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# Solutions of the congruence $1 + 2^{f(n)} + \dots + n^{f(n)} \equiv 0 \pmod{n}$

José María Grau<sup>\*1</sup>, Pieter Moree<sup>\*\*2</sup>, and Antonio M. Oller-Marcén<sup>\*\*\*3</sup>

<sup>1</sup> Departamento de Matemáticas, Universidad de Oviedo, Avda. Calvo Sotelo, s/n, 33007 Oviedo, Spain

<sup>2</sup> Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

<sup>3</sup> Centro Universitario de la Defensa de Zaragoza, Ctra. de Huesca, s/n, 50090 Zaragoza, Spain

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In this paper we characterize, in terms of the prime divisors of  $n$ , the pairs  $(k, n)$  for which  $n$  divides  $\sum_{j=1}^n j^k$ . As an application, we derive some results on the sets

$$\mathcal{M}_f := \left\{ n \geq 1 : f(n) > 1 \text{ and } \sum_{j=1}^n j^{f(n)} \equiv 0 \pmod{n} \right\}$$

for some choices of  $f$ .

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## 1 Introduction

In the literature on power sums  $S_k(n) := \sum_{j=1}^n j^k$  the following congruence is well known

**Proposition 1.1** (von Staudt [19], 1840) *Let  $k, n \geq 1$  be integers with  $k$  even. We have*

$$S_k(n) \equiv - \sum_{p|n, p-1|k} \frac{n}{p} \pmod{n}.$$

This result motivates us to study  $S_k(n) \pmod{n}$  and, more generally, to study  $S_{f(n)}(n) \pmod{n}$  for different arithmetic functions  $f$  (see [11] for some results in this spirit). Thus, if  $p-1 \mid f(p)$ , for every prime  $p$ , we have that the congruence  $S_{f(n)}(n) \equiv -1 \pmod{n}$  holds for every  $n = p$  prime and it is interesting to find the composite numbers which also satisfy it. In this direction we have the *Giuga numbers* (see [1]), which are composite numbers such that  $S_{\phi(n)}(n) \equiv -1 \pmod{n}$ , the *strong Giuga numbers*, which are composite numbers such that  $S_{n-1}(n) \equiv -1 \pmod{n}$  (Giuga's conjecture [3] states that there are no strong Giuga numbers. Tipu [20] estimates the number of strong Giuga numbers up to  $x$  to be  $O(x^{1/2} \log x)$  while Luca, Pomerance and Shparlinski [9] improve this to  $O(x^{1/2}/\log^2 x)$ ), or the *K-strong Giuga numbers*, which are composite numbers such that  $S_{K(n-1)}(n) \equiv -1 \pmod{n}$  (see [5]).

In this paper we characterize, in terms of the prime divisors of  $n$ , the pairs  $(k, n)$  for which  $n$  divides  $S_k(n)$ . This characterization is given in the following theorem.

**Theorem 1.2** *Let  $k, n \geq 1$  be integers. Then,  $n \mid S_k(n)$  if and only if one of the following holds:*

- i)  $n$  is odd and  $p-1 \nmid k$  for every prime divisor  $p$  of  $n$ .
- ii)  $n$  is a multiple of 4 and  $k > 1$  is odd.

\* e-mail: grau@uniovi.es, Phone: +34 985 182 094, Fax: +34 985 103 354

\*\* e-mail: moree@mpim-bonn.mpg.de, Phone: +49 228 402 232, Fax: +49 228 402 277

\*\*\* Corresponding author: e-mail: oller@unizar.es, Phone: +34 976 739 611, Fax: +34 976 739 824

Moreover, inspired by Giuga's ideas we investigate the congruence  $S_{f(n)}(n) \equiv 0 \pmod{n}$  for some functions  $f$ . This work started in [4], where the case  $f(n) = (n-1)/2$  was considered. The case of arithmetic functions  $f$  such that  $p-1 \nmid f(p)$  for every prime  $p$  is of special interest.

In what follows we will consider the natural numbers

$$\mathcal{M}_f := \left\{ n \geq 1 : f(n) > 1 \text{ and } S_{f(n)}(n) = \sum_{j=1}^n j^{f(n)} \equiv 0 \pmod{n} \right\} \quad (1.1)$$

associated to an arbitrary function  $f : \mathbb{N} \rightarrow \mathbb{N}$ . The reader might wonder why the definition involves  $f(n) > 1$ , rather than  $f(n) \geq 1$ . The reason for this is that by Theorem 1.2 the case  $f(n) = 1$  is somewhat exceptional.

Here we study the sets  $\mathcal{M}_f$  in the case  $f(n) = an + b$ , the affine case, and in some cases such that  $\mathcal{M}_f$  contains all prime numbers. We have characterized the elements of these sets and, in some cases, we have computed their asymptotic density.

In [6] the related problem of studying the sets  $\{n : S_{Qn}(n) \equiv n \pmod{Qn}\}$  for certain very special  $Q$  ("weak primary pseudoperfect numbers") is studied

## 2 A proof of Theorem 1.2

In this section we will establish Theorem 1.2. It will be convenient to work with

$$S_k(n) := \sum_{j=1}^n j^k \quad \text{and} \quad S_k^*(n) := \sum_{j=1}^{n-1} j^k.$$

In particular we will characterize the pairs  $(k, n)$  such that  $n$  divides  $S_k(n)$ . If  $k = 0$ , clearly  $S_k(n) = n$  and there is no problem to study. Thus, in what follows we will assume  $k > 0$ .

We will start this section with three simple lemmas.

**Lemma 2.1** *Let  $p$  be a prime and let  $k > 0$  be an integer. Then, we have*

$$S_k(p) \equiv \begin{cases} -1 \pmod{p} & \text{if } p-1 \mid k; \\ 0 \pmod{p} & \text{if } p-1 \nmid k. \end{cases}$$

*Proof.* See [7] for the standard proof using primitive roots, or [10] for a recent elementary proof.  $\square$

The next lemma extends Lemma 2 in Moree [17], where it is proved that (2.1) holds if  $p$  is odd or  $p = 2$  and  $r$  is even.

**Lemma 2.2** *Let  $\lambda$  and  $r$  be positive integers and  $p$  be a prime. We have*

$$S_r(p^{\lambda+1}) \equiv p S_r(p^\lambda) \pmod{p^{\lambda+1}}, \quad (2.1)$$

*unless  $\lambda = 1$ ,  $p = 2$ ,  $r$  is odd and  $r \geq 3$  in which case we have  $0 \equiv S_r(4) \not\equiv 2S_r(2) \equiv 2 \pmod{4}$ .*

*Proof.* Note that it is equivalent to prove the statement with  $S_r(\cdot)$  replaced by  $S_r^*(\cdot)$ . Since the statement clearly holds for  $r = 1$  we may assume that  $r \geq 2$ . Every  $0 \leq j < p^{\lambda+1}$  can be uniquely written as  $j = \alpha p^\lambda + \beta$  with  $0 \leq \alpha < p$  and  $0 \leq \beta < p^\lambda$ . Hence we obtain by invoking the binomial theorem

$$S_r^*(p^{\lambda+1}) = \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{p^\lambda-1} (\alpha p^\lambda + \beta)^r \equiv p \sum_{\beta=0}^{p^\lambda-1} \beta^r + r p^\lambda \sum_{\alpha=0}^{p-1} \alpha \sum_{\beta=0}^{p^\lambda-1} \beta^{r-1} \pmod{p^{2\lambda}}.$$

Since the first single sum equals  $S_r^*(p^\lambda)$ , we see that (2.1) holds if and only if  $\frac{r}{2} p(p-1) S_{r-1}^*(p^\lambda) \equiv 0 \pmod{p}$ . Now suppose that the latter congruence does not hold. Then we must have  $p = 2$ ,  $2 \nmid r$  and  $r \geq 3$ . Since  $2 \mid S_{r-1}^*(2^\lambda)$  for  $\lambda \geq 2$  we must have  $\lambda = 1$ . The proof is easily completed on noting that for  $r \geq 3$  and odd we have  $S_r(4) \equiv 1^r + 3^r \equiv 0 \pmod{4}$ .  $\square$

As so often in number theory, "two is the oddest of primes" and needs special treatment

**Lemma 2.3** *Let  $e, k \geq 1$ . We have  $2^e \mid S_k(2^e)$  if and only if  $k \geq 3$  is odd and  $e \geq 2$ .*

**Proof.** Follows on combining the previous two lemmas.  $\square$

In fact, using Lemma 2.2 it is easy to evaluate  $S_k(2^e)$  modulo  $2^e$  (where we ignore the trivial case  $e = 1$ ). We give the result for completeness' sake.

**Lemma 2.4** *Let  $e > 1$ . Then*

$$S_k(2^e) \equiv \begin{cases} 0 \pmod{2^e} & \text{if } k \text{ is odd;} \\ 2^{e-1} \pmod{2^e} & \text{if } k > 1 \text{ is even.} \end{cases}$$

**Proof of Theorem 1.2** If  $b \mid n$ , then clearly  $S_k(n) \equiv \frac{n}{b} S_k(b) \pmod{b}$ . Now let  $n = \prod_{i=1}^s p_i^{e_i}$  be the canonical prime factorisation of  $n$ . Noting that  $p_i \nmid n p_i^{-e_i}$  we infer from  $S_k(n) \equiv \frac{n}{p_i^{e_i}} S_k(p_i^{e_i}) \pmod{p_i^{e_i}}$  that

$$n \mid S_k(n) \text{ if and only if } p_i^{e_i} \mid S_k(p_i^{e_i}), \text{ for } i = 1, 2, \dots, s. \quad (2.2)$$

If  $p_i$  is odd, then it follows by combining Lemma 2.1 and Lemma 2.2 that

$$p_i^{e_i} \mid S_k(p_i^{e_i}) \text{ if and only if } p_i - 1 \nmid k. \quad (2.3)$$

Using this and Lemma 2.3 we see that  $n \mid S_k(n)$  if and only if

- i)  $n$  is odd and  $p - 1 \nmid k$  for every odd prime divisor  $p$  of  $n$ ;
- or
- ii)  $n$  is a multiple of 4,  $k > 1$  is odd and  $p - 1 \nmid k$  for every odd prime divisor  $p$  of  $n$ .

Note that in i) the second “odd” is a consequence of the first “odd”. Likewise in ii) the condition that  $k$  is odd implies that  $p - 1 \nmid k$  for every odd prime divisor  $p$  of  $n$ . On leaving out the redundant parts of i) and ii) the proof is completed.  $\square$

### 3 Some remarks concerning Theorem 1.2

#### 3.1 The Erdős-Moser equation

Erdős conjectured around 1950 that the Diophantine equation

$$S_k(n-1) = n^k \quad (3.1)$$

has only the solution  $1 + 2 = 3$  corresponding to  $(k, n) = (1, 3)$ . Note that if  $(k, n)$  satisfies  $S_k(n-1) = n^k$ , then  $n \mid S_k(n)$ . The first results on this problem were obtained by original but entirely elementary methods by Leo Moser [18], cf. [17]. He showed that if (3.1) has a further solution with  $k > 1$ , then  $k$  is even and  $n > 10^{10^6}$ . He showed that either  $n \equiv 0 \pmod{8}$  or  $n \equiv 3 \pmod{8}$ . Note that by Theorem 1.2 we can actually deduce that  $n \equiv 3 \pmod{8}$  and  $p \mid n$  implies  $p - 1 \nmid k$ . A slightly improved and extended version of Moser's results was given by the second author as Theorem 4 in [16]. This also incorporates that  $n \equiv 3 \pmod{8}$  (explicitly) and  $p \mid n$  implies  $p - 1 \nmid k$  (implicitly). The implicit fact follows from [16, (8)] which states that

$$\sum_{(p-1) \mid k, p \mid n} \frac{1}{p} \in \mathbb{Z} \quad (3.2)$$

and the remark that a sum of reciprocals of distinct primes can never be a positive integer. Moser's proof rests on deriving four equations similar to (3.2) (these are the four mathematical rabbits in the title of [16]). The baby mathematical rabbit (3.2) he apparently overlooked.

Theorem 1.2 can also be used to get some information on the generalized Erdős-Moser equation  $S_k(n-1) = an^k$ , with  $a$  a fixed positive integer. Here it is not difficult to show that if there is a solution with  $k > 1$ , then  $k$  must be even. By Theorem 1.2 we then infer that if  $(a, n, k)$  is a solution with  $k > 1$ , then  $n$  is odd and  $p \mid n$  implies  $p - 1 \nmid k$ . These are known results, see Moree [13].

### 3.2 The Carlitz-von Staudt theorem

Proposition 1.1 deals only with the case  $k$  even. Carlitz [2] considered the case  $k$  is odd and claimed that  $n \mid S_k(n)$  in that case. The second author [12] pointed out that this is false. It is true, however, that  $S_k(n) = rn/2$  with  $r$  an integer. The following lemma from a preprint of Kellner [8] gives the parity of  $r$ .

**Lemma 3.1** *Let  $k \geq 3$  be odd. We have  $S_k(n) = rn/2$  with  $r$  an integer. Here  $r$  is odd if  $n \equiv 2 \pmod{4}$  and  $r$  is even otherwise.*

**Proof.** Since  $k$  is odd, we have  $j^k \equiv -(n-j)^k \pmod{n}$  for every integer  $j$ .

Case  $n$  is even: All terms of the sum cancel each other modulo  $n$  except for the middle term  $(n/2)^k$ . We infer that  $S_k(n) = rn/2$  with  $r \equiv (n/2)^{k-1} \pmod{n}$ . It follows that  $r$  is even if  $4 \mid n$  and  $r$  is odd if  $n \equiv 2 \pmod{4}$ .

Case  $n$  is odd: The sum  $S_k(n)$ , having no middle term, vanishes modulo  $n$  and hence  $r$  is even.  $\square$

Using this lemma we can give a general version of Proposition 1.1.

**Proposition 3.2** *Let  $k, n \geq 1$  be integers, then*

$$S_k(n) \equiv \begin{cases} -\sum_{\substack{p \mid n \\ p-1 \mid k}} \frac{n}{p} \pmod{n}, & \text{if } k \text{ is even;} \\ n/2 \pmod{n}, & \text{if } k = 1 \text{ and } n \text{ is even;} \\ n/2 \pmod{n}, & \text{if } k > 1 \text{ is odd and } n \equiv 2 \pmod{4}; \\ 0 \pmod{n}, & \text{otherwise.} \end{cases}$$

**Proof.** If  $k$  is even this is the classical result given in Proposition 1.1. If  $k = 1$  it is clear that  $S_k(n) = n(n+1)/2$  so,  $S_k(n) \equiv n/2 \pmod{n}$  if  $n$  is even and  $S_k(n) \equiv 0 \pmod{n}$  if  $n$  is odd. The remaining cases follow immediately from Lemma 3.1.  $\square$

Lemma 3.1 can be sharpened. In [16] the second author showed that in fact  $S_k(n) = tn(n+1)/2$ . We now determine the parity of  $t$ .

**Proposition 3.3** *Let  $k \geq 3$  be odd. We have  $S_k(n) = tn(n+1)/2$  with  $t$  an integer. Here  $t$  is odd if  $n \equiv 1, 2 \pmod{4}$  and  $t$  is even otherwise.*

**Proof.** Since  $k$  is odd, we have  $j^k \equiv -(n-j)^k \pmod{n}$  and  $j^k \equiv -(n-j+1)^k \pmod{n+1}$  for every integer  $j$ .

Case  $n$  is even: In this case we have  $S_k(n) \equiv (n/2)^k \pmod{n}$  and  $S_k(n) \equiv 0 \pmod{n+1}$ . Since  $\gcd(n, n+1) = 1$ , we infer that  $S_k(n) = tn(n+1)/2$  with  $t \equiv (n/2)^{k-1} \pmod{2}$ . It follows that  $t$  is even if  $4 \mid n$  and  $t$  is odd if  $n \equiv 2 \pmod{4}$ .

Case  $n$  is odd: In this case we have that  $S_k(n) \equiv 0 \pmod{n}$  and  $S_k(n) \equiv ((n+1)/2)^k \pmod{n+1}$ . Since  $\gcd(n, n+1) = 1$ , we infer that  $S_k(n) = tn(n+1)/2$  with  $t \equiv ((n+1)/2)^{k-1} \pmod{2}$ . It follows that  $t$  is even if  $4 \mid n+1$  and  $t$  is odd if  $n \equiv 1 \pmod{4}$ .  $\square$

## 4 The affine case

In this section we will focus on the case where  $f$  is an *affine function*; i.e., a linear function. In what follows we will denote  $an + b$  by  $f_{a,b}(n)$ . Recall the definition (1.1) of  $\mathcal{M}_f$ . In what follows it will be easier to characterize  $\mathbb{N}_f \setminus \mathcal{M}_f$  instead of  $\mathcal{M}_f$  itself, where

$$\mathbb{N}_f = \{n \in \mathbb{N} : f(n) > 1\}.$$

Let us introduce some further notation. Given  $(a, b) \in \mathbb{N} \times \mathbb{Z}$ , we will consider the set

$$\mathcal{P}_{a,b} := \{p \text{ odd prime} : b \equiv 0 \pmod{\gcd(a, p-1)}\},$$

and if  $(a, b, p) \in \mathbb{N} \times \mathbb{Z} \times \mathcal{P}_{a,b}$  we define

$$\mu_{a,b}(p) := \min\{x \in \mathbb{N} : xpa \equiv -b \pmod{p-1}\}.$$

Note that in case  $p$  is an odd prime the equation  $xpa \equiv -b \pmod{p-1}$  has a solution if and only if  $p$  is in  $\mathcal{P}_{a,b}$ . For notational convenience we shorten  $\{n \in \mathbb{N} : n \equiv c \pmod{d}\}$  to  $\{c \pmod{d}\}$ . The intersection of a set  $S$  with  $\mathbb{N}_f$  will be denoted by  $S_f$ . With this notation in mind we can prove the following result.

**Theorem 4.1** *Let  $(a, b) \in \mathbb{N} \times \mathbb{Z}$ . Put  $p_a := (p-1)/\gcd(a, p-1)$  and  $f(n) := a + bn$ . Then:*

i) *If  $a$  and  $b$  are even,*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2\mathbb{N}\}_f \cup \bigcup_{p \in \mathcal{P}_{a,b}} \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f.$$

ii) *If  $a$  and  $b$  are odd,*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2 \pmod{4}\}_f \cup \bigcup_{p \in \mathcal{P}_{a,b}} \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f.$$

iii) *If  $a$  is even and  $b$  is odd, then*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2 \pmod{4}\}_f.$$

iv) *If  $a$  is odd and  $b$  is even, then*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2\mathbb{N}\}_f.$$

**Proof.** Suppose that  $n \in \mathbb{N}_f$ . Then  $f(n) > 1$ . By Theorem 1.2 we have  $n \nmid S_f(n)$  if and only if

- a)  $n$  is odd and  $p-1 \mid f(n)$  for some odd prime divisor  $p$  of  $n$ ;
- b)  $n \equiv 2 \pmod{4}$ ;
- or
- c)  $n$  is a multiple of 4,  $f(n)$  is even.

We will give a complete proof of i), the other cases being similar.

Since by assumption  $a$  and  $b$  are even,  $f(n)$  is even and hence, by b) and c), we have that  $\{2\mathbb{N}\}_f \subseteq \mathbb{N}_f \setminus \mathcal{M}_f$ . Now, assume that  $n \notin \mathcal{M}_f$  is odd. Then by a) there must exist an odd prime  $p \mid n$  such that  $p-1 \mid an+b$ . Since  $an \equiv 0 \pmod{ap}$  and  $an \equiv -b \pmod{p-1}$  it follows that  $p$  is in  $\mathcal{P}_{a,b}$  and  $an \in \{A + s \cdot \text{lcm}(ap, p-1) : s \geq 0\}$  with

$$A = \min\{x \in \mathbb{N} : x \equiv 0 \pmod{ap}, x \equiv -b \pmod{p-1}\}.$$

Using that  $A = ap\mu_{a,b}(p)$  we find that

$$n \in \left\{ \frac{A}{a} + \frac{s}{a} \text{lcm}(ap, p-1) : s \geq 0 \right\} = \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f.$$

On taking the requirement  $f(n) > 1$  into account we obtain that  $n \in \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f$  for some  $p \in \mathcal{P}_{a,b}$  is necessary and sufficient for an odd  $n$  to be in  $\mathbb{N}_f \setminus \mathcal{M}_f$ .  $\square$

Here and throughout, we denote by  $\delta(A)$  (resp.  $\underline{\delta}(A)$ ,  $\overline{\delta}(A)$ ) the asymptotic (resp. lower, upper asymptotic) density of an integer sequence  $A$ . Recall that

$$\delta(A) = \lim_{N \rightarrow \infty} \frac{\text{card}([0, N] \cap A)}{N},$$

while  $\underline{\delta}(A)$  and  $\overline{\delta}(A)$  are obtained using the lower or upper limit in the previous expression.

We will be interested in computing the asymptotic density of the sets  $\mathcal{M}_{f,a,b}$ , at least for some particular values of  $a$  and  $b$ . To do so we must first show that this density exists and the following lemma will be our main tool.

**Lemma 4.2** *Let  $\mathcal{A} := \{a_k\}_{k \in \mathbb{N}}$  and  $\{c_k\}_{k \in \mathbb{N}}$  be two sequences of positive integers and*

$$\mathcal{B}_k := \{a_k + (s-1)c_k : s \in \mathbb{N}\}.$$

If  $\sum_{k=1}^{\infty} c_k^{-1}$  is convergent and  $\mathcal{A}$  has zero asymptotic density, then  $\bigcup_{k=1}^{\infty} \mathcal{B}_k$  has an asymptotic density with  $\delta(\bigcup_{k=1}^{\infty} \mathcal{B}_k) = \lim_{n \rightarrow \infty} \delta(\bigcup_{k=1}^n \mathcal{B}_k)$  and

$$\delta\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) - \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right) \leq \sum_{i=n+1}^{\infty} \frac{1}{c_i}.$$

**Proof.** Let us denote  $B_n := \bigcup_{k=n+1}^{\infty} \mathcal{B}_k$  and  $\vartheta(n, N) := \text{card}([0, N] \cap B_n)$ . Then

$$\vartheta(n, N) \leq \text{card}([0, N] \cap \mathcal{A}) + N \sum_{k=n+1}^{\infty} \frac{1}{c_k}.$$

From this, we get

$$\bar{\delta}(B_n) = \limsup \frac{\vartheta(n, N)}{N} \leq \limsup \frac{\text{card}([0, N] \cap \mathcal{A})}{N} + \sum_{k=n+1}^{\infty} \frac{1}{c_k} = \sum_{k=n+1}^{\infty} \frac{1}{c_k}.$$

Now, for every  $n$ ,  $\bigcup_{k=1}^n \mathcal{B}_k$  has an asymptotic density and the sequence  $\delta_n := \delta(\bigcup_{k=1}^n \mathcal{B}_k)$  is non-decreasing and bounded (by 1), thus convergent. Consequently

$$\delta_n \leq \underline{\delta}\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) \leq \bar{\delta}\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) = \bar{\delta}\left(\bigcup_{k=1}^n \mathcal{B}_k \cup B_n\right) \leq \delta_n + \bar{\delta}(B_n) \leq \delta_n + \sum_{k=n+1}^{\infty} c_k^{-1},$$

and taking into account that  $\sum_{k=n+1}^{\infty} c_k^{-1}$  converges to zero, it is enough to take limits in order to finish the proof.  $\square$

With the help of this lemma the following proposition is easy to be proved.

**Proposition 4.3** *If  $(a, b) \in \mathbb{N} \times \mathbb{Z}$ , then the set  $\mathcal{M}_{f_{a,b}}$  has an asymptotic density  $\delta(\mathcal{M}_{f_{a,b}})$ .*

**Proof.** As  $\delta(\mathbb{N}_{f_{a,b}}) = 1$  it is enough to see that  $\mathbb{N}_{f_{a,b}} \setminus \mathcal{M}_{f_{a,b}}$  has an asymptotic density.

Cases iii) and iv) above are obvious. In cases i) and ii) it is enough to apply the previous lemma since  $\mathbb{N}_{f_{a,b}} \setminus \mathcal{M}_{f_{a,b}}$  is a countable union of arithmetic progressions modulo  $p \cdot p_a$  whose initial terms,  $p \cdot \mu_{a,b}(p)$ , form a set of zero asymptotic density, and the associated series of reciprocal moduli

$$\sum_{p \text{ prime}} \frac{1}{p \cdot p_a} = \sum_{p \text{ prime}} \frac{\gcd(a, p-1)}{p(p-1)}$$

is convergent.  $\square$

The rest of this section will be devoted to the study of  $\delta(\mathcal{M}_{f_{1,b}})$ . If  $b$  is even,  $\mathcal{M}_{f_{1,b}}$  is exactly the set of odd positive integers  $> 1 - b$  and its asymptotic density is  $\frac{1}{2}$ . The case when  $b$  is odd is much more interesting. In particular we will see that, in this case, the asymptotic density of  $\mathcal{M}_{f_{1,b}}$  is slightly greater than  $\frac{1}{2}$ . Our density computation will be based on the following corollary of Theorem 4.1.

**Corollary 4.4** *Put*

$$\mathcal{G}_p^b := \{-bp \pmod{p(p-1)}\}. \quad (4.1)$$

*If  $b \in \mathbb{Z}$  is odd, then  $\mathbb{N}_{f_{1,b}} \setminus \mathcal{M}_{f_{1,b}} = \bigcup_{p \geq 3} \{\mathcal{G}_p^b\}_{f_{1,b}} \cup \{2 \pmod{4}\}_{f_{1,b}}$ .*

We note that  $\delta(\bigcup_{p \geq 3} \mathcal{G}_p^0)$  is the density of the set of integers such that  $p(p-1) \mid m$  for some  $p \mid m$  with  $p \geq 3$ . Note that, for  $b$  odd,

$$\delta(\mathcal{M}_{f_{1,b}}) = 1 - \delta(\mathbb{N}_{f_{1,b}} \setminus \mathcal{M}_{f_{1,b}}) = 1 - \delta\left(\bigcup_{p \geq 3} \mathcal{G}_p^b \cup \{2 \pmod{4}\}\right) = \frac{3}{4} - \delta\left(\bigcup_{p \geq 3} \mathcal{G}_p^b\right), \quad (4.2)$$

where we used the observation that  $\mathcal{G}_p^b$  consists of odd integers only. The final density in (4.2) can be computed using the inclusion-exclusion principle. For this it will be necessary to have a good criterion to determine when

the intersection of  $\mathcal{G}_p^b$  for various odd primes  $p$  is empty. For  $m$  square-free we have  $\text{lcm}[p-1 : p \mid m] = \lambda(m)$ , with  $\lambda$  the Carmichael function.

**Proposition 4.5** *Let  $\mathcal{P}$  be a finite set of odd primes and put  $m := \prod_{p \in \mathcal{P}} p$ . Then  $\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b$  is non-empty if and only if  $\gcd(m, \phi(m)) \mid b$ . If the intersection is non-empty, then the set  $\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b$  is an arithmetic progression having modulus  $\text{lcm}(m, \lambda(m))$ .*

**Proof.** It is clear that  $\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b$  is non-empty if and only if there exists  $n$  such that  $n/p \equiv n \equiv -b \pmod{p-1}$  and  $p \mid n$  for every  $p \in \mathcal{P}$ . This happens if and only if there exists  $n$  such that  $n \equiv -b \pmod{\lambda(m)}$  and  $n \equiv 0 \pmod{m}$ . Note that the latter congruences have a solution if and only if  $\gcd(m, \lambda(m))$  divides  $b$ . To finish the proof it is enough to observe that,  $m$  being square-free,  $\gcd(m, \lambda(m)) = \gcd(m, \phi(m))$  and to apply the Chinese remainder theorem.  $\square$

To compute the density of the set  $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$  we define, given  $\epsilon > 0$ ,  $k := k(\epsilon)$  to be the smallest integer such that

$$\sum_{j \geq k} \frac{1}{p_j(p_j - 1)} < \epsilon.$$

Thus, with an error of at most  $\epsilon$ , the density of the set  $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$  is the same as the density of  $\bigcup_{j < k} \mathcal{G}_{p_j}^b$ :

$$\delta \left( \bigcup_{j < k} \mathcal{G}_{p_j}^b \right) < \delta(\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}) < \delta \left( \bigcup_{j < k} \mathcal{G}_{p_j}^b \right) + \epsilon$$

and, by the inclusion-exclusion principle, we find

$$\delta \left( \bigcup_{j < k} \mathcal{G}_{p_j}^b \right) = \sum_{s \geq 1} \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq k-1} \frac{\alpha_{i_1, i_2, \dots, i_s}}{\text{lcm}[p_{i_1}(p_{i_1} - 1), \dots, p_{i_s}(p_{i_s} - 1)]},$$

with the coefficient  $\alpha_{i_1, i_2, \dots, i_s}$  being zero if  $\bigcap_{l=1}^s \mathcal{G}_{p_{i_l}}^b = \emptyset$ , and being  $(-1)^{s-1}$  otherwise. Alternatively we can write this as

$$\delta \left( \bigcup_{j < k} \mathcal{G}_{p_j}^b \right) = - \sum_{\substack{m > 1, m \mid p_2 p_3 \dots p_{k-1} \\ \gcd(m, \varphi(m)) \mid b}} \frac{\mu(m)}{\text{lcm}(m, \lambda(m))}. \quad (4.3)$$

It is not difficult to see, cf. [4], that the series

$$\sum_{\gcd(m, \varphi(m)) \mid b} \frac{\mu(m)}{\text{lcm}(m, \lambda(m))}$$

converges absolutely. Using this, (4.2) and (4.3), we then obtain the following result.

**Theorem 4.6** *If  $b \in \mathbb{Z}$  is odd, then*

$$\delta(\mathcal{M}_{f_{1,b}}) = \frac{3}{4} + \sum_{\substack{m > 1, 2 \nmid m \\ \gcd(m, \varphi(m)) \mid b}} \frac{\mu(m)}{\text{lcm}(m, \lambda(m))}.$$

**Corollary 4.7** *We have*

$$\delta(\mathcal{M}_{f_{1,\pm 1}}) = \frac{3}{4} + \sum_{\substack{m > 2 \\ \gcd(m, \varphi(m)) = 1}} \frac{(-1)^{\omega(m)}}{m\lambda(m)},$$

where  $\omega(m)$  is the number of distinct prime factors of  $m$ .

**Proof.** Note that  $m > 1$ ,  $2 \nmid m$  and  $\gcd(m, \varphi(m)) \in \{-1, 1\}$  if and only if  $m > 2$  and  $\gcd(m, \varphi(m)) = 1$ . The  $m$  satisfying these conditions are odd and square-free and thus we have  $\gcd(m, \varphi(m)) = \gcd(m, \lambda(m)) = 1$  and hence  $\text{lcm}(m, \lambda(m)) = m\lambda(m)$  and  $\mu(m) = (-1)^{\omega(m)}$ .  $\square$



The asymptotic density of  $\mathcal{M}_{f_{1,\pm 1}}$  is closely related to that of the set

$$\mathfrak{P} := \left\{ n \geq 1 : 2 \nmid n, S_{\frac{n-1}{2}} \equiv 0 \pmod{n} \right\},$$

which was defined and studied in [4] and where it is shown that

$$\delta(\mathfrak{P}) = \frac{1}{2} + \sum_{\substack{m \geq 2 \\ \gcd(m, \varphi(m))=1}} \frac{(-1)^{\omega(m)}}{2m\lambda(m)} \in [0.379005, 0.379826].$$

By combining this with Corollary 4.7 we reach the following conclusion.

**Proposition 4.8** *We have  $\delta(\mathcal{M}_{f_{1,\pm 1}}) = 2\delta(\mathfrak{P}) - 1/4 \in [0.50801, 0.50966]$ .*

Recall that a Carmichael number  $n$  is a positive composite integer that satisfies Fermat's Little Theorem:  $a^{n-1} \equiv 1 \pmod{n}$  for every  $a$  coprime to  $n$ . It follows that a Carmichael number  $n$  meets Korselt's criterion: it must be square-free with  $p-1$  dividing  $n-1$  for each prime factor  $p$  of  $n$ . We will say that a positive integer  $n$  is an *anti-Korselt number* if for every  $p$  prime divisor of  $n$ ,  $p-1$  does not divide  $n-1$ .

**Lemma 4.9**

- i) *An integer  $n$  is an anti-Korselt number if and only if  $2 \nmid n$  and  $n \mid S_{n-1}(n)$ .*
- ii) *The set of anti-Korselt numbers  $\mathfrak{K}$  has an asymptotic density  $\delta(\mathfrak{K})$  satisfying*

$$\delta(\mathfrak{K}) = \delta(\mathcal{M}_{f_{1,-1}}) - \frac{1}{4} = 2\delta(\mathfrak{P}) - \frac{1}{2} \in [0.25801, 0.259652].$$

**Proof.**

- i) This follows from Theorem 1.2 and the observation that anti-Korselt numbers are odd.
- ii) The density  $\delta(\mathfrak{K})$  equals that of the odd integers in  $\mathcal{M}_{f_{1,-1}}$ , and hence, keeping in mind that the sets  $G_p^{-1}$ ,  $p \geq 3$ , consist of odd numbers only, we infer from Corollary 4.4 that

$$\delta(\mathfrak{K}) = \delta(\{n : 2 \nmid n\}) - \delta\left(\bigcup_{p \geq 3} G_p^{-1}\right) = \frac{1}{2} - \delta\left(\bigcup_{p \geq 3} G_p^{-1}\right).$$

By (4.2) we see that  $\delta(\mathfrak{K}) = \delta(\mathcal{M}_{f_{1,-1}}) - 1/4$ . Now invoke Proposition 4.8. □

**Lemma 4.10** *Let  $\{A_i\}_{i=1}^n$  and  $\{B_i\}_{i=1}^n$  be two families of sets such that:*

- i)  $\delta(A_i) = \delta(B_i)$ .
- ii)  $\delta(A_i \cap A_j) \geq \delta(B_i \cap B_j)$ .

*Then*

$$\delta\left(\bigcup_{i=1}^n A_i\right) \leq \delta\left(\bigcup_{i=1}^n B_i\right),$$

*with the inequality being strict if any of the inequalities in ii) is strict.*

**Proof.** We proceed by induction on  $n$ . The result for  $n = 2$  is trivial. Now, assume that

$$\delta\left(\bigcup_{i=1}^n A_i\right) \leq \delta\left(\bigcup_{i=1}^n B_i\right).$$

Note that, from condition ii) it follows that

$$\delta\left(A_{n+1} \cap \left(\bigcup_{i=1}^n A_i\right)\right) \geq \delta\left(B_{n+1} \cap \left(\bigcup_{i=1}^n B_i\right)\right).$$

Hence, we have that

$$\begin{aligned} \delta\left(\bigcup_{i=1}^{n+1} A_i\right) &= \delta\left(\bigcup_{i=1}^n A_i\right) + \delta(A_{n+1}) - \delta\left(A_{n+1} \cap \left(\bigcup_{i=1}^n A_i\right)\right) \\ &\leq \delta\left(\bigcup_{i=1}^n B_i\right) + \delta(B_{n+1}) - \delta\left(B_{n+1} \cap \left(\bigcup_{i=1}^n B_i\right)\right) = \delta\left(\bigcup_{i=1}^{n+1} B_i\right), \end{aligned}$$

and the result follows.  $\square$

**Lemma 4.11** *Let  $b \mid b'$  and suppose that  $m$  is an odd integer. We have*

$$\delta\left(\bigcup_{p|m} G_p^b\right) \geq \delta\left(\bigcup_{p|m} G_p^{b'}\right) \geq \delta\left(\bigcup_{p|m} G_p^0\right) = - \sum_{d|m, d>1} \frac{\mu(d)}{\text{lcm}(d, \lambda(d))}.$$

**Proof.** We consider the families  $\{\mathcal{G}_p^{b'}\}$ ,  $\{\mathcal{G}_p^b\}$  and  $\{\mathcal{G}_p^0\}$  (recall Corollary 4.4). Since  $\mathcal{G}_p^{b'}$ ,  $\mathcal{G}_p^b$  and  $\mathcal{G}_p^0$  are arithmetic progressions of the same modulus  $p(p-1)$ , it follows that  $\delta(\mathcal{G}_p^{b'}) = \delta(\mathcal{G}_p^b) = \delta(\mathcal{G}_p^0)$ . Also observe that, if  $p \neq q$  are primes and  $\mathcal{G}_p^{b'} \cap \mathcal{G}_q^{b'} = \emptyset$ , then also  $\mathcal{G}_p^b \cap \mathcal{G}_q^b = \emptyset$ . On the other hand, if  $\mathcal{G}_p^{b'} \cap \mathcal{G}_q^{b'} \neq \emptyset$ , then  $\mathcal{G}_p^b \cap \mathcal{G}_q^b$  is either empty or has the same density as  $\mathcal{G}_p^{b'} \cap \mathcal{G}_q^{b'}$ . Note that the intersection  $\mathcal{G}_p^0 \cap \mathcal{G}_q^0$  is never empty. On applying Lemma 4.10 the two inequalities are established. The final identity holds by an argument similar to the one used to establish Equation (4.3), where we use again that the intersection  $\mathcal{G}_p^0 \cap \mathcal{G}_q^0$  is never empty.  $\square$

**Corollary 4.12** *If  $b \mid b'$  and  $m$  is odd, then*

$$\delta(\mathcal{M}_{f_{1,b}}) \leq \delta(\mathcal{M}_{f_{1,b'}}) \leq \sum_{d|m} \frac{\mu(d)}{\text{lcm}(d, \lambda(d))} - \frac{1}{4}.$$

By applying Proposition 4.8 and Corollary 4.12 with  $m$  the product of the first 22 odd primes, we obtain

$$0.508 < \delta(\mathcal{M}_{f_{1,1}}) < \delta(\mathcal{M}_{f_{1,b}}) < 0.647.$$

It is easy to observe using Lemma 4.10 that if  $b$  is odd and  $|b| > 1$ , then  $\delta(\mathcal{M}_{f_{1,b}}) > \delta(\mathcal{M}_{f_{1,1}})$ . Let  $\kappa(n) = \prod_{p|n} p$  denote the squarefree kernel of  $n$ . By Theorem 4.6 it follows that if  $\kappa(b) = \kappa(b')$ , then  $\delta(\mathcal{M}_{f_{1,b'}}) = \delta(\mathcal{M}_{f_{1,b}})$ . Moreover, if  $\kappa(b) \mid \kappa(b')$  and  $\kappa(b) < \kappa(b')$ , then  $\delta(\mathcal{M}_{f_{1,b'}}) > \delta(\mathcal{M}_{f_{1,b}})$ .

## 5 $\mathcal{M}_f$ containing the prime numbers

In this section we will characterize the set  $\mathcal{M}_f$  for some functions  $f$  such that  $f(p) = \frac{p-1}{2}$  for every odd prime. Note that in this case  $\mathcal{M}_f$  contains all odd primes  $p > 3$ . In particular, we will focus on  $f = \frac{\varphi}{2}$  and  $f = \frac{\lambda}{2}$ , where  $\varphi$  and  $\lambda$  denote the Euler and Carmichael function, respectively.

**Proposition 5.1** *We have  $\mathcal{M}_{\frac{\varphi}{2}} = \{p^k : p \text{ odd prime}\} \setminus \{3\}$ .*

**Proof.** Note that  $\frac{\varphi(p^k)}{2} > 1$  if and only if  $p^k \neq 3$ . Hence,  $3 \notin \mathcal{M}_{\frac{\varphi}{2}}$  and in what follows we assume that  $p^k \neq 3$ .

If  $p$  is an odd prime and  $k \in \mathbb{N}$ ,  $\frac{\varphi(p^k)}{2} = \frac{p^{k-1}(p-1)}{2}$  and  $\gcd\left(\frac{p^{k-1}(p-1)}{2}, p-1\right) < p-1$ . Consequently we can apply Theorem 1.2 to get  $p^k \in \mathcal{M}_{\frac{\varphi}{2}}$ .

Now, if  $n$  is odd and there exist distinct odd primes  $p, q$  dividing  $n$ , it readily follows that  $p-1$  divides  $\frac{\varphi(n)}{2}$  so Theorem 1.2 i) applies and it follows that  $n \notin \mathcal{M}_{\frac{\varphi}{2}}$ . Thus, if there is an odd  $n \in \mathcal{M}_{\frac{\varphi}{2}}$  it must be a prime power exceeding 3.

Finally, if  $n \in \mathcal{M}_{\frac{\varphi}{2}}$  is even, Theorem 1.2 ii) implies that 4 divides  $n$  and also that  $\frac{\varphi(n)}{2}$  is odd and exceeding 1. Since these statements are contradictory the result follows.  $\square$

In what follows we will use the notation  $v_2(m) := \max\{k \in \mathbb{N} : 2^k \text{ divides } m\}$ .

**Proposition 5.2** Let  $n = 2^m p_1^{r_1} \cdots p_s^{r_s}$  with  $s > 0$ . Then  $3 \neq n \in \mathcal{M}_{\frac{\lambda}{2}}$  if and only if one of these conditions holds:

- i)  $m = 0$  and  $v_2(p_i - 1) = v_2(p_j - 1)$  for every  $i, j$ .
- ii)  $m = 2$  or  $3$ ,  $v_2(p_i - 1) = 1$  for every  $i$  and  $\frac{n}{2^m} \neq 3$ .

**Proof.** Note that  $\frac{\lambda(n)}{2} > 1$  if and only if  $n \neq 3$ . Hence,  $3 \notin \mathcal{M}_{\frac{\lambda}{2}}$  and in what follows we assume that  $n \neq 3$ .

If condition i) holds,  $n = p_1^{r_1} \cdots p_s^{r_s}$  and  $p_i = 2^t q_i + 1$  with  $q_i$  even and  $t$  not depending on  $i$ . In this case  $\lambda(n) = \text{lcm}(\varphi(p_1^{r_1}), \dots, \varphi(p_s^{r_s})) = 2^t \text{lcm}(p_1^{r_1-1} q_1, \dots, p_s^{r_s-1} q_s) = 2^t L$  with  $L$  odd. Consequently  $\frac{\lambda(n)}{2} = 2^{t-1} L$  and since  $L$  is odd it follows that  $p_i - 1$  does not divide  $\frac{\lambda(n)}{2}$  and Theorem 1.2 i) implies that  $n \in \mathcal{M}_{\frac{\lambda}{2}}$ .

If condition ii) holds, it follows that  $\lambda(n) = 2L$  with  $L > 1$  odd. Consequently  $\lambda(n)/2 = L > 1$  is odd and by Theorem 1.2 ii) we conclude that  $n \in \mathcal{M}_{\frac{\lambda}{2}}$ .

Finally, assume that  $n = 2^m p_1^{r_1} \cdots p_s^{r_s}$  with  $s > 0$  and  $p_i = 2^{m_i} q_i + 1$  with  $q_i$  odd is such that  $n \in \mathcal{M}_{\frac{\lambda}{2}}$ . First of all, Theorem 1.2 implies that  $m = 0$  or  $m > 1$ .

If  $m > 1$ , Theorem 1.2 ii) implies that  $\frac{n}{2^m} \neq 3$  and also that  $\frac{\lambda(n)}{2}$  is odd so  $m = 2$  or  $3$  and  $p_i^{r_i-1}(p_i - 1) = \varphi(p_i^{r_i}) = 2L_i$  with  $L_i$  odd; i.e.,  $p_i - 1 = 2q_i$  with  $q_i$  odd as claimed.

If, on the other hand,  $m = 0$ , Theorem 1.2 i) implies that  $p_i - 1$  does not divide  $\frac{\lambda(n)}{2}$  for any  $i$ . But if  $m_i > m_j$  for some  $i \neq j$  we have that  $2^{m_i-1} q_j$  divides  $\frac{\lambda(n)}{2}$  and, consequently,  $p_j - 1$  divides  $\frac{\lambda(n)}{2}$ . A contradiction.  $\square$

Now, given a positive integer  $k$  we define the set

$$\Upsilon_k := \{n \text{ odd} : v_2(p - 1) = k \text{ for every } p \mid n\}.$$

Note that if  $k \neq j$ , then  $\Upsilon_k$  and  $\Upsilon_j$  are disjoint. With this notation, Proposition 5.2 can be stated as

$$\mathcal{M}_{\frac{\lambda}{2}} = \left( \bigcup_{k=1}^{\infty} \Upsilon_k \cup 4\Upsilon_1 \cup 8\Upsilon_1 \right) \setminus \{3, 12, 24\}. \quad (5.1)$$

Let  $\mathcal{M}_{\frac{\lambda}{2}}(x)$  denote the number of integers  $\leq x$  in the set  $\mathcal{M}_{\frac{\lambda}{2}}$  and  $\Upsilon_j(x)$  the number of integers  $\leq x$  in the set  $\Upsilon_j$ .

**Proposition 5.3** Let  $k \geq 1$  be an arbitrary integer. We have

$$\mathcal{M}_{\frac{\lambda}{2}}(x) = \frac{x}{\log x} \left( c_1 \log^{1/2} x + \sum_{j=2}^k c_j \log^{2-k} x + O_k(\log^{2-k-1} x) \right),$$

with

$$c_1 = \frac{11}{16} \prod_{p \equiv 1 \pmod{4}} \left( 1 - \frac{1}{p^2} \right)^{1/2} = 0.66896484 \dots$$

and all constants  $c_2, \dots, c_k$  positive. The implied constant in the error term depends at most on  $k$ .

**Proof.** For positive coprime integers  $a$  and  $d$ , let  $N_{a,d}(x)$  denote the number of integers  $n \leq x$  that are composed only of primes  $p \equiv a \pmod{d}$ . It is a standard result, cf. [15], that

$$N_{a,d}(x) = \frac{c_{a,d} x}{\log^{1-1/\varphi(d)} x} \left( 1 + O_d \left( \frac{1}{\log x} \right) \right), \quad (5.2)$$

with  $c_{a,d}$  a positive constant. For  $j \geq 1$  we have, by (5.2),

$$\Upsilon_j(x) = N_{1+2^j, 2^{j+1}}(x) = \frac{d_j x}{\log^{1-2^{-j}} x} \left( 1 + O_j \left( \frac{1}{\log x} \right) \right), \quad (5.3)$$

with  $d_j$  a positive constant. One has, cf. [15, p. 235],

$$d_1 = \frac{1}{2} \prod_{p \equiv 1 \pmod{4}} \left( 1 - \frac{1}{p^2} \right)^{1/2} = 0.4865198883 \dots \quad (5.4)$$

Note that

$$\sum_{j=k+1}^{\infty} \Upsilon_j(x) \leq N_{1,2^{k+1}}(x) = O(x \log^{2^{-k-1}-1} x). \quad (5.5)$$

Since the infinite sets in the decomposition (5.1) are pairwise disjoint we see from (5.1) that

$$\mathcal{M}_{\frac{1}{2}}(x) = \Upsilon_1(x) + \Upsilon_1\left(\frac{x}{4}\right) + \Upsilon_1\left(\frac{x}{8}\right) + \sum_{j=2}^k \Upsilon_j(x) + \sum_{j=k+1}^{\infty} \Upsilon_j(x) + O(1).$$

The result now follows from (5.3) and (5.5) with  $c_1 = 11d_1/8$  and  $c_j = d_j$  for  $j \geq 2$ .  $\square$

**Remark 5.4** By Satz 1 of Wirsing [22] we have

$$N_{3,4}(x) \sim \frac{e^{-\gamma/2}}{\sqrt{\pi}} \frac{x}{\log x} \prod_{\substack{p \leq x \\ p \equiv 3 \pmod{4}}} (1 - 1/p)^{-1}.$$

On inserting Uchiyama's asymptotic for the latter product (see [21]) and using  $\prod_p (1 - 1/p^2) = 1/\zeta(2) = 6/\pi^2$ , one finds that  $N_{3,4}(x) \sim d_1 x (\log x)^{-1/2}$  with  $d_1$  as in (5.4).

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