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On the Zero Divisor Graphs of the Ring of Lipschitz Integers Modulo n

José María Grau, Celino Miguel, Antonio M. Oller-Marcén*

Abstract. This article studies the zero divisor graphs of the ring of Lipschitz integers modulo n. In particular we focus on the number of vertices, the diameter and the girth. We also give some results regarding the domination number of these graphs.

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1. Introduction

The idea of studying the interplay between ring-theoretic properties of a ring R and graph-theoretic properties of a graph defined after it, is quite recent. It was first introduced for commutative rings by Beck in 1988 [7]. In Beck's definition the vertices of the graph are the elements of the ring and two distinct vertices x and y are adjacent if and only if $xy = 0$. Later, Anderson and Livingston [5] slightly modified this idea, considering only the non-zero zero divisors of R as vertices of the graph with the same adjacency condition. Redmond [24] extended this notion of zero-divisor graph to noncommutative rings.

Given a ring R with identity we denote the set of zero divisors of R by $Z(R)$, and the set of non-zero zero divisors by $Z^*(R)$. The set of units in R, that is, the set of invertible elements, is denoted by $\mathcal{U}(R)$. As is well known in a finite ring every element is either a unit or a zero divisor. As usual we use $|S|$ to denote the cardinality of the set S.

For a noncommutative ring R with identity, we define two different graphs associated to R. The directed zero divisor graph, $\Gamma(R)$, and the undirected zero divisor graph, $\overline{\Gamma}(R)$. Both graphs share the same vertex set, namely, the set $Z^*(R)$ of non-zero zero divisors of R. In $\Gamma(R)$, given two distinct vertices x and y, there is a directed edge of the form $x \to y$ if and only if $xy = 0$. On the other hand, two distinct vertices x and y of $\overline{\Gamma}(R)$ are

*Corresponding author.

connected by an edge if and only if either $xy = 0$ or $yx = 0$. Note that, by definition, both $\Gamma(R)$ and $\overline{\Gamma}(R)$ are simple graphs, so there are no loops, thus the existence of self-annihilating elements of R is not encoded in the graph. Several properties of zero divisor graphs of different general classes of rings are studied in [3,4,16,24,26].

Recall that, if $n > 1$ is a rational integer and $\langle n \rangle$ is the ideal in the Gaussian integers generated by n, then the factor ring $\mathbb{Z}[i]/\langle n \rangle$ is isomorphic to the Gaussian integers modulo n

$$\mathbb{Z}_n[i] := \{a + bi : a, b \in \mathbb{Z}_n\}.$$

The zero divisor graph of the ring of Gaussian integers modulo n has recently received great attention [1,2,23].

The algebraic construction defined above for the Gaussian integers can be easily extended to the ring $\mathbb{Z}[i, j, k]$ of Lipschitz integer quaternions. Indeed, let again be $n > 1$ a rational integer and denote by $\langle n \rangle$ the principal ideal in $\mathbb{Z}[i, j, k]$ generated by n. Then, the factor ring $\mathbb{Z}[i, j, k]/\langle n \rangle$ is isomorphic to

$$\mathbb{Z}_n[i, j, k] := \{a + ib + cj + dk : a, b, c, d \in \mathbb{Z}_n\},$$

which is called the ring of Lipschitz quaternions modulo n.

The aim of this paper is to study the zero divisor graphs of the ring of Lipschitz quaternions modulo n, both the directed $\Gamma(\mathbb{Z}_n[i, j, k])$ and the undirected $\overline{\Gamma}(\mathbb{Z}_n[i, j, k])$.

If $n = p_1^{r_1} \dots p_k^{r_k}$, is the prime power factorization of n, the Chinese remainder theorem induces a natural isomorphism

$$\mathbb{Z}_n[i, j, k] \cong \mathbb{Z}_{p_1^{r_1}}[i, j, k] \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}[i, j, k]. \tag{1}$$

Therefore, in order to study the structure of the rings $\mathbb{Z}_n[i, j, k]$ we can restrict ourselves to the prime power case.

If p is an odd prime and l is a positive integer, then $\mathbb{Z}_{p^l}[i, j, k]$ is isomorphic to the full matrix ring $M_2(\mathbb{Z}_{p^l})$ [13,25]. Consequently, for an odd positive integer n the ring $\mathbb{Z}_n[i, j, k]$ is isomorphic to the matrix ring $M_2(\mathbb{Z}_n)$. Hence, in this case we can use known results about the zero divisor graph of matrix rings over commutative rings.

Unfortunately, if n is even it is no longer true that $\mathbb{Z}_n[i, j, k]$ is isomorphic to the matrix ring $M_2(\mathbb{Z}_n)$. In fact, note that an element $z = z_0 + z_1 i + z_2 j + z_3 k \in \mathbb{Z}_n[i, j, k]$ is a unit if and only if its norm $\|z\| = z_0^2 + z_1^2 + z_2^2 + z_3^2$ is a unit in $\mathbb{Z}_n$. Since

$$\|z + w\| = \|z\| + \|w\| + 2\mathrm{Re}(z\overline{w}),$$

the sum of two units of $\mathbb{Z}_n[i, j, k]$ is never a unit. This fact is clearly false in $M_2(\mathbb{Z}_n)$ and the claim holds.

2. The Number of Vertices

Recall that both graphs $\Gamma(\mathbb{Z}_n[i,j,k])$ and $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ share the same set of vertices. Namely, the non-zero zero divisors of $\mathbb{Z}_n[i,j,k]$. Due to the isomorphism (1) we can focus on the case when n is a prime power.

Proposition 1. *Let p be an odd prime number and $t \geq 1$. Then, the number of vertices of the graph $\overline{\Gamma}(\mathbb{Z}_{p^t}[i,j,k])$ is*

$$p^{4t-1} + p^{4t-2} - p^{4t-3} - 1.$$

Proof. We have the isomorphism $\mathbb{Z}_{p^t}[i,j,k] \cong M_2(\mathbb{Z}_{p^t})$. Thus, it is enough to determine the number of non-zero zero divisors in the matrix ring $M_2(\mathbb{Z}_{p^t})$.

Following [15] there are $p^{4(t-1)}(p^2-1)(p^2-p)$ units in $M_2(\mathbb{Z}_{p^t})$. Since in a finite ring every element is either a unit or a zero divisor it follows that there are $p^{4t-1} + p^{4t-2} - p^{4t-3} - 1$ non-zero divisors in $M_2(\mathbb{Z}_{p^t})$, as claimed. $\square$

If $p = 2$ the isomorphism $\mathbb{Z}_{2^t}[i,j,k] \cong M_2(\mathbb{Z}_{2^t})$ is no longer true. Consequently, we must use a different approach in order to compute the number of non-zero zero divisors of $\mathbb{Z}_{2^t}[i,j,k]$.

Proposition 2. *The number of vertices of the graph $\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])$ is $2^{4t-1}-1$.*

Proof. Note that an element $z = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{Z}_{2^t}[i,j,k]$ is a unit if and only if and only if its norm $\|x\| = x_0^2 + x_1^2 + x_2^2 + x_3^2$ is a unit in $\mathbb{Z}_{2^t}$. On the other hand, an element is a unit in $\mathbb{Z}_{2^t}$ if its reduction modulo 2 is a unit. Since $x_0^2+x_1^2+x_2^2+x_3^2 \equiv 0 \pmod 2$ if and only if $x_0^2+x_1^2+x_2^2+(x_3+1)^2 \equiv 1 \pmod 2$, it follows that

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto x_0 + x_1 i + x_2 j + (x_3+1)k$$

defines a one-to-one correspondence between the set of all zero divisors and the set of all units in $\mathbb{Z}_{2^t}[i,j,k]$. Again, by the fact that every element is either a unit or a zero divisor we obtain $|\mathcal{U}(\mathbb{Z}_{2^t}[i,j,k])| = |Z(\mathbb{Z}_{2^t}[i,j,k])| = 2^{4t-1}$. Hence, there are exactly $2^{4t-1}-1$ non-zero zero divisors in $\mathbb{Z}_{2^t}[i,j,k]$, which is the number of vertices of the graph $\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])$ as claimed. $\square$

Remark 1. Observe that, if $x = x_0+x_1i+x_2j+x_3k$ and $y = y_0+y_1i+y_2j+y_3k$ are two zero divisors, then $x_0^2+x_1^2+x_2^2+x_3^2 \equiv y_0^2+y_1^2+y_2^2+y_3^2 \equiv 0 \pmod 2$. It is clear that

$$(x_0+y_0)^2 + (x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 \equiv 0 \pmod 2$$

and hence the sum of two zero divisors is a zero divisor. Therefore, the set of all zero divisors forms an ideal. This ideal is necessarily the unique maximal ideal of $\mathbb{Z}_{2^t}[i,j,k]$, since a proper ideal can not contain a unit.

Recall that, given a direct sum of rings $R = R_1 \oplus \ldots \oplus R_k$, an element $r \in R$ is a unit if and only if every projection of r in R_i is a unit in R_i. Hence, if $n = 2^t p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime power decomposition of n, the isomorphism (1) together with the fact that an element in a finite ring is either a unit or a zero divisor leads to:

$$|\mathcal{U}(\mathbb{Z}_n[i,j,k])| = \begin{cases} \prod_{i=1}^k \left(p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}\right), & \text{if } t=0; \\ 2^{4t-1}\prod_{i=1}^k (p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}), & \text{if } t>0. \end{cases}$$

As a consequence of the previous work we have the main result of this section.

Theorem 1. *Let $n = 2^t p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime power decomposition of n. Then, the number of vertices in the graph $\Gamma(\mathbb{Z}_n[i,j,k])$ or $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ is:*

$$|V(\Gamma(\mathbb{Z}_n[i,j,k]))| = \begin{cases} n^4 - \prod_{i=1}^k \left(p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}\right) - 1, & \textit{if } t = 0; \\ n^4 - 2^{4t-1} \prod_{i=1}^k (p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}) - 1, & \textit{if } t > 0. \end{cases}$$

Proof. Just apply the previous observation recalling that the non-zero elements of a finite ring are either units or zero-divisors. □

3. The Diameter

We recall that the *distance* between two distinct vertices a and b of a graph, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting them (the distance being infinity if no such path exists). The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is given by

$$\mathrm{diam}(G) = sup\{\mathrm{d}(a,b) : a, b \text{ distinct vertices of } G\},$$

if G has at least two distinct vertices and $\mathrm{diam}(G) = 0$ otherwise.

Our objective in this section is to find the diameter of the directed zero divisor graph $\Gamma(\mathbb{Z}_n[i,j,k])$ and of the undirected zero divisor graph $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$.

Recall that $Z_L(R)$ and $Z_R(R)$ denote, respectively, the set of left and right zero divisors of R. The following result was proved in [24].

Theorem 2. *Let R be a noncommutative ring, with $Z^*(R) \neq \emptyset$. Then $\Gamma(R)$ is connected if and only if $Z_L(R) = Z_R(R)$. If $\Gamma(R)$ is connected, then $diam(\Gamma(R)) \leq 3$.*

Note that in any finite ring R we have $Z_L(R) = Z_R(R)$. Hence, the previous theorem implies that the directed zero divisor graph $\Gamma(\mathbb{Z}_n[i,j,k])$ is connected and $\mathrm{diam}(\Gamma(\mathbb{Z}_\mathrm{n}[\mathrm{i},\mathrm{j},\mathrm{k}])) \leq 3$. We will now see that, in many cases, the equality holds. To do so, we first need a technical result involving the direct sum of finite unital noncommutative rings. A commutative version was established in [6].

Lemma 3. *Let $R = R_1 \oplus R_2$, where R_1 and R_2 are finite unital noncommutative rings. Then.* $\mathrm{diam}(\Gamma(R)) = 3$.

Proof. First note that, since R_1 and R_2 are finite and noncommutative if follows by the Wedderburn's little theorem that both R_1 and R_2 have nonzero zero divisors.

On the other hand, since R_1 and R_2 are rings with identity we can choose a unit u_1 from R_1 and a unit u_2 from R_2. Let $x \in Z^*(R_1)$ and $y \in Z^*(R_2)$ and consider the elements $(x, u_2), (u_1, y) \in Z^*(R)$. We will prove that the distance between the vertices (x, u_2) and (u_1, y) is 3. Indeed $(x, u_2)(u_1, y) =$

$(xu_1, u_2y) \neq (0,0)$. Hence $d((x,u_2),(u_1,y)) > 1$. On the other hand if $(a,b) \in Z^*(R)$ satisfies

$$(x,u_2)(a,b) = (a,b)(u_1,y) = (0,0),$$

then we have $u_2b = 0$ and $au_1 = 0$ implying $a = b = 0$, a contradiction. Therefore, $d((x,u_2),(u_1,y)) > 2$. Finally, using Theorem 2 we get the result. □

As a consequence, we have the following result.

Proposition 3. *Let n be an integer divisible by at least two primes. Then,*

$$\operatorname{diam}(\Gamma(\mathbb{Z}_n[i,j,k])) = 3.$$

Proof. It is enough to apply the isomorphism (1) together with Lemma 3. □

Remark 2. It is clear that $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_n[i,j,k])) \leq \operatorname{diam}(\Gamma(\mathbb{Z}_n[i,j,k]))$. Now, if n is divisible by at least two primes, there exist vertices in $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ that are not at distance 2. Hence, we obtain the equality $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_n[i,j,k])) = \operatorname{diam}(\Gamma(\mathbb{Z}_n[i,j,k]))$ in this case.

Now, we must focus on the prime power case. We first look at the odd case, where the following technical lemma is useful [9, Lem. 4.2; Cor. 4.1].

Lemma 4. *Let R be a commutative ring and $n \geq 2$. If every finite set of zero divisors from R has a non-zero annihilator, then* $\operatorname{diam}(\Gamma(M_n(R)) = 2$. *In particular, if F is a field, then* $\operatorname{diam}(\Gamma(M_n(F)) = 2$.

Proposition 4. *Let $t \geq 1$ and let p be an odd prime. Then,*

$$\operatorname{diam}(\Gamma(\mathbb{Z}_{p^t}[i,j,k])) = 2.$$

Proof. If $t = 1$, $\mathbb{Z}_p$ is a field and the result follows from the second part of Lemma 4. Now assume that $t > 1$. In this case the maximal ideal of the local ring $\mathbb{Z}_{p^t}$ is the principal ideal generated by p. This maximal ideal is nilpotent with index of nilpotence t. Therefore, the element p^{t-1} belongs to the annihilator of every zero divisor in $\mathbb{Z}_{p^l}$ and Lemma 4 applies again. □

Remark 3. Again, $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_{p^t}[i,j,k])) \leq \operatorname{diam}(\Gamma(\mathbb{Z}_{p^t}[i,j,k]))$. Now, if p is an odd prime, there exist vertices in $\overline{\Gamma}(\mathbb{Z}_{p^t}[i,j,k])$ that are not at distance 1. Hence, we obtain the equality $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_{p^t}[i,j,k])) = \operatorname{diam}(\Gamma(\mathbb{Z}_{p^t}[i,j,k]))$ in this case.

Finally, we turn to the $p = 2$ case. The following result computes the diameter of the undirected zero divisor graph.

Proposition 5. *Let $t \geq 1$. Then* $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])) = 2$.

Proof. The graph $(\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])$ is not complete since the vertices $1+i$ and $1+j$ are not adjacent. On the other hand, the vertex $2^{t-1}(1+i+j+k)$ is adjacent to all other vertices; i.e., every zero divisor of the ring $R = \mathbb{Z}_{2^t}[i,j,k]$ is annihilated on both sides by $2^{t-1}(1+i+j+k)$.

To see this, first of all note that the result is clearly true for $t = 1$, because there are only eight zero divisors in $R' = \mathbb{Z}_2[i,j,k]$. Now, if $t > 1$, let y be a zero divisor in R and let us prove that $2^{t-1}(1+i+j+k)y = 2^{t-1}y(1+i+j+k) = 0$. Indeed, if we consider the projection $\pi : R \to R'$ it is clear that $y' = \pi(y)$ is also a zero divisor; therefore y' is annihilated on both sides by $1+i+j+k$. Hence, $(1+i+j+k)y$ and $y(1+i+j+k)$ belong to the kernel of π, which is the ideal generated by 2. Thus, there are z_1 and z_2 in R such that $(1+i+j+k)y = 2z_1$ and $y(1+i+j+k) = 2z_2$, and the conclusion follows. □

Recall that a directed graph G is called symmetric if, for every directed edge $x \to y$ that belongs to G, the corresponding reversed edge $y \to x$ also belongs to G. We are going to prove that the directed zero divisor graph $\Gamma(\mathbb{Z}_{2^t}[i,j,k])$ is symmetric. As a consequence, it will follow that $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])) = \operatorname{diam}(\Gamma(\mathbb{Z}_{2^t}[i,j,k]))$.

A ring R is called reversible [10] if, for every $a, b \in R$, $ab = 0$ implies that $ba = 0$. Clearly, a ring R is reversible if and only if its directed zero divisor graph $\Gamma(R)$ is symmetric. Thus, to prove that $\Gamma(\mathbb{Z}_{2^t}[i,j,k])$ is symmetric we will prove that $\mathbb{Z}_{2^t}[i,j,k]$ is reversible. To do so, we need a series of technical lemmata.

Lemma 5. *Let $w \in \mathbb{Z}[i,j,k]$. If $\|w\| \equiv 0 \pmod 4$, then either all the components of w are even or all of them are odd.*

Proof. Put $w = a_1 + a_2 i + a_3 j + a_4 k$ and denote by $n_w := \operatorname{card}\{i : a_i \text{ is even}\}$. Since $a_i^2 \equiv 0, 1 \pmod 4$, it follows that $0 \equiv \|w\| \equiv n_w \pmod 4$ and hence the result. □

Lemma 6. *Let $w \in \mathbb{Z}[i,j,k]$. If $\|w\| \equiv 0 \pmod 8$, then all the components of w are even.*

Proof. Put $w = a_1 + a_2 i + a_3 j + a_4 k$. Since $\|w\| = a_1^2 + a_2^2 + a_3^2 + a_4^2 \equiv 0 \pmod 4$ the previous lemma implies that either all the components of w are even or all of them are odd. Assume that all of them are odd and put $a_i = 2a_i' + 1$ for every i. Then, $\|w\| = a_1^2 + a_2^2 + a_3^2 + a_4^2 = 4((a_1')^2 + a_1' + (a_2')^2 + a_2' + (a_3')^2 + a_3' + (a_4')^2 + a_4) + 4$. Hence, $((a_1')^2 + a_1' + (a_2')^2 + a_2' + (a_3')^2 + a_3' + (a_4')^2 + a_4') + 1 \equiv 0 \pmod 2$. This is clearly a contradiction and the result follows. □

Proposition 6. *Let $w, z \in \mathbb{Z}[i,j,k]$. If $wz \equiv 0 \pmod{2^t}$, then $zw \equiv 0 \pmod{2^t}$. In other words, the ring $\mathbb{Z}_{2^t}[i,j,k]$ is reversible.*

Proof. We will proceed by induction on t.

The case $t = 1$ is obvious since $\mathbb{Z}[i,j,k]/2\mathbb{Z}_2[i,j,k]$ is trivially commutative.

Let us consider $t = 2$ and assume that $wz \equiv 0 \pmod 4$. Hence, $\|w\|\|z\| = \|wz\| \equiv 0 \pmod{16}$. If $\|w\| \equiv 0 \mod 8$ we can apply Lemma 6 to conclude that $w = 2w'$ for some $w' \in \mathbb{L}$. Hence (using the case $t = 1$) we have,

$$wz \equiv 0 \pmod 4 \Leftrightarrow w'z \equiv 0 \pmod 2 \Leftrightarrow zw' \equiv 0 \pmod 2 \Leftrightarrow zw \equiv 0 \pmod 4.$$

the same holds if $\|z\| \equiv 0 \pmod 8$. Finally, if both $\|w\|, \|z\| \equiv 0 \pmod 4$ we apply Lemma 5 to conclude that all the components of w and z are odd.

But in this case it can be easily seen that $zw - wz \in 4\mathbb{Z}[i,j,k]$ and the result follows.

Now, assume that $t > 2$ and that $wz \equiv 0 \pmod{2^t}$. In this case $\|w\|\|z\| = \|wz\| \equiv 0 \pmod{2^{2t}}$ and, since $t > 2$ it follows that either $\|w\| \equiv 0 \pmod 8$ or $\|z\| \equiv 0 \pmod 8$. If, for instance, $\|w\| \equiv 0 \pmod 8$ we apply Lemma 6 again to conclude that $w = 2w'$ for some $w' \in \mathbb{Z}[i,j,k]$ and we can proceed like in the previous paragraph. The same holds if $\|z\| \equiv 0 \pmod 8$ and the proof is complete. □

Remark 4. The concept of *symmetric* ring was defined by Lambek in [19]: a ring R is symmetric if, for every $a, b, c \in R$, $abc = 0$ implies that also $acb = 0$. It has sometimes been erroneously asserted (and even "proved") that reversible and symmetric are equivalent conditions. If a unital ring is symmetric, then it is also reversible. But this is no longer true for non-unital rings, as illustrated by an example of Birkenmeier [8]. In the case of unital rings, the smallest known reversible non-symmetric ring was given in [21]. Namely, it is the group algebra $\mathbb{F}_2Q_8$ where $Q_8$is the quaternion group. In [14] it was proved that this is in fact the smallest reversible group algebra over a field which is not symmetric. In [20] it was also confirmed that $\mathbb{F}_2Q_8$ is indeed the smallest reversible group ring which is not symmetric. Note that $\mathbb{Z}_4[i,j,k]$ is a reversible ring due to Proposition 6 which is trivially non-symmetric. In fact, it is the smallest known ring with characteristic different from 2 with this property, having the same number of elements (256) as the aforementioned example $\mathbb{F}_2Q_8$.

Since $\mathbb{Z}_{2^t}[i,j,k]$ is reversible, $\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])) = \operatorname{diam}(\Gamma(\mathbb{Z}_{2^t}[i,j,k]))$, so from all the previous work we obtain the following result.

Theorem 7. *Let $n \geq 2$ be any integer. Then*

$$\operatorname{diam}(\overline{\Gamma}(\mathbb{Z}_n[i,j,k])) = \operatorname{diam}(\Gamma(\mathbb{Z}_n[i,j,k])) = \begin{cases} 2, & \text{if } n \text{ is a prime power;} \\ 3, & \text{otherwise.} \end{cases}$$

Recall that a graph G is complete provided every pair of distinct vertices is connected by a unique edge. In [2, Theorem15] it was proved that the undirected zero divisor graph for the ring of Gaussian integers modulo n, $\overline{\Gamma}(\mathbb{Z}_n[i])$, is complete if and only if $n = q^2$, where q is a rational prime such $q \equiv 3 \pmod 4$. In our case we have the following.

Corollary 1. *For an integer $n \geq 2$ the graph $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ is never complete.*

Proof. The diameter of a complete graph is 1. Since this is not possible due to Theorem 7, the result follows. □

4. The Girth

A *cycle* in a graph is a path starting and ending at the same vertex. The *girth* of G, denoted by $\mathrm{g}(G)$, is the length of the shortest cycle contained in G. If the graph does not contain any cycle, its girth is defined to be infinity. All the previous concepts can be defined for directed graphs just considering directed paths.

Let us consider the directed zero divisor graph $\Gamma(\mathbb{Z}_n[i,j,k])$. If $n \geq 2$ is odd then $g(\Gamma(\mathbb{Z}_n[i,j,k])) = 2$, since

$$\begin{bmatrix} 1\,0 & \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $n = 2^t$ the ring $\mathbb{Z}_{2^t}[i,j,k]$ is reversible and consequently $g(\Gamma(\mathbb{Z}_n[i,j,k])) = 2$.

Now, we turn to the undirected case. It is clear that $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ is a simple graph; i.e., it does not contain loops and two vertices are not connected by more that one edge. Thus, it follows that $g(\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])) \geq 3$. We will now see that the equality holds.

To compute the girth of $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ for odd n, we recall the following result [9, Prop.3.2]

Proposition 7. *Let R be a commutative ring and $n \geq 2$. Then* $\mathrm{g}(\overline{\Gamma}(M_n(R))) = 3$.

This proposition clearly implies that $\mathrm{g}(\overline{\Gamma}(\mathbb{Z}_n[i,j,k]) = 3$ because, for odd n, we have that $\mathbb{Z}_n[i,j,k] \cong M_2(\mathbb{Z}_n)$.

The case $n = 2^t$ is analyzed in the following result.

Proposition 8. *Let $t \geq 1$. Then* $\mathrm{g}(\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])) = 3$.

Proof. If $t = 1$, we have the cycle (see the previous remark) $(1+i)$—$(j+k)$—$(1+i+j+k)$—$(1+i)$, for instance. If $t = 2$, we have the cycle 2—$2i$—$(2+2i)$—2, for instance. Finally, if $t > 2$ we can consider the cycle 2^{t-1}—2—$2^{t-1}i$—2^{t-1}. This proves the result. □

Finally, if we recall the isomorphism (1), the previous discussion leads to the following.

Theorem 8. *For every integer $n \geq 2$,* $\mathrm{g}(\overline{\Gamma}(\mathbb{Z}_n[i,j,k])) = 3$.

A graph G is complete bipartite if its vertices can be partitioned into two subsets such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. In [2, Theorem17] it was proved that the undirected zero divisor graph for the ring of Gaussian integers modulo n, $\overline{\Gamma}(\mathbb{Z}_n[i])$, is complete bipartite if and only if $n = p^2$, where p is a rational prime such $p \equiv 1 \pmod 4$ or $n = q_1 q_2$, with q_1, q_2 rational primes such that $q_1 \equiv q_2 \equiv 3 \pmod 4$. In our case we have the following.

Corollary 2. *For an integer $n > 2$, the graph $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$ is never complete bipartite.*

Proof. The girth of a complete bipartite graph is 4. Since this is not possible due to Theorem 8, the result follows. □

5. The Domination Number

A *dominating set* for a graph G is a subset of vertices D, such that every vertex not in D is adjacent to at least one member of D. The *domination number*, denoted by $\gamma(G)$, is the number of vertices in a minimal dominating set.

The problem of determining the domination number of an arbitrary graph is NP-complete [11]. Nevertheless, particular cases have been recently studied. In [2], for instance, the domination number of the zero divisor graph of the ring of Gaussian integers modulo n was studied. In particular, the authors characterized the values of n for which the domination number of $\Gamma(\mathbb{Z}_n[i])$ is 1 or 2.

This section is devoted to study the domination number of the undirected zero divisor graph $\overline{\Gamma}(\mathbb{Z}_n[i,j,k])$. The easiest case arises when n is a power of 2.

Theorem 9. *The domination number of the undirected graph $\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])$ is 1 for every $t \geq 1$.*

Proof. We have seen in the proof of Proposition 5 that $2^{t-1}(1+i+j+k)$ is adjacent to all other vertices. This proves that $\{2^{t-1}(1+i+j+k)\}$ is a dominating set of the graph $\overline{\Gamma}(\mathbb{Z}_{2^t}[i,j,k])$. □

The rest of the section will be devoted to study de case when n is an odd prime. In particular we will prove that, for an odd prime number p, the domination number of $\overline{\Gamma}(\mathbb{Z}_p[i,j,k])$ is $p+1$.

Theorem 10. *The domination number of the zero divisor graph $\overline{\Gamma}(\mathbb{Z}_p[i,j,k])$, where p is an odd prime number, is $p+1$.*

Proof. Again we use the fact that $\mathbb{Z}_p[i,j,k] \cong M_2(\mathbb{Z}_p)$. Note that left ideals and right ideals of $M_2(\mathbb{Z}_p)$ are in bijection with subspaces of the vector space $\mathbb{Z}_p^2$. In fact, if S is a subspace of $\mathbb{Z}_p^2$, then

$$I = \{A \in M_2(\mathbb{Z}_p) \ : \ Ker(A) \supseteq S\},$$

is a left ideal of $M_2(\mathbb{Z}_p)$. On the other hand,

$$J = \{A \in M_2(\mathbb{Z}_p) \ : \ Im(A) \subseteq S\},$$

is a right ideal of $M_2(\mathbb{Z}_p)$.

Now, since the order of a subspace divides the order of the space, it follows that the subspaces of $\mathbb{Z}_p^2$ are of order 1, p or p^2. The proper subspaces are the subspaces of order p. If S_1 and S_2 are two proper subspaces, then either $S_1 \cap S_2 = \{0\}$ or $S_1 = S_2$. Consequently, there are $p+1$ proper subspaces of $\mathbb{Z}_p$. Let

$$S_0, S_1, \ldots, S_p$$

be the $p+1$ distinct proper subspaces of $\mathbb{Z}_p$. Consider the set of non-zero zero divisors

$$D = \{D_0, D_1, \ldots, D_p\} \subseteq M_2(\mathbb{Z}_p),$$

where, for $i = 0, \ldots, p$ we have $Ker(D_i) = S_i$.

Let $M \in M_2(\mathbb{Z}_p)$ be a non-zero zero divisor. Then, $Im(M)$ is a proper subspace, and hence $Im(M) = S_k$, for some $k \in \{0, \dots, p\}$. Consequently, $D_k M = 0$, and so D is a dominating set for $\overline{\Gamma}(M_2(\mathbb{Z}_p))$.

Let $s < p$ and consider the set

$$E = \{E_0, E_1, \dots, E_s\} \subseteq M_2(\mathbb{Z}_p),$$

where, for $i = 0, \dots, s$, each E_i is a non-zero zero divisor. Then, we can choose a subspace S_n such that $S_n \neq Ker(E_i)$, for $i = 0, 1, \dots, s$ and a subspace S_m such that $S_m \neq Im(E_i)$, for $i = 0, 1, \dots, s$.

Now, let B be a non-zero zero divisor such that $Ker(B) = S_m$ and $Im(B) = S_n$. Since $BE_i \neq 0$ and $E_i B \neq 0$, for $i = 0, 1, \dots, s$, it follows that E is not a dominating set and the claim follows. □

As a consequence of the previous result we can easily compute the domination number of $\overline{\Gamma}(\mathbb{Z}_n[i, j, k])$ when n is an odd square-free integer.

Theorem 11. *Let $n = p_1 \cdots p_k$ with p_i prime for every i. Then, the domination number of the zero divisor graph $\overline{\Gamma}(\mathbb{Z}_n[i, j, k])$ is $k + p_1 + \cdots + p_k$.*

Proof. Let $C_i := \{M_{i,1}, \dots, M_{i,1+p_i}\}$ be the dominating set for the graph $\overline{\Gamma}(\mathbb{Z}_{p_i}[i, j, k])$ given by Theorem 10. Now, it is easy to see that the set

$$\bigcup_{i=1}^{k} \{(0, 0, \dots, M_{i,1}, \dots 0, 0..), \dots, (0, 0, \dots, M_{i,1+p_i}, \dots 0, 0 \dots)\}$$

is a minimal dominating set of $\overline{\Gamma}(\mathbb{Z}_n[i, j, k])$ and hence the result. □

In a similar way, we can proof the following result.

Theorem 12. *Let $n = 2^s p_1 \cdots p_k$ with p_i is prime for every i and $s > 0$. Then, the domination number of the zero divisor graph $\overline{\Gamma}(\mathbb{Z}_n[i, j, k])$ is $1 + k + p_1 + \cdots + p_k$.*

We end this section presenting an open problem: For an odd prime number p and a positive integer t, what is the domination number of $\overline{\Gamma}(\mathbb{Z}_{p^t}[i, j, k])$?

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José María Grau
Departamento de Matemáticas
Universidad de Oviedo
Avda. Calvo Sotelo s/n
33007 Oviedo
Spain
e-mail: grau@uniovi.es

Celino Miguel
Department of Mathematics
Instituto de Telecomunicações
Beira Interior University
Covilhã
Portugal
e-mail: celino@ubi.pt

Antonio M. Oller-Marcén
Centro Universitario de la Defensa de Zaragoza
Ctra. Huesca s/n
50090 Saragossa
Spain
e-mail: oller@unizar.es

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