ON THE DIAMETER OF THE COMMUTING GRAPH OF THE FULL MATRIX RING OVER THE REAL NUMBERS

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ABSTRACT. In a recent paper C. Miguel proved that the diameter of the commuting graph of the matrix ring $\mathrm{M}_n(\mathbb{R})$ is equal to 4 if either $n = 3$ or $n \geq 5$. But the case $n = 4$ remained open, since the diameter could be 4 or 5. In this work we close the problem showing that also in this case the diameter is 4.
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1. Introduction

For a ring R, the *commuting graph* of R, denoted by $\Gamma(R)$, is a simple undirected graph whose vertices are all non-central elements of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. The commuting graph was introduced in [1] and has been extensively studied in recent years by several authors [2, 3, 4, 5, 6, 7, 12, 13].

In a graph G, a path $\mathcal{P}$ is a sequence of distinct vertices $(v_1, \dots, v_k)$ such that every two consecutive vertices are adjacent. The number $k-1$ is called the length of $\mathcal{P}$. For two vertices u and v in a graph G, the distance between u and v, denoted by $d(u, v)$, is the length of the shortest path between u and v, if such a path exists. Otherwise, we define $d(u, v) = \infty$. The diameter of a graph G is defined

$$\mathrm{diam}(G) = \sup\{d(u, v) : \ u \text{ and } v \text{ are vertices of } G\}.$$

A graph G is called connected if there exists a path between every two distinct vertices of G, equivalently, $\mathrm{diam}(G) < \infty$.

Much research has been conducted regarding the diameter of commuting graphs of certain classes of rings [3, 7, 8, 9, 10]. Here, we deal with the full matrix rings over fields. Let $\mathbb{F}$ be an arbitrary field. We known that $\Gamma(\mathrm{M}_2(\mathbb{F}))$ is never connected. It was proved in [4] that $\Gamma(\mathrm{M}_n(\mathbb{F}))$ is connected if and only if every field extension of $\mathbb{F}$ of degree n contains a proper intermediate field. Moreover, it was shown in [3] that if $\Gamma(\mathrm{M}_n(\mathbb{F}))$ is connected, then $4 \leq \mathrm{diam}(\Gamma(\mathrm{M}_n(\mathbb{F}))) \leq 6$ and it is conjectured that $\mathrm{diam}(\Gamma(\mathrm{M}_n(\mathbb{F}))) \leq 5$. Let $\mathbb{Q}$ and $\mathbb{R}$ be the fields of rational and real numbers, respectively. We know from [3, 4] that $\Gamma(\mathrm{M}_n(\mathbb{Q}))$ is non-connected for any $n \geq 2$ and $\mathrm{diam}(\Gamma(\mathrm{M}_n(\mathbb{F}))) = 4$ for every algebraically closed field $\mathbb{F}$

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and $n \geq 3$. Quite recently, C. Miguel [11] has verified this conjecture for $\mathbb{R}$, proving the following result.

Theorem 1.1. *Let $n \geq 3$ be any integer. Then, diam$(\Gamma(\mathrm{M}_n(\mathbb{R}))) = 4$ for $n \neq 4$ and $4 \leq$ diam$(\Gamma(\mathrm{M}_4(\mathbb{R}))) \leq 5$.*

Unfortunately, this result left open the question wether $\mathrm{diam}(\Gamma(\mathrm{M}_4(\mathbb{R})))$ is 4 or 5. In this paper we solve this open problem. Namely we will prove the following result.

Theorem 1.2. *The diameter of $\Gamma(\mathrm{M}_4(\mathbb{R}))$ is equal to 4.*

2. On the diameter of $\Gamma(\mathrm{M}_n(\mathbb{R})$

Before we proceed, let us introduce some notation. If $a, b \in \mathbb{R}$, we define the matrix $A_{a,b}$ as

$$A_{a,b} := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Now, given two matrices $X, Y \in M_n(\mathbb{R})$, we define

$$X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R}).$$

Finally, two matrices $A, B \in \mathrm{M}_n(\mathbb{R})$ are called *similar* and are written as $A \sim B$ if there exists an invertible matrix P such that $P^{-1}AP = B$.

The proof of Theorem 1 given in [11] relies on the possible forms of the Jordan canonical form of a real matrix. In particular, it is proved that the distance between two matrices $A, B \in \mathrm{M}_4(\mathbb{R})$ is at most 4 unless we are in the situation where A and B have no real eigenvalues and only one of them is diagonalizable over $\mathbb{C}$. In other words, the case when

$$A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix}, \quad B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix}.$$

The following result will provide us the main tool to prove that the distance between A and B is at most 4 also in the previous setting. It is true for any division ring D. In what follows, given a matrix A, L_A and R_A will denote the left and right multiplication by A, respectively.

Proposition 2.1. *Let $A, B \in \mathrm{M}_n(D)$ matrices such that $A^2 = A$ and $B^2 = 0$. Then, there exists a non-scalar matrix commuting with both A and B.*

Proof. Since $A^2 = A$; i.e., $A(I - A) = (I - A)A = 0$, then one of nullity A or nullity $(I - A)$ is at least $n/2$. Since $I - A$ is also idempotent and a matrix commutes with A if and only if it commutes with $I - A$ we can assume that nullity $A \geq n/2$. Moreover, since $B^2 = 0$, it follows that nullity $B \geq n/2$.

Now, if $\mathrm{Ker} L_A \cap \mathrm{Ker} L_B \neq \{0\}$ and $\mathrm{Ker} R_A \cap \mathrm{Ker} R_B \neq \{0\}$ we can apply [3, Lemma 4] and the result follows. Hence, we assume that $\mathrm{Ker} L_A \cap \mathrm{Ker} L_B = \{0\}$, since in the case $\mathrm{Ker} R_A \cap \mathrm{Ker} R_B = \{0\}$ we can consider the transposes of A and B instead of A and B, respectively. Note that, in these conditions, $n = 2r$ and the nullities of A and B are equal to r.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for $\mathrm{Ker}L_A$ and $\mathrm{Ker}L_B$, respectively, and consider $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ a basis for D^n. Since A is idempotent, it follows that $D^n = \mathrm{Ker}L_A \oplus \mathrm{Im}L_A$.

We want to find the representation matrix of L_A in the basis $\mathcal{B}$. To do so, if $v \in \mathcal{B}_2$, we write $v = a + a'$ with $a \in \mathrm{Ker}L_A$ and $a' \in \mathrm{Im}L_A$. If $a' = Aa''$ for some $a'' \in D^n$, then $Av = Aa + Aa' = 0 + A(Aa'') = Aa'' = a' = -a + v$. Since $Av = 0$ for every $v \in \mathcal{B}_1$, we get that the representation matrix of L_A in the basis $\mathcal{B}$ is of the form

$$\begin{pmatrix} 0 & A' \\ 0 & I_r \end{pmatrix},$$

with $A' \in \mathrm{M}_r(D)$.

Now, we want to find the representation matrix of L_B in the basis $\mathcal{B}$. Clearly, $Bv = 0$ for every $v \in \mathcal{B}_2$. Let $w \in \mathcal{B}_1$. Then, $Bw = w_1 + w_2$ with $w_1 \in \mathrm{Ker}L_A$ and so $w_2 \in \mathrm{Ker}L_B$. Hence, $0 = B^2w = Bw_1$ and $w_1 \in \mathrm{Ker}L_A \cap \mathrm{Ker}L_B = \{0\}$. Thus, the representation matrix of L_B in the basis $\mathcal{B}$ is of the form

$$\begin{pmatrix} 0 & 0 \\ B' & 0 \end{pmatrix},$$

with $B' \in \mathrm{M}_r(D)$.

As a consequence of the previous work we can find a regular matrix P such that:

$$PAP^{-1} = \begin{pmatrix} 0 & A' \\ 0 & I_r \end{pmatrix}, \quad PBP^{-1} = \begin{pmatrix} 0 & 0 \\ B' & 0 \end{pmatrix}.$$

Now, if $A'B' \neq B'A'$, then $P^{-1}(A'B' \oplus B'A')P$ is a non-scalar matrix commuting with A and B. If A' and B' commute, we can find a non-scalar matrix $S \in \mathrm{M}_r(D)$ commuting with both A' and B'. Therefore $P^{-1}(S \oplus S)P$ commutes with both A and B and the proof is complete. □

We are now in the condition to prove the main result of the paper.

Theorem 2.2. *The diameter of* $\Gamma(\mathrm{M}_4(\mathbb{R}))$ *is four.*

Proof. In [11] it was proved that $d(A, B) \leq 4$ for every $A, B \in \mathrm{M}_4(\mathbb{R})$, unless

$$A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix} \text{ and } B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix},$$

for some real numbers a, b, c, d, r, s. Hence, we only focus on this case. Assume that

$$A = P^{-1} \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix} P \text{ and } B = Q^{-1} \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix} Q,$$

for some invertible matrices P and Q. Let

$$M = P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} P \text{ and } N = Q^{-1} \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix} Q.$$

It is straightforwardly checked that $M^2 = M$, $N^2 = 0$, $AM = MA$, and $BN = NB$. Furthermore, Proposition 2.1 implies that there exists a non-scalar matrix X that commutes both with M and N.

Thus, we have found a path (A, M, X, N, B) of length 4 connecting A and B and the result follows. $\square$

Corollary 2.3. *For every $n \geq 3$, diam$(\Gamma(\mathrm{M}_4(\mathbb{R}))) = 4$.*

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