq \\ &\leq \varepsilon \cdot \sup_{t \in (t_0 - \varepsilon, t_0)} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] < \\ &< \varepsilon \cdot \inf_{t \in (t_0, t_0 + \varepsilon)} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] \leq \\ &\leq \int_{t_0}^{t_0 + \varepsilon} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] dt = I_2. \end{aligned}$$

Let us now take into account that

$$h_\varepsilon^{t_0}(t) = 0, \quad \forall t \in [0, t_0 - \varepsilon] \cup [t_0 + \varepsilon, T]; \quad (h_\varepsilon^{t_0})'(t) = 0, \quad \forall t \in [0, t_0 - \varepsilon] \cup (t_0 + \varepsilon, T],$$

then

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) &:= \lim_{x \rightarrow 0^+} \frac{F(q + x h_\varepsilon^{t_0}) - F(q)}{x} \\ &= \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} [h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) + (h_\varepsilon^{t_0})'(t) \cdot L_{z'}(t, q(t), q'(t))] dt, \end{aligned}$$

and hence

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) &= \int_{t_0 - \varepsilon}^{t_0} [h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) + 1 \cdot L_{z'}(t, q(t), q'(t))] dt + \\ &+ \int_{t_0}^{t_0 + \varepsilon} [h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) + (-1) \cdot L_{z'}(t, q(t), q'(t))] dt, \end{aligned}$$

and we have that

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) &= \int_{t_0 - \varepsilon}^{t_0} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] dt \\ &- \int_{t_0}^{t_0 + \varepsilon} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] dt = I_1 - I_2 < 0, \end{aligned}$$

which contradicts the assumption that  $q$  is a minimum of  $F$ .

If  $q'(t) > \phi(t, q(t))$  in some interval to the left of  $t_0$  and  $q'(t) < \phi(t, q(t))$  in some interval to the right of  $t_0$ , the proof would be analogous, taking  $\delta^+ F(q, -h_\varepsilon^{t_0})$ .

We therefore conclude that  $q'(t) = \phi(t, q(t))$  in some interval that contains  $t_0$ .

Let us also see that  $q'$  is continuous at  $t_0$ . We shall proceed by contradiction.

Let us assume that  $q'(t_0^-) < q'(t_0^+)$  (if we assume that  $q'(t_0^-) > q'(t_0^+)$ , the proof will be analogous). Bearing in mind what was demonstrated above,  $q'$  is discontinuous at  $t_0$  only in the following cases:

- (a)  $q'(t_0^-) < \phi(t_0, q(t_0)); \quad q'(t_0^+) = \phi(t_0, q(t_0)),$
- (b)  $q'(t_0^-) = \phi(t_0, q(t_0)); \quad q'(t_0^+) > \phi(t_0, q(t_0)).$

For (a), there will exist an  $\varepsilon > 0$  such that  $q'(x) = \phi(x, q(x))$  at  $[t_0, t_0 + \varepsilon]$ . We may choose  $\varepsilon$  such that  $q'(x) < \phi(x, q(x))$  at  $[t_0 - \varepsilon, t_0]$ . In this case

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) &= \int_{t_0 - \varepsilon}^{t_0} [h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) + (h_\varepsilon^{t_0})'(t) \cdot L_{z'}(t, q(t), q'(t))] dt + \\ &+ \int_{t_0}^{t_0 + \varepsilon} [h_\varepsilon^{t_0}(t) \cdot L_z^-(t, q(t), q'(t)) + (h_\varepsilon^{t_0})'(t) \cdot L_{z'}^-(t, q(t), q'(t))] dt \end{aligned}$$

and, by identical reasoning to that used above, we shall have that  $\delta^+ F(q, h_\varepsilon^{t_0}) < 0$ , which once more means a contradiction of the fact that  $q$  is a minimum of  $F$ .

Finally, for (b), there will exist an  $\varepsilon > 0$  such that  $q'(x) = \phi(x, q(x))$  at  $[t_0 - \varepsilon, t_0]$ . We may choose  $\varepsilon$  such that  $q'(x) > \phi(x, q(x))$  at  $(t_0, t_0 + \varepsilon]$ . In this case,

$$\begin{aligned} \delta^+ F(q, -h_\varepsilon^{t_0}) &= \int_{t_0 - \varepsilon}^{t_0} [-h_\varepsilon^{t_0}(t) \cdot L_z^+(t, q(t), q'(t)) - (h_\varepsilon^{t_0})'(t) \cdot L_{z'}^+(t, q(t), q'(t))] dt + \\ &+ \int_{t_0}^{t_0 + \varepsilon} [-h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) - (h_\varepsilon^{t_0})'(t) \cdot L_{z'}(t, q(t), q'(t))] dt, \end{aligned}$$

where, by identical reasoning to that used in (a), we shall once more have the contradiction

$$\delta^+ F(q, -h_\varepsilon^{t_0}) < 0.$$

□

This result has a very clear interpretation: under optimum operating conditions, thermal plants never switch brusquely from one generating power zone to another, but rather carry out a smooth transition, remaining on the boundary  $q'(t) \equiv \phi(t, q(t))$  for a certain interval.

**4. Application to a hydrothermal problem.** A program that solves the optimization problem was elaborated using the Mathematica package. The Optimization Algorithm, briefly described below, is very similar to the algorithm that we present in [4]. The solution to the problem consists in finding for each  $K$  the function  $q_K$  that satisfies the conditions of Theorem 2 and, from among these functions, an admissible function  $q_K \in \Theta$ . Theorem 2 allows the extremals  $q_K$  to be constructed in a simple way:

Stage 1) For each  $K$  we construct  $q_K$ , where  $q_K$  satisfies the conditions of Theorem 2 and the initial condition  $q_K(0) = 0$ . In general, the construction of  $q_K$  cannot be carried out all at once over all the interval  $[0, T]$ . The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs ( $q'(t) \neq \phi(t, q(t))$ ) and boundary arcs ( $q'(t) = \phi(t, q(t))$ ) until completing the interval  $[0, T]$ . This is relatively simple to implement using a discretized version of Equations (2.7).

To calculate the exit instant of each smooth transition zone, we shall verify the conditions in Theorem 2. That is, the exit time will be the first value of  $t$  in which the following condition is no longer fulfilled for  $K$ :

$$\mathbb{Y}_q^+(t) \leq K \leq \mathbb{Y}_q^-(t).$$

Stage 2)  $K$  is calculated such that  $q_K \in \Theta$ . The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions. Effectively, we may consider the function  $\varphi(K) := q_K(T)$  and calculate the root of  $\varphi(K) - b = 0$ , which may be realized approximately using elemental procedures like the secant method.

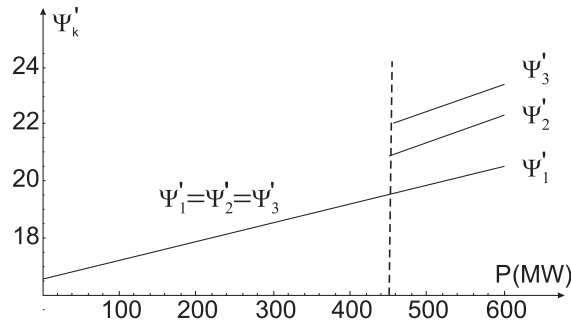


Fig. 3. Slopes of the cost curves of the thermal plant.

The program was applied to an example of a hydrothermal system made up of one thermal plant and one hydroplant. We shall analyze the behavior of the thermal plant assuming the existence of a valve point ( $P_1 = 450$ ). This study may be easily extended to the remaining valve points.

To improve the quality of the study, we shall analyze three cases (Fig. 3): In the first case, we shall assume that the valve point does not suppose a discontinuity in the derivative  $\Psi'$ . In the second, we shall assume that it implies a moderate jump in the slope, and in the third, a more pronounced jump. The cost function in each case is:

$$\text{Case 1: } \Psi_1(P) = 10696.0977 + 16.5477P + 0.003299P^2, \quad P_{\min} \leq P < P_{\max}$$

$$\text{Case 2: } \Psi_2(P) = \begin{cases} 10696.0977 + 16.5477P + 0.003299P^2, & P_{\min} \leq P \leq 450 \\ 10365.3252 + 16.6496P + 0.004706P^2, & 450 < P < P_{\max} \end{cases}$$

$$\text{Case 3: } \Psi_3(P) = \begin{cases} 10696.0977 + 16.5477P + 0.003299P^2, & P_{\min} \leq P \leq 450 \\ 9897.21269 + 17.6678P + 0.004755P^2, & 450 < P < P_{\max} \end{cases}$$

with  $P_{\min} = 0$ ;  $P_{\max} = 600$ .

For the power production of the hydroplant (variable-head), we consider a function of  $z(t)$  and  $z'(t)$  defined as

$$P_h(t, z(t), z'(t)) := A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t),$$

with

$$A(t) = \frac{B_y}{G}(S_0 + t \cdot i), B = \frac{B_y}{G}.$$

So, the function of effective hydraulic generation is

$$H(t, z(t), z'(t)) := P_h(t, z(t), z'(t)) - b_H P_h^2(t, z(t), z'(t)),$$

where  $b_H = 0.000220$  ( $1/MW$ ) is the loss coefficient. The values for the coefficients of the hydroplant are: the efficiency  $G = 526315$  ( $m^4/h.MW$ ), the restriction on the volume  $b = 941.6 \cdot 10^5$  ( $m^3$ ), the natural inflow  $i = 101.952 \cdot 10^5$  ( $m^3/h$ ), the initial volume  $S_0 = 203.904 \cdot 10^9$  ( $m^3$ ) and the coefficient  $B_y = 149.510^{-12}$  ( $m^{-2}$ ) (a parameter that depends on the geometry of the tanks).

Next, we present the optimal solution. Table 1 shows the power demand  $P_d(t)$  ( $MW$ ), and the optimal solution, i.e. the state variable:  $q(t)(m^3 \cdot 10^6)$ , and the optimal thermal power:  $P(t)(MW)$  for the three cases. Although the discretization used comprised 48 instants for the optimization interval  $[0, 24]$ , for the sake of brevity we present only the integer values of  $t(h)$ . The objective value is 448699 (*Euros*) in Case1, 448970 (*Euros*) in Case 2, and 449339 (*Euros*) in Case 3.

Table I. Optimal Solution and Power Demand.

| $t$ | $P_d$ | Case 1<br>$q$ | $P$     | Case 2<br>$q$ | $P$     | Case 3<br>$q$ | $P$     |
|-----|-------|---------------|---------|---------------|---------|---------------|---------|
| 0   | 1980  | 0             | 455.776 | 0             | 450.    | 0             | 450.    |
| 1   | 1916  | 3.6972        | 416.407 | 3.1952        | 450.    | 3.1952        | 450.    |
| 2   | 1871  | 7.0746        | 389.072 | 5.6752        | 440.572 | 5.5889        | 450.    |
| 3   | 1739  | 9.6070        | 310.749 | 7.2546        | 363.322 | 6.5251        | 399.559 |
| 4   | 1688  | 11.5648       | 281.154 | 8.2633        | 334.122 | 6.8933        | 370.616 |
| 5   | 1681  | 13.3826       | 277.071 | 9.1330        | 330.09  | 7.1233        | 366.619 |
| 6   | 1765  | 15.6948       | 325.782 | 10.4938       | 378.142 | 7.8419        | 414.24  |
| 7   | 1795  | 18.3556       | 343.400 | 12.2010       | 395.516 | 8.9054        | 431.454 |
| 8   | 1847  | 21.3486       | 374.301 | 14.2384       | 425.988 | 10.4027       | 450.    |
| 9   | 1930  | 24.9080       | 424.499 | 17.0774       | 450.    | 13.2328       | 450.    |
| 10  | 2024  | 29.1617       | 482.595 | 21.6465       | 455.497 | 17.8562       | 450.    |
| 11  | 2060  | 33.8033       | 505.155 | 26.8617       | 475.791 | 23.4603       | 456.422 |
| 12  | 2022  | 38.2901       | 481.232 | 31.8924       | 454.277 | 28.7071       | 450.    |
| 13  | 1989  | 42.4978       | 460.62  | 36.3819       | 450.    | 33.1966       | 450.    |
| 14  | 2015  | 46.7903       | 476.746 | 41.0932       | 450.256 | 37.9103       | 450.    |
| 15  | 2039  | 51.2823       | 491.716 | 46.1291       | 463.702 | 43.1624       | 450.    |
| 16  | 2034  | 55.7887       | 488.523 | 51.1818       | 460.833 | 48.4509       | 450.    |
| 17  | 2040  | 60.3068       | 492.232 | 56.2480       | 464.17  | 53.7687       | 450.    |
| 18  | 2074  | 65.0220       | 513.613 | 61.5484       | 483.431 | 59.4563       | 464.257 |
| 19  | 2116  | 70.0490       | 540.272 | 67.2193       | 507.533 | 65.5073       | 488.904 |
| 20  | 2084  | 74.9751       | 519.840 | 72.7700       | 489.055 | 71.4404       | 470.011 |
| 21  | 2082  | 79.8099       | 518.516 | 78.2116       | 487.861 | 77.2666       | 468.791 |
| 22  | 2113  | 84.8279       | 538.191 | 83.8709       | 505.654 | 83.3063       | 486.985 |
| 23  | 2090  | 89.8056       | 523.484 | 89.4819       | 492.352 | 89.2987       | 473.386 |
| 24  | 1980  | 94.1600       | 454.422 | 94.1600       | 450.    | 94.1600       | 450.    |

In Case 1, there is no smooth transition; hence  $K$  has the same value in all the instants:  $-p(0) = K = 0.00101472651608$ . In Cases 2 and 3, a smooth transition does occur. In the free instants,  $K$  takes the values:  $K = 0.00106026202804$  and  $K = 0.00109182791977$  in Case 2 and Case 3, respectively. In the smooth transition instants, the value of  $K$  remains within the interval  $[\mathbb{Y}_q^+(t), \mathbb{Y}_q^-(t)]$ . To see this in more detail, in Table II we present the entry and exit times and the value of  $K$  and the values of  $\mathbb{Y}_q^+(t)$ , and  $\mathbb{Y}_q^-(t)$ , which, as can be easily proven, verify Theorem 2.

Table II. Entry and Exit times. The smooth transition.

| $t$ | Case 2<br>$[\mathbb{Y}_q^+(t), \mathbb{Y}_q^-(t)]$ | Case 3<br>$[\mathbb{Y}_q^+(t), \mathbb{Y}_q^-(t)]$ |
|-----|----------------------------------------------------|----------------------------------------------------|
| 0   | [0.0010095373118, 0.0010803096183]                 | [0.0010095373118, 0.00113525876399]                |
| 1   | [0.0010446241595, 0.0011178550438]                 | [0.0010446241595, 0.00117471308241]                |
| 2   | 0.00106026202804                                   | [0.0010686245231, 0.00120170075024]                |
| 3   | 0.00106026202804                                   | 0.00109182791977                                   |
| 4   | 0.00106026202804                                   | 0.00109182791977                                   |
| 5   | 0.00106026202804                                   | 0.00109182791977                                   |
| 6   | 0.00106026202804                                   | 0.00109182791977                                   |
| 7   | 0.00106026202804                                   | 0.00109182791977                                   |
| 8   | 0.00106026202804                                   | [0.001081504409, 0.00121618141338]                 |
| 9   | [0.00103747734733, 0.001110202169]                 | [0.0010374776647, 0.0011666700436]                 |
| 10  | 0.00106026202804                                   | [0.00098522584484, 0.001107908297]                 |
| 11  | 0.00106026202804                                   | 0.00109182791977                                   |
| 12  | 0.00106026202804                                   | [0.0009864688517, 0.0011092992284]                 |
| 13  | [0.0010051729880, 0.0010756262490]                 | [0.0010051707305, 0.0011303273776]                 |
| 14  | 0.00106026202804                                   | [0.0009905539685, 0.0011138874728]                 |
| 15  | 0.00106026202804                                   | [0.0009768688274, 0.0010984949299]                 |
| 16  | 0.00106026202804                                   | [0.0009797949547, 0.0011017823000]                 |
| 17  | 0.00106026202804                                   | [0.0009763900265, 0.0010979501699]                 |
| 18  | 0.00106026202804                                   | 0.00109182791977                                   |
| 19  | 0.00106026202804                                   | 0.00109182791977                                   |
| 20  | 0.00106026202804                                   | 0.00109182791977                                   |
| 21  | 0.00106026202804                                   | 0.00109182791977                                   |
| 22  | 0.00106026202804                                   | 0.00109182791977                                   |
| 23  | 0.00106026202804                                   | 0.00109182791977                                   |
| 24  | [0.0010107682640, 0.0010815928155]                 | [0.0010107639988, 0.0011365782603]                 |

The graphs of the optimal solution can be seen in Figs. 4 and 5. Note that, as Theorem 3 guarantees, when the valve point supposes a discontinuity in  $\Psi'$ , the optimum thermal power presents a constant interval. Furthermore, it can be appreciated (Fig. 4) that the more pronounced the jump in the derivative, the more prolonged this interval will be, being inexistent in Case 1 (continuous  $\Psi'$ ).

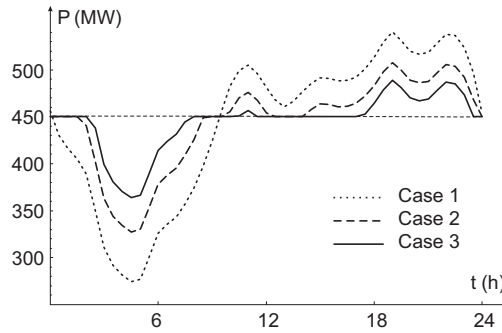


Fig. 4. Optimal Thermal Power.

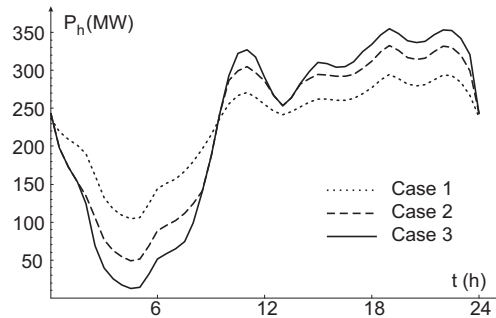


Fig. 5. Optimal Hydro-Power.

An optimization interval of  $T = 24$  h. was considered, with a discretization of  $24 \cdot 2$  subintervals. The secant method was used to calculate the approximate value of  $K$  for which  $q_K(T) - b = 0$ . In 4 iterations:  $|q_K(T) - b| < 10^{-2}(m^3)$ . The CPU time employed was 12.0 sec.

**5. Conclusions.** This paper proposes a new approach for solving economic dispatch problems with valve-point effects. The ED problem with valve-point effects is represented as a nonsmooth optimization problem in which the cost function for each thermal plant is approximately represented by several piecewise quadratic functions. We establish a necessary minimum condition for this problem with non-smooth Lagrangian, employing nonsmooth analysis for this purpose. Furthermore, we present a qualitative aspect of the solution, the smooth transition: under optimum operating conditions, thermal plants never switch brusquely from one generating power zone to another, remaining above the boundary for a certain interval. The feasibility of our method was illustrated by a hydro-thermal system and we compared the results obtained using three different valve-point effects for the thermal plant.

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Received September 2012; 1st revision April 2013; 2nd revision June 2013.

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