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VARIATIONS ON GIUGA NUMBERS AND GIUGA'S CONGRUENCE

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A k-strong Giuga number is a composite integer such that $\sum_{j=1}^{n-1} j^{n-1} \equiv -1 \pmod{n}$. We consider the congruence $\sum_{j=1}^{n-1} j^{k(n-1)} \equiv -1 \pmod{n}$ for each $k \in \mathbb{N}$ (thus extending Giuga's ideas for $k = 1$). In particular, it is proved that a pair (n,k) with composite n satisfies this congruence if and only if n is a Giuga number and $\lambda(n) \mid k(n-1)$. In passing, we establish some new characterizations of Giuga numbers and study some properties of the numbers n satisfying $\lambda(n) \mid k(n-1)$.

1. Preliminaries

As the starting point of the present paper, we use the following definition:

Definition 1.

(i) A Giuga number is a composite integer n such that $p \mid (n/p - 1)$ for every p, prime divisor of n.

(ii) A strong Giuga number is a composite integer n such that $p(p-1) \mid (n/p-1)$ for every p, prime divisor of n.

Strong Giuga numbers are counterexamples to Giuga's conjecture [8], i.e., they are composite integers such that

$$\sum_{j=1}^{n-1} j^{n-1} \equiv -1 \pmod{n}. \tag{1}$$

There are several equivalent ways to define Giuga numbers [1, 5, 8, 10]. In particular, we focus on the following characterization of Giuga numbers [5, p. 41], which, in a certain sense, is an analog of the characterization given in (1) for strong Giuga numbers.

Proposition 1. *Let n be an integer. Then n is a Giuga number if and only if*

$$\sum_{j=1}^{n-1} j^{\phi(n)} \equiv -1 \pmod{n},$$

where ϕ is Euler's totient function.

Clearly, strong Giuga numbers are also Giuga numbers and, in fact, by using the so-called Korselt's criterion [14], we can show that the following assertion is true:

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Proposition 2. *An integer n is a strong Giuga number if and only if it is both a Giuga number and a Carmichael number.*

Much work has been done regarding these numbers [1, 3, 10, 13, 16, 17] and several generalizations and/or variations are possible [6, 9]. In the present paper, some new characterizations of Giuga numbers are established. These characterizations lead to a generalization of both strong Giuga numbers and Carmichael numbers.

2. New Characterizations of Giuga Numbers

This section is devoted to the description of a family of characterizations of Giuga numbers that includes Proposition 1. The following lemma is our main tool:

Lemma 1. *For any natural numbers $A,\ B$ and N,*

$$\sum_{j=1}^{N-1} j^{A\lambda(N)} \equiv \sum_{j=1}^{N-1} j^{B\phi(N)} \pmod{N}.$$

Proof. We set

$$N = 2^a p_1^{r_1} \dots p_s^{r_s}$$

with p_i distinct odd primes and choose $i \in \{1, \dots, s\}$. Thus, we get

$$\sum_{j=1}^{N-1} j^{A\lambda(N)} \equiv \frac{N}{p_i^{r_i}} \sum_{j=1}^{p_i^{r_i}-1} j^{A\lambda(N)} \pmod{p_i^{r_i}},$$

$$\sum_{j=1}^{N-1} j^{B\phi(N)} \equiv \frac{N}{p_i^{r_i}} \sum_{j=1}^{p_i^{r_i}-1} j^{B\phi(N)} \pmod{p_i^{r_i}}.$$

Further, since both $A\lambda(N),\ B\phi(N) \geq r_i$, we find

$$\sum_{j=1}^{p_i^{r_i}-1} j^{A\lambda(N)} = \sum_{\substack{1\leq j\leq p_i^{r_i}-1 \\ (p_i,j)=1}} j^{A\lambda(N)} + \sum_{\substack{1\leq j\leq p_i^{r_i}-1 \\ p_i|j}} j^{A\lambda(N)} \equiv \phi(p_i^{r_i}) + 0 \pmod{p_i^{r_i}},$$

$$\sum_{j=1}^{p_i^{r_i}-1} j^{B\phi(N)} = \sum_{\substack{1\leq j\leq p_i^{r_i}-1 \\ (p_i,j)=1}} j^{B\phi(N)} + \sum_{\substack{1\leq j\leq p_i^{r_i}-1 \\ p_i|j}} j^{A\lambda(N)} \equiv \phi(p_i^{r_i}) + 0 \pmod{p_i^{r_i}}.$$

Consequently,

$$\sum_{j=1}^{N-1} j^{A\lambda(N)} \equiv \sum_{j=1}^{N-1} j^{B\phi(N)} \pmod{p_i^{r_i}} \quad \text{for every} \quad i = 1, \dots, s.$$