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Optimal control of counter-terrorism tactics

L. Bayón*, P. Fortuny Ayuso, P.J. García-Nieto, J.M. Grau, M.M. Ruiz

Department of Mathematics, University of Oviedo, E.P.I. Campus of Viesques, Gijón 33203, Spain

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ABSTRACT

This paper presents an optimal control problem to analyze the efficacy of counter-terrorism tactics. We present an algorithm that efficiently combines the Minimum Principle of Pontryagin, the shooting method and the cyclic descent of coordinates. We also present a result that allows us to know a priori the steady state solutions. Using this technique we are able to choose parameters that reach a specific solution, of which there are two. Numerical examples are presented to illustrate the possibilities of the method. Finally, we study the sufficient conditions for optimality and suggest an improvement on the functional which also guarantees local optimality.

1. Introduction

Modeling a "stock" of terrorists, is not common, but has precedents, especially after September 11, 2001 [1]. In this sense [2] presents an intelligent ecological metaphor to analyze actions by Governments and citizens against terror. In [3] a model for the transmission dynamics of extreme ideologies in vulnerable populations is presented. In [4] the authors propose a terror-stock model that treats the suicide bombing attacks in Israel. In other countries like, for example, Spain or Ireland, the problem has also been analyzed.

Several papers develop dynamical models of terrorism. In [5] the authors incorporate the effects of both military/police and nonviolent/persuasive intervention to reduce the terrorist population. This idea is widely developed in [6] where the controls are two types of counter-terror tactics: "water" and "fire", which is the model we shall consider in this paper. The effect of intelligence (water tactics) in counter-terrorism is analyzed also in [7]. Nowadays, it is agreed that counter-terrorism policies have the potential to generate positive support for terrorism [8]. Recently, in [9] a model with two-states (undetected and detected terrorists) and only one control variable (the number of undercover intelligence agents) is considered.

In this context we present in this work a new approach to analyze the efficacy of counter-terrorism tactics. We state an optimal control problem that attempts to minimize the total cost of terrorism. An excellent summary of optimal control application in these issues can be consulted in [10] and its economic implications in [11].

The optimization criterion is to minimize the discounted damages created by terror attacks plus the costs of counter-terror efforts. The underlying mathematical problem is complicated. It constitutes a multi-control, constrained problem where the optimization interval is infinite. An important feature is that the time t is not explicitly present in the problem (hence, it is a time-autonomous problem), except in the discount factor. Using Pontryagin's Minimum Principle, the shooting method and the cyclic descent of coordinates we develop an optimization algorithm. We also present a method (based

* Corresponding author.
E-mail address: bayon@uniovi.es (L. Bayón).

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upon [12]) for computing the optimal steady-states in multi-control, infinite-horizon, autonomous models. This method does not require the solution of the dynamic optimization problem. Using it, we can choose parameters that reach a desirable steady-state solution. The problem presents two steady-states, albeit one of them in a region where it becomes effectively one-dimensional. We focus mainly on the multi-control problem.

The paper is organized as follows. Section 2 presents the mathematical model. The optimization algorithm is developed in Section 3, and the method for computing the optimal steady-states is analyzed in Section 4. Section 5 presents several numerical examples which illustrate the performance of the algorithm under different conditions. In Section 6 we discuss Arrow's sufficient conditions for optimality in our problem. We also suggest an improvement on the functional in which the cost function is convex in the number of terrorists (due to the value added by information sharing, interactions, etc.) and show how the solution found by our method in this case satisfies the sufficient conditions locally. Finally, the main conclusions of our work are discussed in Section 7.

2. Mathematical model

We use the excellent model provided by [6], which classifies counter-terrorism tactics into two categories:

- “Fire” strategies are tactics that involve significant collateral damage. They include, for example, the killing of terrorists through drones, the use of indiscriminate checkpoints or the aggressive blockade of roads.
- “Water” strategies, on the other hand, are counter-measures that do not affect innocent people, like intelligence arrests against suspect individuals.

The fire and water strategies will be denoted by the control variables $v(t)$ and $u(t)$, respectively. Both controls have their advantages, and their drawbacks. For example $v(t)$ has the direct benefit of eliminating current terrorists but the undesirable indirect effect of stimulating recruitment rates (and the possible harm to innocent bystanders). On the other hand, $u(t)$ is more expensive and more difficult to be applied than $v(t)$.

The strength or size of the terrorists is represented by the state variable $x(t)$. This includes not only the number of active terrorists, but also the organization's total resources including financial ones, weapons, etc. [2]. Its value changes over time and we distinguish two inflows and three outflows in it:

$$\dot{x} = \tau + I(v, x) - O_1(x) - O_2(u, x) - O_3(v, x) \tag{1}$$

We include first of all a term τ, accounting for a small constant recruitment rate. Second, following [3], the model considers that new terrorists are recruited by existing terrorists. So the inflow $I(v, x)$ is increasing in proportion to the current number of terrorists x. But this growth is bounded and should also slow down. Moreover, the aggressive control v, also increases recruitment. In summary the form of the model is:

$$I(v, x) = (1 + \rho v)kx^{\alpha} \tag{2}$$

with $\tau,\ \rho \geq 0$, $k > 0$ and $0 \leq \alpha \leq 1$.

On the other hand, we consider three outflows: The first one, $O_1(x)$, represents the rate at which people leave the organization by several reasons not related with the controls. This natural outflow is assumed linear in x:

$$O_1(x) = \mu x \tag{3}$$

with $\mu > 0$. The second outflow, $O_2(u, x)$, reflects the effect of water strategies. This outflow is assumed to be concave in x because there is a limited number of units that conduct water operations:

$$O_2(u, x) = \beta(u)x^{\theta} \tag{4}$$

with $\theta \leq 1$. The third outflow $O_3(v, x)$ is due to fire strategies. This is modeled as linear in x, because the methods are perceived to be “direct attack”:

$$O_3(v, x) = \gamma(v)x \tag{5}$$

The functions $\beta(u)$ and $\gamma(v)$ should be concave; Caulkins [6] uses the same functional form for both: a logarithmic function. The water function is pre-multiplied by a constant β smaller than the corresponding constant γ for fire operations. These two constants reflect the “efficiency” of the two types of operations.

Finally, the costs of terrorism are assumed to be linear in the number of terrorists, that is, of the form cx. We also model the cost control function as separable, and the costs of employing the water and fire strategies are modeled as quadratic. Over a infinite planning horizon, the objective is to minimize the sum of both costs (terrorism and counter-terror operations). We also assume that outcomes are discounted by a constant rate r. In brief, the control problem we pose can be written as:

$$\min_{u,v\geq 0} J = \min_{u,v\geq 0} \int_0^{\infty} (cx + u^2 + v^2)e^{-rt}dt \tag{6}$$

$$\dot{x} = \tau + (1 + \rho v)kx^{\alpha} - \mu x - \beta \ln(1 + u)x^{\theta} - \gamma \ln(1 + v)x; \quad x(0) = x_0$$

$$u(t) \geq 0; \quad v(t) \geq 0$$

where x_0 is the initial stock level and we impose also control constraints.

3. Optimization algorithm

The above problem (68), is an Optimal Control Problem (OCP) where the total costs have to be minimized, given the state dynamics and the control constraints. Denoting $\mathbf{u}(t) = (u(t), v(t)) = (u_1(t), u_2(t))$ we have:

$$\min_{\mathbf{u}(t)} J = \int_0^\infty F(t, x(t), \mathbf{u}(t))dt \tag{7}$$

subject to satisfying:

$$\dot{x}(t) = f(t, x(t), \mathbf{u}(t)), \ \ 0 \le t < \infty; \quad x(0) = x_0 \tag{8}$$

$$\mathbf{u}(t) \in \mathbf{U}(t), \ \ 0 \le t < \infty \tag{9}$$

The problem presents several noteworthy features. First, the optimization interval is infinite. Second, the time t is not explicitly present in the problem (time-autonomous problem), except in the discount factor. Third, we impose constraints on the control and, fourth, it constitutes a multi-control problem.

3.1. Multi-control problem

To solve the multi-control variational problem, we propose a numerical algorithm which uses a particular strategy related to the cyclic coordinate descent (CCD) method [13]. The classic CCD method minimizes a function of n variables cyclically with respect to the coordinates. With our method, the problem can be solved like a sequence of problems whose error functional converges to zero. The algorithm (with $i = 1, 2$) carries out several iterations and at each *jth* iteration it calculates 2 stages, one for each i. At each stage, it computes the optimal of $u_i(t)$, assuming the other variable is fixed.

Beginning with some admissible $\mathbf{u}^0$, we construct a sequence of ($\mathbf{u}^j$) and the algorithm will search:

$$\lim_{j\to\infty} \mathbf{u}^j \tag{10}$$

It is easy to justify the convergence of the algorithm taking into account Zangwill's global convergence Theorem [14].

The whole problem is not multi-control. We remark here, in passing, that, as we shall see later, there is a value x^S, which we call a *switching point*, such that the optimal solution $(x^*(t), u^*(t), v^*(t))$ satisfies $v^*(t) = 0$ for all t such that $x^*(t) \le x^S$. This switching point is independent of the solution; therefore, for $x \le x^S$, the problem becomes essentially uni-control. However, for $x > x^S$ the multi-control techniques are required.

3.2. The one-dimensional reduced problem

Based upon the previous statement, we present now the solution for the one-dimensional case, using Pontryagin's Minimum Principle (PMP) (see, for example, [15–17]).

$$\begin{aligned} &\min_{u(t)} J = \int_0^\infty F(t, x(t), u(t))dt \\ &\dot{x}(t) = f(t, x(t), u(t)), \ \ 0 \le t < \infty; \quad x(0) = x_0 \\ &u(t) \ge 0, \ \ 0 \le t < \infty \end{aligned} \tag{11}$$

Our integrand takes the form:

$$F(t, x(t), u(t)) = G(t, x(t), u(t))e^{-rt} \tag{12}$$

where r is the positive rate of discount, and G is a function bounded from above. Under these conditions, the integral is found to be convergent for each admissible control. Let H be the associated Hamiltonian:

$$H(t, x, u, \lambda) = \lambda^0 F(t, x, u) + \lambda \cdot f(t, x, u) \tag{13}$$

where λ is the co-state variable. Using PMP, the optimal solution can be obtained from a two-point boundary value problem. In order for $u^* \in U$ to be optimal, there must exist a function $\lambda(t)$ such that, for almost every $t \in [0, \infty)$:

$$\dot{x} = H_\lambda = f; \quad x(0) = x_0 \tag{14}$$

$$\dot{\lambda} = -H_{\mathbf{x}}; \quad \lim_{t\to\infty} H(t) = 0 \tag{15}$$

$$H(t, x, u^*, \lambda) = \min_{u(t)\in U} H(t, x, u, \lambda) \tag{16}$$

An elementary argument shows that, in the problem under consideration, any optimal path $(x^*(t), u^*(t), v^*(t))$ satisfies that $0 < u^*(t) < M$ for some $M > 0$. This, together with the transversality condition $\lim_{t\to\infty} H(t) = 0$ implies that both $\lambda^0 \neq 0$ and $\lim_{t\to\infty} \lambda(t) = 0$ (which is sometimes stated as a transversality condition). The latter is the property we shall use later on in order to compute an optimal path. Also, as $\lambda^0 \neq 0$, we normalize it to 1.

Notice that, frequently, the condition

$$\lim_{t\to\infty} \lambda(t) = 0 \tag{17}$$

is stated as a necessary condition for optimality in infinite-time problems. In [18], it is proved that, under certain hypotheses, for problems with time discount, it is necessary. However, those hypotheses can be (as in this case) not easily checked. Notice also, that there are examples of infinite-time problems with discount in which $\lambda(t)$ does not converge to 0 [19].

Due to the non-linearity of the system dynamics, the optimal solution can only be computed numerically. In this paper we propose an efficient method which adapts the shooting method, Euler's method, and numerical integration. All the calculations are carried out in the Mathematica environment.

We will denote by $\mathbb{Y}_x(t)$ the function:

$$\mathbb{Y}_x(t) = -\frac{F_u}{f_u} \cdot e^{\int_0^t f_x ds} + \int_0^t F_x \cdot e^{\int_0^s f_x dz} ds \tag{18}$$

The algorithm is based upon the following theorem.

Theorem 1. *A necessary maximum condition*

Let u^* be the optimal control, let $x^* \in \widehat{C}^1$ be a solution of the above problem (11). Then there exists a constant $K \in \mathbb{R}^+$such that:

$$\begin{array}{ll} If\ u^*(t) > 0 & \Longrightarrow \mathbb{Y}_{x^*}(t) = K \\ If\ u^*(t) = 0 & \Longrightarrow \mathbb{Y}_{x^*}(t) \geq K \end{array} \tag{19}$$

Proof. In virtue of PMP there exists a piece-wise C^1 function λ that satisfies the linear differential equation:

$$\dot{\lambda}(t) = -H_x = -F_x - \lambda(t) \cdot f_x \tag{20}$$

Denoting $K = \lambda(0)$, we have:

$$\lambda(t) = \left[K - \int_0^t F_x e^{\int_0^s f_x dz} ds\right] e^{-\int_0^t f_x ds} \tag{21}$$

For each t, $u(t)$ minimizes H. Using the Kuhn-Tucker Theorem, there exists for each t a real non negative number, $\delta(t)$, such that $u(t)$ is a critical point of the augmented Hamiltonian:

$$\mathbb{H}(u(t)) = F(t) + \lambda(t) \cdot f(t) + \delta(t) \cdot (-u(t)) \tag{22}$$

If $u(t) > 0$, then $\delta(t) = 0$. In this case:

$$F_u + \lambda(t) \cdot f_u = 0 \tag{23}$$

Notice that $f_u \neq 0$ in our problem, as $f_u = -\beta(1+u)^{-1} x^\theta$. Using this, from (21) and (23), we obtain:

$$K = -\frac{F_u}{f_u} \cdot e^{\int_0^t f_x ds} + \int_0^t F_x \cdot e^{\int_0^s f_x dz} ds \tag{24}$$

and the following relation holds:

$$\mathbb{Y}_x(t) = K \tag{25}$$

If $u(t) = 0$, then $\delta(t) \geq 0$. By a similar argument and bearing in mind that $f_u \leq 0$, we get that

$$\mathbb{Y}_x(t) \geq K \tag{26}$$

□

We have performed the numerical (and approximate) construction of x^* using a discretized version of Eq. (24). For each K, we construct x^* using (25) and (26) to impose the constraint.

The calculation of the optimal K has been achieved by means of an adaptation of the shooting method. Varying the constant, K, we search for the extremal that fulfills the transversality condition (17). Using the secant method (starting out from two values $K_{\min}$ and $K_{\max}$), our algorithm converges satisfactorily (see Section 5).

Finally: an adapted version of the CCD method for functionals (whose detailed description can be found in [20], applied to hydrotermal problems), has been used in order to solve the multi-control problem. The classic CCD method minimizes a function of n variables cyclically with respect to the coordinate variables. With our method, the problem is solved as a sequence of problems whose error functional converges to zero.

4. Steady-state solutions

In [12] a method for computing the optimal steady-state in infinite-horizon one-dimensional problems is presented which does not require the solution of the dynamic optimization problem, in which the bounds $U(t)$ do not play any role. Tsur considers a one-dimensional version of our problem:

$$\min_{\mathbf{u}(t)} J = \int_0^{\infty} G(x(t), \mathbf{u}(t))e^{-rt}dt \tag{27}$$

$$\dot{x}(t) = f(x(t), \mathbf{u}(t)), \quad x(0) = x_0 \tag{28}$$

We propose another adaptation of the CCD method. Beginning with some admissible $\mathbf{u}^0$, we construct a sequence $(\mathbf{u}^j)$ and at each stage, we compute the optimal steady state of $u_i(t)$, assuming the other variable is fixed. At each ith stage (for $i = 1, 2$), we consider that the steady-state solution is $u_i = R_i(x)$ and we define the *evolution function*:

$$L_i(x) = r\left(\frac{G_{u_i}(x, R_i(x))}{f_{u_i}(x, R_i(x))} + \dot{W}_i(x)\right) \tag{29}$$

with:

$$W_i(x) = \frac{1}{r}G(x, R_i(x)) \tag{30}$$

A necessary condition for an optimal steady state x_s is:

$$L_i(x_s) = 0 \tag{31}$$

The algorithm shows a fast convergence to the optimal values of $u_i(t)$ and also gives the unique value of $x_s > x^S$. It is noteworthy that for the dynamic equation model presented in (68) with $\mathbf{u}(t) = (u(t), v(t)) = (u_1(t), u_2(t))$:

$$\dot{x} = \tau + (1 + \rho v)kx^{\alpha} - \mu x - \beta \ln(1 + u)x^{\theta} - \gamma \ln(1 + v)x \tag{32}$$

the function $u = R_1(x)$, obtained by imposing $\dot{x} = 0$, is:

$$R_1(x) = -1 + \exp\left(\frac{x^{-\theta}(\tau + (1 + \rho v)kx^{\alpha} - \mu x - \gamma \ln(1 + v)x)}{\beta}\right) \tag{33}$$

Nevertheless, the value of $v = R_2(x)$, is not so easy to obtain. It turns out to be

$$R_2(x) = -1 - \frac{x^{1-\alpha}\gamma}{k\rho}W\left(-\frac{1}{\gamma}\exp\left(\frac{\tau - \mu x - kx^{\alpha}(\rho - 1)}{x\gamma}\right)k(1 + u)^{\frac{-x^{\theta-1}\beta}{\gamma}}x^{\alpha-1}\rho\right) \tag{34}$$

where W is the Lambert W-function. Remember that $W(z)$ is a set of functions which are the branches of the inverse of the function:

$$z = f(W) = We^{W} \tag{35}$$

where W is a complex variable. In this work we are interested in real-valued $W(x)$, which, adding the condition $W(x) \geq -1$, gives a single-valued function $W_0(x)$: the principal branch of the W-function. We refer the reader to [21] for a survey on existing results on this function.

In the next section we will see the behavior of this algorithm and its application to calculate the optimal steady-state of our problem, for $x > x^S$.

Steady-states before the switching point. As remarked above, the problem becomes effectively one-dimensional when $x \leq x^S$: namely $v = 0$ for $x \leq x^S$. This simplifies the problem in this case, obviously. As we shall see, Tsur's method provides the existence of two steady states in $x \in [0, x^S]$: one of them stable and the other one unstable, without recoursing to multi-control techniques. Therefore, the optimization problem presents two stable steady states, one to the left of the switching point and another one to the right.

5. Numerical examples

5.1. Base case

We examine now the behavior of our approach in several examples. For the sake of comparison, we use the (carefully chosen) parameters used in [6]. The discount rate is a typical $r = 0.05$. The outflow rate is assumed to be 5% and the constant inflow rate term is small $\tau = 10^{-5}$. The parameter k is chosen such that the steady state is normalized to 1 in the absence of counter-terrorism tactics, and neglecting τ. This way, x is measured as a percentage of the steady-state size of

Table 1
Parameters for the fire-and-water model [6].

Par.	Description	Value
r	Discount rate	0.05
c	Costs per unit of terrorists	1
τ	Constant inflow	10^{-5}
ρ	Contribution of fire control to recruitment	1
k	Normalization factor	0.05
α	Influence of actual state on recruitment	0.75
μ	"Natural" per capita outflow	0.05
β	Efficiency of water operations	0.01
θ	Diminishing returns from water operations	0.1
γ	Maximum efficiency of fire operations	0.1

the terrorist organization when the government does not use counter-terror operations. The uncontrolled dynamics is given by:

$$\dot{x} = kx^{\alpha} - \mu x \tag{36}$$

Integrating this Bernoulli differential equation, and computing the limit when $t \to \infty$ we have:

$$x = \sqrt[1-\alpha]{\frac{k}{\mu}} \tag{37}$$

We see that the normalization of x leads to $k = \mu$. The influence of x on recruitment, α, is a value between 0 and 1. The efficiency parameters γ and β are chosen assuming fire strategies are approximately 10 times more powerful than water strategies for the maximum size of x. Nevertheless, for small values of x, fire and water strategies can be equally effective. In this case, we choose the parameter θ such that (for small $x \simeq 1/10$):

$$\beta \ln(1+u)x^{\theta} = \gamma \ln(1+v)x \to \frac{\gamma}{\beta} = x^{1-\theta} = 1 \tag{38}$$

The parameter ρ measures the influence of fire strategies on recruitment. It is chosen so that the product $\rho v(t)$ represents the 20 – 30% of the recruitment rate. Finally, the costs c per terrorist are assumed to be 1 in the base case. Table 1 summarizes the values of the parameters for the model.

Switching point. From the equations found in [6] (which we omit for brevity), the value of the optimal control $v^*(t)$ is, without active restrictions (and using our notation, in which we do not remove the discount factor)

$$v^*(t) = \frac{1}{4}\left(-\rho kx(t)^{\alpha}\lambda(t)e^{-rt} - 2 + \sqrt{(\rho kx(t)^{\alpha}\lambda(t)e^{-rt} - 2)^2 + 8\lambda(t)e^{-rt}\nu x(t)}\right). \tag{39}$$

Let us remark that the equation given for v^* in Eq. (3) in [6] lacks a factor λ in the first term out of the square root, which prevents the authors from exactly computing the switching point. Eq. (39) has the form (omitting the factor 1/4)

$$v^*(t) = -(a+b) + \sqrt{(a-b)^2 + 8c} = -(a+b) + \sqrt{(a+b)^2 - 4ab + 8c}. \tag{40}$$

Rewriting (39) in this form, we get

$$v^*(t) = -(2 + \rho kx(t)^{\alpha}\lambda(t)e^{-rt}) + \sqrt{(2 + \rho kx(t)^{\alpha}\lambda(t)e^{-rt})^2 - 8\lambda(t)e^{-rt}(\rho kx(t)^{\alpha} - \nu x(t))} \tag{41}$$

so that (as $\lambda(t) > 0$) whenever $\rho kx(t)^{\alpha} > \nu x(t)$, the optimal solution has $v^*(t) < 0$, which is not possible (as $v(t)$ denotes a positive *action*). Hence, if

$$\rho kx(t)^{\alpha} > \nu x(t) \tag{42}$$

the restriction $v(t) = 0$ is activated. This is the same as

$$x(t)^{\alpha}(\rho k - \nu x(t)^{1-\alpha}) > 0 \tag{43}$$

with $x(t) \geq 0$. That is, whenever

$$x(t) < x^S = \left(\frac{\rho k}{\nu}\right)^{\frac{1}{1-\alpha}} \tag{44}$$

one has to impose $v^*(t) = 0$. For the given values of the parameters, we obtain that this switching point is $x^S = 0.0625$ *exactly*.

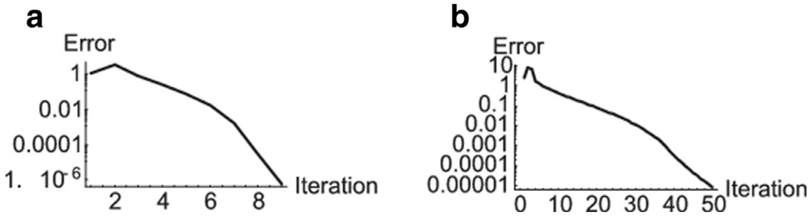

Fig. 1. Convergence of the algorithm for each control.

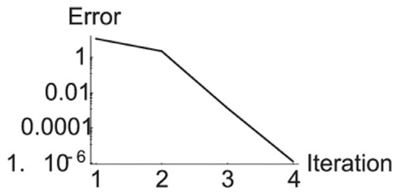

Fig. 2. Convergence of the CCD algorithm.

5.2. High-value steady state

We implemented a Mathematica program to apply the results obtained in this paper. As noted in the previous section, the steady-state solutions can be computed *a priori*. We focus on the high value of the steady state, x_s, as this is the one in which the system is truly multi-control. As there are no more steady states with $x > x_s$, we shall compute it starting from $x(0) > x_s$. The algorithm, adapting the CCD method, solves equation (31) in x_s. Starting from two initial values $u^0(t) = v^0(t) = 0$, we obtain the following steady-state values:

$$x_s \simeq 0.61773 \tag{45}$$

$$u_s \simeq 0.06834 \tag{46}$$

$$v_s \simeq 0.14605 \tag{47}$$

They were also obtained in [6] (but only approximately: he gives $x_s \simeq 0.62$). However, it should be recalled that the determination of this value is as important as the computation of the dynamics of the process leading towards that steady-state.

In order to compute the dynamic solution, we use the optimization algorithm based on Theorem 1. The following results were obtained using a discretization in t with 250 sub-intervals. Starting with $u^0(t) = v^0(t) = 0$ and two initial values of K for each control, namely $K_{\min}$ and $K_{\max}$, we compute the discretized solution x^* step by step with:

$$\mathbb{Y}_x(t) = K \text{ or } \mathbb{Y}_x(t) \geq K \tag{48}$$

Using the secant method, we achieve the prescribed tolerance (10^{-5}) in the transversality condition

$$\lim_{t\to\infty} \lambda(t) = 0 \tag{49}$$

in only 9 iterations for $u(t)$ (see Fig. 1(a)), and in 50 iterations for $v(t)$ (see Fig. 1(b)). Once these values are computed, the algorithm alternates the two controls in a cyclic way, retro-feeding the values $u^j(t), v^j(t)$ computed in each iteration. The CCD algorithm shows a fast convergence: in just 4 iterations the tolerance imposed for the error is reached: in our case, the difference between two consecutive values of K to be less than 10^{-3} (see Fig. 2).

Our results show the double convergence of the algorithm: on one hand, of each control to the transversality condition (17) and, on the other, of the CCD method. We note this latter result especially, as it not only shows the fast convergence but it also opens up the possibility of applying our method to problems with more dimensions.

The optimal dynamical evolution of the strength or size of the terrorists, $x(t)$, is shown in Fig. 3 and of the fire-control $v(t)$, is shown in Fig. 4. The optimal path has been obtained with an initial value $x(0) = 0.95$. The plot of the water-control $u(t)$ is not shown because its value is practically constant in time. As a matter of fact, it starts at 0.066 for $t = 0$ and increases slowly towards its steady state value: 0.068.

The CPU time required by the program was 121 sec on a personal computer (Intel Core 2/2.66 GHz). The optimal value of K is 14.16651221278533. The evolution of $\lambda(t)$ and its asymptotic behavior towards zero, can be seen in Fig. 5.

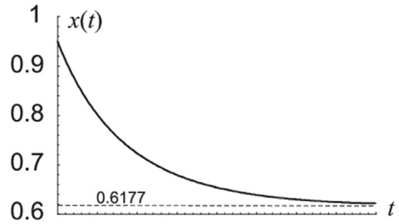

Fig. 3. Optimal $x(t)$.

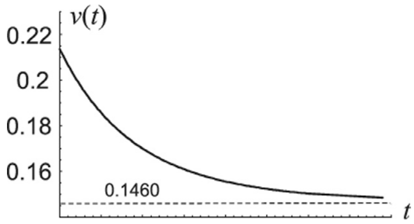

Fig. 4. Optimal $v(t)$.

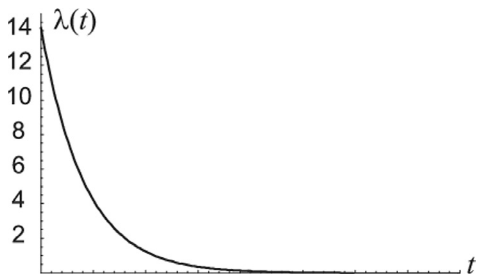

Fig. 5. Optimal $\lambda(t)$.

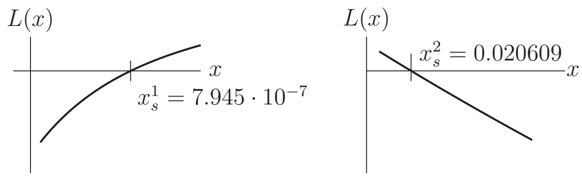

Fig. 6. Stability and instability of steady states x_s^1 and x_s^2.

5.3. Low-value steady state

Using Tsur's method straightforwardly, we can compute the steady states for $x < x^S$, as the problem is one-dimensional ($v(t) = 0$ for these values of x). Our computations give the same results as those of [6], namely: if x_s^1 and u_s^1 and x_s^2 and u_s^2 are the x- and u-values at these steady states, then

$$\begin{array}{ll} x_s^1 \simeq 7.94549 \cdot 10^{-7} & x_s^2 \simeq 0.0206096. \\ u_s^1 \simeq 0.0046106 & u_s^2 \simeq 0.284695 \end{array} \tag{50}$$

Tsur's method provides also the nature of these steady states: the first one is stable (so that solutions tend to it) and the second one is unstable (so that solutions *never* end on it). These values agree with those of [6]. Tsur's method gives the steady states as zeros of the evolution function $L(x)$. The stability/instability is provided, in Tsur's method, by the slope of $L(x)$ at those points. Fig. 6 shows how the slope of $L(x)$ is positive at x_s^1, which implies (in a minimization problem) that x_s^1 is stable. On the other hand, the slope of $L(x)$ at x_s^2 is negative, which implies that this is an unstable steady state.

We compute now the dynamic solution approaching the low-value steady state. As Caulkins [6] demonstrates, there is a DNS point $x_D \simeq 0.013$ at which the optimal paths converging towards the high-value steady state and towards the low-value one have the same cost. We compute our path starting at x_D in order to verify that our method provides the same solution as that of [6].

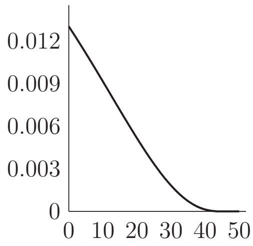

Fig. 7. Opt. values of $x(t)$.

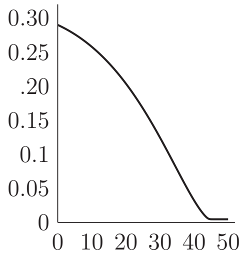

Fig. 8. Opt. values of $u(t)$.

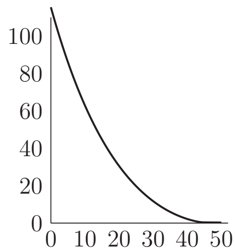

Fig. 9. Opt. values of $\lambda(t)$.

Our computations provide, after 6 seconds, the plots in Figs. 7–9 for $x(t)$, $u(t)$ and $\lambda(t)$. After 50 years we obtain

$$x_s^1 \simeq 7.9445 \cdot 10^{-7}, u_s^1 \simeq 0.004612 \tag{51}$$

and a total cost $\simeq 1.10532$, all in agreement with Caulkins [6] and the values given by Tsur's algorithm.

5.4. Parametric study near the high-value steady state

After solving the base case, we are going to study the influence of the model parameters on the steady state solution. In [6], the author presents an extensive sensitivity analysis of each parameter and their influence both in the state-control space and in the state-co-state space. We pursue a different objective. Using the Algorithm presented in Section 4, we analyze the influence of the parameters which we deem susceptible of modification by the government: the efficiency parameters γ and β.

We shall set the steady state of the terrorists' stock as the government expectation and find the different combinations of γ and β which lead to it. First, we perform the sensitivity analysis of each efficiency. The results are shown in Figs. 10 and 11. We note that in these plots, the cost shown is not the cost J of the functional over all the time span but the cost by unit of time of the optimal solution (the integrand). We deem this measure more adjusted to reality, as once the steady state has been reached, that is the true value that will set the cost along time.

As expected, the influence on the steady state and on cost is greater for γ, as it is related to the fire actions. The most remarkable fact, in our view, is that increasing β leads to a steady increase of its associated control u whereas increasing γ leads to v reaching a maximum and then decreasing. The explanation of this lies in the model: for water tactics, the concave character is included in the model via the x^θ term while the model for the fire tactics is linear in x. Concavity seems to appear when performing the sensitivity analysis: increasing the efficiency does not necessarily imply increasing the use of fire tactics.

The joint influence of both efficiencies is studied using successive applications of our algorithm, using brute force and varying:

$$\beta \in [0.01, 0.02], \ \text{step} = 0.001; \quad \gamma \in [0.1, 0.2], \ \text{step} = 0.01 \tag{52}$$

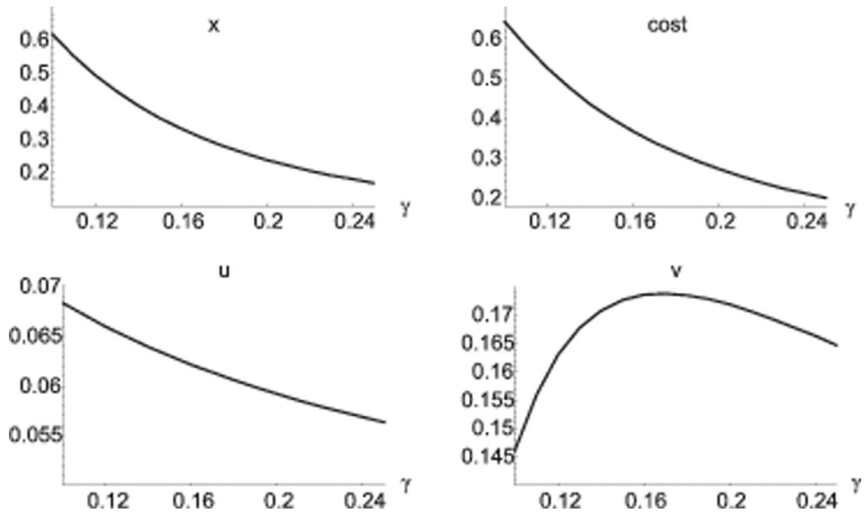

Fig. 10. Sensitivity analysis of γ.

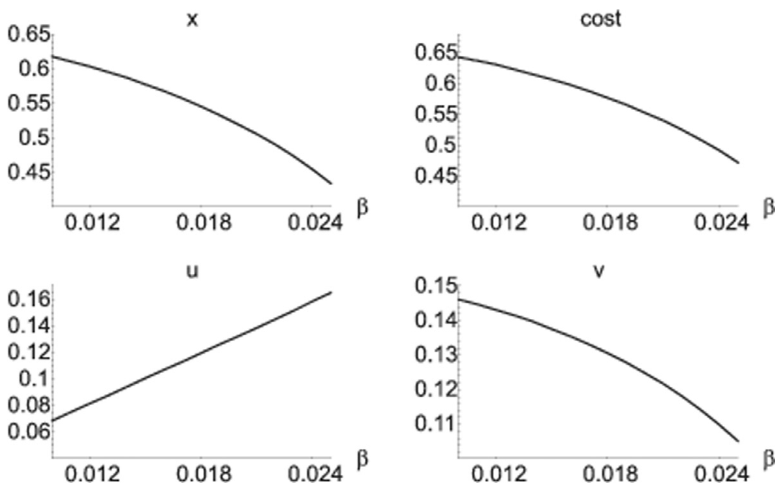

Fig. 11. Sensitivity analysis of β.

Our algorithm computes the optimal steady states of the 121 combinations in just 33.6 sec on a personal computer (Intel Core 2/2.66 GHz). This shows the computational advantage of this method against the one used in the previous section, which requires the computation of the whole dynamic solution. Results are shown in Fig. 12.

Adjusting using least squares, we obtain the function:

$$x_s(\gamma, \beta) = 1.43186 - 10.5508\gamma + 23.4843\gamma^2 + 1.97376\beta - 301.821\beta^2 \tag{53}$$

with $r^2 = 0.998$. Using this function we can solve the problem stated at the beginning: if the government wishes to reach a steady-state of, say, $x_s = 0.40$, what are the efficiencies it must get for each tactic? By imposing

$$x_s(\gamma, \beta) = 0.4 \tag{54}$$

we get the contour curve shown in Fig. 13. Solving the problem for the values given by this curve gives the results shown in Table 2.

Notice first of all that, due to the approximations, x_s is not exactly 0.4 but the precision is enough. Secondly, observe that modifying the efficiencies causes (logically) a noticeable change in the associated controls. However, the most remarkable finding is in the last column: the cost per time unit of the solution. We see (apart from approximation errors) that its value is essentially constant on the whole contour curve. This means that the government has freedom to choose any of the possible combinations so that the decision should lie on principles other than cost. For example, it may be easier to increase the efficiency of water strategies rather than fire, due to ethical or social causes, as these do not require patently bellicose actions.

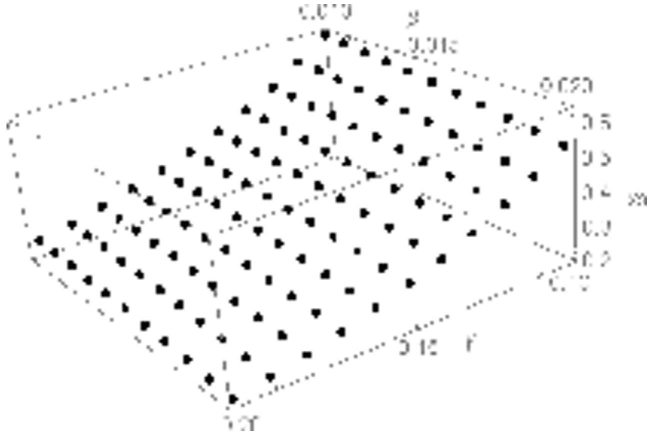

Fig. 12. Steady-state as function of γ and β.

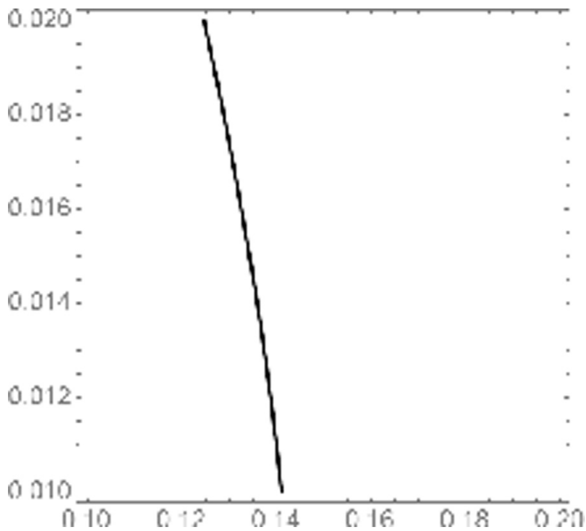

Fig. 13. Contour Line $x_s(\gamma, \beta) = 0.4$.

Table 2
Combinations of optimal solutions for $x_s(\gamma, \beta) = 0.4$.

γ	β	x_s	u_s	v_s	*cost*
0.141	0.010	0.399	0.0638	0.1710	0.432
0.140	0.011	0.398	0.0700	0.1693	0.432
0.139	0.012	0.398	0.0763	0.1673	0.432
0.137	0.013	0.398	0.0826	0.1652	0.432
0.136	0.014	0.398	0.0889	0.1628	0.432
0.134	0.015	0.397	0.0953	0.1601	0.432
0.133	0.016	0.397	0.1017	0.1573	0.432
0.131	0.017	0.396	0.1082	0.1541	0.431
0.129	0.018	0.395	0.1147	0.1506	0.431
0.127	0.019	0.394	0.1213	0.1469	0.430
0.125	0.020	0.392	0.1280	0.1427	0.429

6. Sufficient conditions

We consider now the multi-control OCP in its general Bolza form:

$$\min_{u(t)} J = \int_0^T F(x(t), u(t), t)dt + B[T, x(T)] \tag{55}$$

subject to:

$$\dot{x}(t) = f(x(t), u(t), t); \;\; x(0) = x_0 \tag{56}$$

$$u(t) \in U(t), \;\; 0 \le t \le T \tag{57}$$

where $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ is the state vector and $u(t) = (u_1(t), \ldots, u_n(t)) \in \mathbb{R}^n$ the control vector. We assume the following:

(i) F and $f = (f_1(t), \ldots, f_n(t))$ are continuous.
(ii) F and f have continuous second derivatives with respect to t and x but their second derivatives with respect to u may be discontinuous.
(iii) The control variable $u(t)$ needs only be piecewise continuous.
(iv) The state variable $x(t)$ is differentiable but its first derivative needs only be piecewise continuous (i.e. the graph of $x(t)$ may have “corners”).
(v) The function B has continuous partial derivatives.

The set of admissible controls is, usually, convex and compact. Pontryagin’s Theorem establishes necessary conditions for the optimum in our problem (55).

In this section, we study the sufficient conditions which guarantee optimality. They, as is usual in optimization theory impose some type of convexity of the functions defining the problem. We concentrate on Arrow’s conditions, in this work.

Consider the “derived Hamiltonian”

$$H(x(t), u(t), \lambda(t), t) = F(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) \tag{58}$$

where $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ is the costate vector. Assume that for the following problem:

$$\min_{u \in U(t)} H(x, u, \lambda, t) \tag{59}$$

the function:

$$u = u^0(x, \lambda, t) \tag{60}$$

is an admissible solution satisfying the PMP. Set:

$$H^0(x, \lambda, t) = H(x, u^0, \lambda, t) \tag{61}$$

The following result provides a sufficient condition for global optimality, assuming U is compact and convex:

Arrow’s Theorem. *Assume $u^*(t)$, $x^*(t)$ and $\lambda^*(t)$ satisfy PMP for all $t \in [0, T]$. If $H^0(x, \lambda^*, t)$ is convex in x for eacht $\in [0, T]$ and B is also convex in x, then u^* is an optimal control for the problem, x^* is the optimal state trajectory andλ^* is the optimal trajectory of the costate variables.*

For local optimality, one only needs to take a compact and convex neighbourhood of (x^*, u^*).

6.1. Infinite-horizon problems

If we consider the problem (55) with infinite horizon, following [22], if the problem includes a discount factor and can be stated as

$$\min_{u(t)} J = \int_0^\infty F(x(t), u(t)) e^{-rt} dt \tag{62}$$

subject to:

$$\dot{x}(t) = f(x(t), u(t), t); \;\; x(0) = x_0 \tag{63}$$

$$u(t) \in U(t) \tag{64}$$

then:

Sufficient conditions for an infinite-horizon problem with discount factor. Given the problem above with the same conditions on F, f, x and u and with H and H^0 defined as before, let$u^*(t)$, $x^*(t)$ and $\lambda^*(t)$ be admissible solutions satisfying PMP for all t. Then, if

$$\lim_{t \to \infty} \lambda^*(t) x(t) \ge 0 \tag{65}$$

for all admissible $x(t)$ and the conditions for Arrow’s Theorem hold, then the pair $(x^*(t), u^*(t))$ is a global minimum.

Of course, the same principle as above serves to ensure a local minimum. In our case, the first inequality

$$\lim_{t \to \infty} \lambda^*(t) x(t) \ge 0 \tag{66}$$

holds by the conditions imposed on the problem.

6.2. A new functional

Some computations show that we cannot guarantee Arrow's conditions with the previous functional for a solution tending to x_s. Despite this, the value of the corresponding second partial derivative is near enough to zero to prevent us from also stating that it is *not* an optimum.

Recall that the functional is given by:

$$(cx + u^2 + v^2)e^{-rt} \tag{67}$$

so that the cost is considered linear in the number of terrorists. This is an assumption which we think does not take into account the value of information sharing, interaction and synergies in groups of people. In our view, a convex function of x provides an improved cost model in terms of the number of terrorists. We are going to perform the study solving (for simplicity we take x^2) the new problem:

$$\min_{u,v\geq 0} J = \min_{u,v\geq 0} \int_0^\infty (cx^2 + u^2 + v^2)e^{-rt}dt \tag{68}$$

$$\dot{x} = \tau + (1+\rho v)kx^\alpha - \mu x - \beta \ln(1+u)x^\theta - \gamma \ln(1+v)x; \quad x(0) = x_0 \tag{69}$$

$$u(t) \geq 0; \quad v(t) \geq 0 \tag{70}$$

With the same data as in the previous case, the solution we find leads to a high-value steady-state, somewhat different from the one obtained before. In this case, we have:

$$x_s = 0.605 \tag{71}$$

$$u_s = 0.081 \tag{72}$$

$$v_s = 0.163 \tag{73}$$

The Hamiltonian is given by:

$$\begin{aligned} H(x,u,v,\lambda,t) \quad &= e^{-rt}\left(cx^2 + u^2 + v^2\right) \\ &+\lambda\left(\tau + (1+\rho v)kx^\alpha - \mu x - \beta \ln(1+u)x^\theta - \gamma \ln(1+v)x\right) \end{aligned} \tag{74}$$

so that:

$$H_u(x,u,v,\lambda,t) = 2ue^{-rt} - \frac{\beta\lambda x^\theta}{u+1} \tag{75}$$

$$H_v(x,u,v,\lambda,t) = 2ve^{-rt} + \lambda\left(k\rho x^\alpha - \frac{\gamma x}{v+1}\right) \tag{76}$$

The Hamiltonian is convex in u and v, as:

$$H_{uu}(x,u,v,\lambda,t) = 2e^{-rt} + \frac{\beta\lambda x^\theta}{(u+1)^2} \geq 0 \tag{77}$$

$$H_{vv}(x,u,v,\lambda,t) = 2e^{-rt} + \frac{\gamma\lambda x}{(v+1)^2} \geq 0 \tag{78}$$

Solving the equations:

$$H_u(x,u,v,\lambda,t) = 0 \tag{79}$$

$$H_v(x,u,v,\lambda,t) = 0 \tag{80}$$

we obtain:

$$u^0(x,\lambda,t) = \frac{1}{2}\left(\sqrt{2\beta\lambda e^{rt}x^\theta + 1} - 1\right) \tag{81}$$

$$v^0(x,\lambda,t) = \frac{1}{4}\left(\sqrt{k^2\lambda^2\rho^2 e^{2rt}x^{2\alpha} - 4k\lambda\rho e^{rt}x^\alpha + 8\gamma\lambda x e^{rt} + 4} + k\lambda\rho\left(-e^{rt}\right)x^\alpha - 2\right) \tag{82}$$

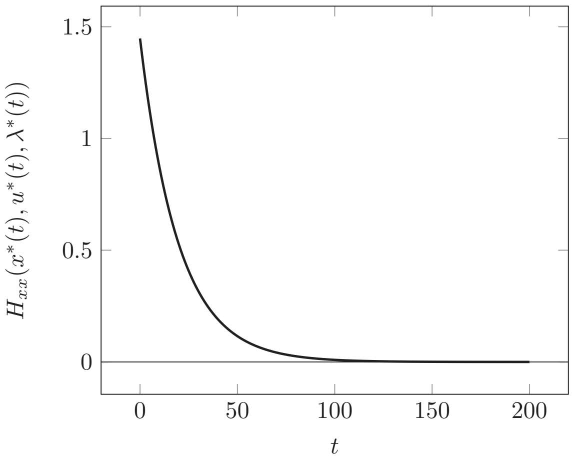

Fig. 14. Convexity of the derived Hamiltonian.

so that the derived Hamiltonian is given by:

$$\begin{aligned}H^0(x,\lambda,t) = {} & e^{-rt}\left(cx^2 + u^0(x,\lambda,t)^2 + v^0(x,\lambda,t)^2\right) \\ & +\lambda\left(\tau + \left(1+\rho v^0(x,\lambda,t)\right)kx^\alpha - \mu x - \beta \ln\left(1+u^0(x,\lambda,t)\right)x^\theta \right. \\ & \left. -\lambda\gamma \ln\left(1+v^0(x,\lambda,t)\right)x\right.\end{aligned} \tag{83}$$

For the $\lambda^*(t)$ obtained, we get the plot of H^0_{xx} as a function of t shown in Fig. 14, as a function of t, which guarantees the local convexity and hence, local optimality of our solution:

7. Conclusions

We consider ways for a government to optimally employ "water" and "fire" strategies for fighting terrorism. The model tries to balance the costs of terror attacks with the cost of terror control. We present two main contributions. First, a new effective algorithm for computing the dynamical solution whose cyclic nature allows its use in models of greater dimension (say with more controls or more state variables) without any conceptual modification. Secondly, we present a method for computing the steady-state solution which can be used to analyze the dependence of the steady-state strategy on several parameters.

We show how this latter method allows us to compute, a priori, the optimal values of the steady-state, without having to solve the whole dynamical problem. This way one can analyze the influence of the parameters in a simpler way, which leads to remarkable computational savings. We can study not just the influence of the parameters on the model but (what is more important), compute what the parameters must be in order to obtain a desired steady-state. We have studied the influence of the efficiency parameters γ and β. We consider that the study of the influence of the other parameters may be interesting from the theoretical point of view but, as they are intrinsic to the model, they cannot be modified (at least in a simple way) by the authorities. However, the efficiency γ of the fire controls is a vital question for any army, as is the efficiency β of the water controls for any government. In the latter case, both the secret services intelligence and even the educational measures taken by the governments are relevant.

Finally, we study a modification of the model in which the cost function is strictly convex in the number of terrorists and verify that the solution found by our method is, indeed, a local optimum.

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