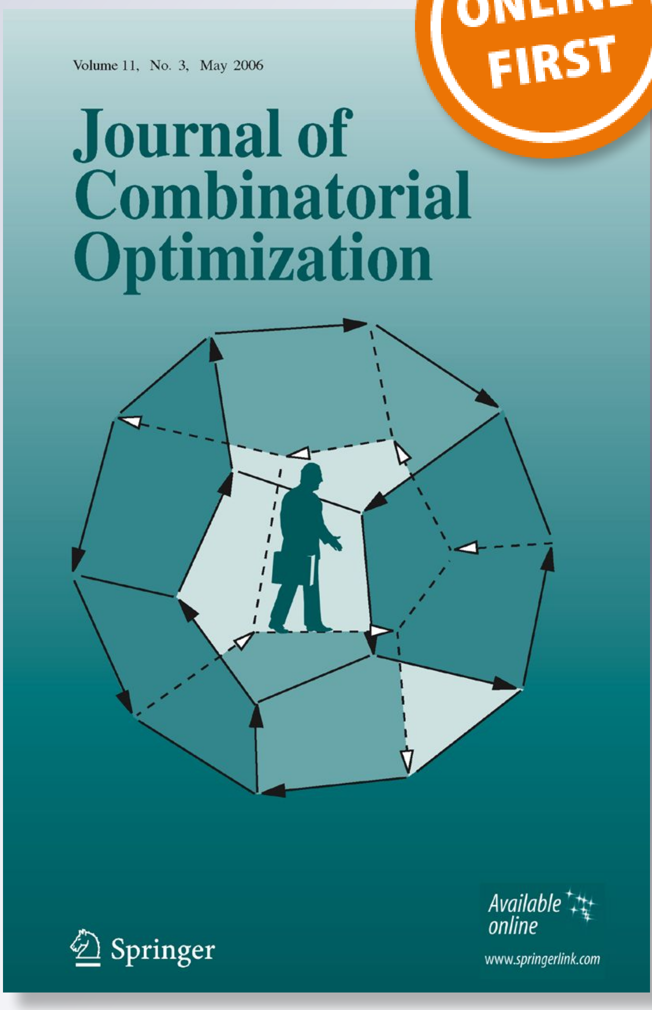
A new look at the returning secretary problem

J. M. Grau Ribas

Journal of Combinatorial Optimization

ISSN 1382-6905

J Comb Optim
DOI 10.1007/s10878-018-0349-8



Journal of Combinatorial Optimization
https://doi.org/10.1007/s10878-018-0349-8

A new look at the returning secretary problem

J. M. Grau Ribas[1]

Abstract
We consider a version of the secretary problem in which each candidate has an identical twin. As in the classical secretary problem, the aim is to choose a top candidate, i.e., one of the best twins, with the highest possible probability. We find an optimal stopping rule for such a choice, the probability of success, and its asymptotic behavior. We use a novel technique that allows the problem to be solved exactly in linear time and obtain the asymptotic values by solving differential equations. Furthermore, the proposed technique may be used to study the variants of the same problem and in other optimal stopping problems.

Keywords Secretary problem · Combinatorial optimization

Mathematics Subject Classification 60G40 · 62L15

1 Introduction

The *secretary problem* is one of many names for a famous problem of optimal stopping theory. This problem can be stated as follows: an employer is willing to hire the best secretary out of n rankable candidates. These candidates are interviewed one by one in random order. A decision about each particular candidate is to be made immediately after the interview. Once rejected, a candidate cannot be called back. During the interview, the employer can rank the candidate among all the preceding ones, but is unaware of the quality of yet unseen candidates. The goal is then to determine the optimal strategy that maximizes the probability of selecting the best candidate.

This problem has a very elegant solution. Dynkin (1963) and Lindley (1961) independently proved that the best strategy consists in a so-called threshold strategy. Namely, in rejecting roughly the first n/e (cutoff value) interviewed candidates and then selecting the first one that is better than all the preceding ones. Following this

✉ J. M. Grau Ribas
grau@uniovi.es

1 Departamento de Matemáticas, Universidad de Oviedo, Avda. Calvo Sotelo s/n, 33007 Oviedo, Spain

Published online: 19 September 2018

strategy, the probability of selecting the best candidate is at least $1/e$, this being its approximate value for large values of n. This well-known solution was later refined by Gilbert and Mosteller (1966), showing that $\lfloor (n-\frac{1}{2})e^{-1}+\frac{1}{2}\rfloor$ is a better approximation than $\lfloor n/e \rfloor$, although the difference is never greater than 1.

The secretary problem has been addressed by many authors in different fields such as applied probability, statistics and decision theory. Extensive bibliographies on the topic can be found in Ferguson (1989), Ferguson et al. (1991) and Szajowski (2009). On the other hand, different generalizations of this classic problem have recently been considered within the framework of partially ordered objects (Freij and Wastlund 2010; Garrod and Morris 2012; Georgiou et al. 2008) or matroids (Babaioff et al. 2007; Soto 2011). The paper by Bearden Bearden (2006) is also worth mentioning, where the author considers a situation in which the employer receives a payoff for selecting a candidate equal to the "score" of the candidate (in the classic problem, the payoff is 1 if the candidate is really the best, and 0 otherwise). In this situation, the optimal cutoff value is roughly the square root of the number of candidates.

A variant introduced relatively recently by Garrod et al. (2012) and included in his doctoral thesis (Garrod 2011) considers a version of the secretary problem in which each candidate has an identical twin. As in the classic problem, the aim is to select a top candidate, i.e. one of the best twins, with the highest possible probability. More recently, Shai Vardi addressed the same problem in Vardi (2014, 2015), where he calls it *the returning secretary problem*. Curiously, he seems to ignore Garrod's papers, as they are not included in his bibliography.

As in the case of the secretary problem, it would be excessive to say that this new variant is motivated by a real-world application scenario. However, it constitutes an interesting mathematical challenge that has led us to introduce a novel methodology based on the resolution of differential equations and that might be useful in the asymptotic analysis of other optimal stopping problems.

In this problem, as in the secretary problem, the optimal stopping strategy is a strategy threshold. Specifically, for each number of pairs of twin secretaries, $n \in \mathbb{N}$, there exists a $\mathbf{k}_n$ such that the optimal strategy is to interview $\mathbf{k}_n$ different secretaries and thereafter select, in your second inspection, the first one that is better than all the previous ones. Both Garrod and Shai Vardi approach this problem as a best-choice problem for partially ordered objects and, in their papers, study asymptotic behaviour as $n \to \infty$ of the optimal stopping threshold, $\mathbf{k}_n$, and of the probability of success, $\mathbf{P}_n$. Both authors deduce, denoting by W the Lambert W-function,

$$\lim_{n\to\infty} \frac{\mathbf{k}_n}{n} = \vartheta := \frac{2}{W(2e^5)} = 0.4709265...$$

For the asymptotic probability of success, both arrive at different expressions of the same value:

Garrod finds that

$$\lim_{n\to\infty} \mathbf{P}_n = \frac{\left(3+4\left(-1+\sqrt{\frac{1}{1-\vartheta}}\right)\left(-1+\frac{1}{\vartheta}\right)^2\right)\vartheta}{3} = 0.767974...$$

and Shai Vardi that

$$\lim_{n\to\infty} \mathbf{P}_n = 2\sqrt{1-\vartheta} - \frac{4(1-\vartheta)}{3} - \frac{\left(1-\sqrt{1-\vartheta}\right)^2 \log(\vartheta)}{3} = 0.767974...$$

In the present paper, we address the problem in a completely different way, obtaining the asymptotic probability of success, which appears with a different expression than those above:

$$\lim_{n\to\infty} \mathbf{P}_n = \frac{-4+6\sqrt{1-\vartheta}+4\vartheta+\left(-2+2\sqrt{1-\vartheta}+\vartheta\right)\log(\vartheta)}{3} = 0.767974...$$

We have thus introduced a novel technique that falls outside the framework of partial ordered objects (which in our opinion do not from part of the nature of this problem) that can also be applied to the study of variants of the problem in which, for example, penalization is considered in the case of failure or costs for the interviews conducted.

The paper is organized as follows. In Sect. 2, we present some technical results. In Sect. 3, the formula that calculates the probability of success, the optimal threshold and its asymptotic values is deduced. In Sect. 4, we reflect on the efficiency of the exact calculation of $\mathbf{P}_n$ and $\mathbf{k}_n$. In Sect. 5, we solve variants of the problem, considering penalties in the case of failure and costs for the interviews. In Sect. 6, we conclude the paper by raising some future challenges related to this work.

2 Some technical results

In optimal stopping problems, where the probability of success is given by a function $F_n(k)$, where k represents the stopping threshold and n the number of stages, it is usual for the asymptotic value of the optimal stopping threshold to be of the type $\beta \cdot n$, where $\beta \in (0, 1)$. Sometimes, the arguments for its calculation are somewhat lax, considering $F_n(k)$ the step to the limit, with the mere substitution $k \to nx$ and subsequent maximization of the limit function $f(x) := \lim_n(F_n(nx))$. Whatever the case, when this convergence is uniform on $[0, 1]$, the reasoning is entirely correct, because we have the following result.

Proposition 1 *Let* $\{F_n\}_{n\in\mathbb{N}}$ *be a sequence of real functions with* F_n *defined by integer values in* $\{0, 1, ..., n\}$*, and let* $\mathcal{M}(n)$ *be the value for which the function* F_n *reaches its maximum. Assume that the sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$ *given by* $f_n(x) := F_n(\lfloor nx \rfloor)$ *converges uniformly on* $[0, 1]$ *to* f *continuous in* $[0, 1]$ *and that* θ *is the only global maximum of* f *in* $[0, 1]$. *Then,*

(i) $\lim_n \mathcal{M}(n)/n = \theta$.
(ii) $\lim_n F_n(\mathcal{M}(n)) = f(\theta)$.
(iii) *If* $\mathfrak{M}(n) \sim \mathcal{M}(n)$ *then* $\lim_n F_n(\mathfrak{M}(n)) = f(\theta)$.

Proof The proof is straightforward and analogous to that presented in Bayón et al. (2017) for the functions F_n with a real variable and domain $[0, n]$ and $f_n(x) = F_n(nx)$. □

The uniform convergence of $f_n(x) := F_n(\lfloor nx \rfloor)$ can sometimes be difficult to establish a priori. However, the nature of the problem, together with a simple inspection of the evolution of the succession curve $f_n(x)$ for moderately large values of n, makes the assumption of the uniformity of convergence entirely reasonable and even the assumption that the function limit is of class C^1. Making this assumption, let us now see a novel procedure for calculating the limit of the sequence f_n when the functions F_n are defined recursively. However, let us first consider some auxiliary results.

Lemma 1 *Consider a sequence of functions* $F_n : [0, n] \cap \mathbb{Z} \to \mathbb{R}$ *each of which is defined recursively by the conditions:*

$$F_n(k) = G_n(k) + H_n(k)F_n(k-1) \text{ and } F_n(0) = \mu$$

If $f_n(x) := F_n(\lfloor nx \rfloor)$ *converges uniformly to* f *on* $[0, 1]$, $\mathfrak{h}_n(x) := H_n(\lfloor nx \rfloor) - 1$ *and* $\mathfrak{g}_n(x) := G_n(\lfloor nx \rfloor)$ *converge to* 0 *on* $(0, 1)$ *and uniformly on* $[a, b]$ *for all* $0 < a < b < 1$, *then* $f_n^*(x) := F_n(\lfloor nx \rfloor - 1)$ *converges to* f *on* $(0, 1)$ *and uniformly on* $[a, b]$ *for all* $0 < a < b < 1$.

Proof It suffices to observe that the sequence

$$f_n^*(x) = \frac{f_n(x) - \mathfrak{g}_n(x)}{\mathfrak{h}_n(x) + 1}$$

converges to f on $(0, 1)$ and uniformly on $[a, b]$ for all $0 < a < b < 1$. □

Lemma 2 *Let* $\{F_n\}_{n\in\mathbb{N}}$ *be a sequence of real functions of real variable with* F_n *with domain* $[0, n]$. *If* $f_n(x) := F_n(nx)$ *converges uniformly on* $[0, 1]$ *to* f, *then also* $f_n^*(x) := F_n(\lfloor nx \rfloor)$.

Proof It is straightforward. □

Lemma 3 *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of real functions such that* $f_n \to f$ *uniformly on* $[0, 1]$, *with* f *being continuous. We define*

$$\mathcal{W}_n(x) := \frac{f_n(x - 1/n) - f_n(x)}{-1/n}$$

and it is known that for all $0 < a < b < 1$, $\mathcal{W}_n(x)$ *is Riemann integrable and that it converges at* $\mathcal{W}(x)$ *uniformly on* $[a, b]$. *It is also known that* $\mathcal{W}$ *is continuous in* $(0, 1)$. *Thus,* $f(x)$ *is derivable in* $(0, 1)$ *and* $f'(x) = \mathcal{W}(x)$.

Proof Consider a compact $[a, b]$ with $0 < a < b < 1$ and $x \in [a, b]$. Due to the uniform convergence of $\mathcal{W}_n$, we may state that

$$\int_a^x \mathcal{W}(t)dt = \lim_{n\to\infty} \int_a^x \mathcal{W}_n(t)dt = (\star).$$

Developing to the limit on the right,

$$(\star) = \lim_{n\to\infty} n\left(\int_a^x f_n(t) - f_n(t-1/n)\,dt\right),$$

from which, due to the change in variables $u = t - 1/n$:

$$(\star) = \lim_{n\to\infty} n\left(\int_{x-1/n}^x f_n(t)dt - \int_{a-1/n}^a f_n(t)dt\right).$$

We shall prove that this limit is exactly $f(x)-f(a)$. Let $\epsilon > 0$. There exists an n_0 such that $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in [a, b]$. Moreover, due to f being continuous, it is uniformly continuous. Hence, there exists an n_0 such that $|f(x) - f(y)| < \epsilon/2$ if $|x - y| < 1/n_0$. Thus,

$$\begin{aligned}&\left|n\left(\int_{x-1/n}^x f_n(t)dt - \int_{a-1/n}^a f_n(t)dt\right) - f(x) + f(a)\right|\\ &= \left|n\left(\int_{x-1/n}^x f_n(t) - f(x)dt - \int_{a-1/n}^a f_n(t) - f(a)dt\right)\right| < n\frac{1}{n}\frac{\epsilon}{2} - n\frac{1}{n}\frac{\epsilon}{2} = \epsilon\end{aligned}$$

Therefore,

$$\int_a^x \mathcal{W}(t)dt = f(x) - f(a),$$

for $x \in [a, b]$, and hence $\mathcal{W}(x) = f'(x)$. Furthermore, as this is true for any $a, b \in (0, 1)$, it follows that $f(x)$ is derivable in $(0, 1)$ with $f'(x) = \mathcal{W}(x)$ for all $x \in (0, 1)$. □

Theorem 1 *Consider a sequence of functions* $F_n : [0, n] \cap \mathbb{Z} \to \mathbb{R}$*, each of which are defined recursively by the conditions:*

$$F_n(k) = G_n(k) + H_n(k)F_n(k-1) \text{ and } F_n(0) = \mu$$

Let $f_n(x) := F_n(\lfloor nx \rfloor)$*,* $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ *and* $g_n(x) := nG_n(\lfloor nx \rfloor)$*. If both* $h_n(x)$ *and* $g_n(x)$ *converge on* $(0, 1)$ *and uniformly on* $[\varepsilon, \varepsilon']$ *for all* $0 < \varepsilon < \varepsilon' < 1$ *to continuous functions in* $(0, 1)$*,* $h(x)$ *and* $g(x)$*, respectively, and* $f_n(x) \to f(x)$ *uniformly on* $[0, 1]$ *with* $f \in C[0, 1]$*, then* $f(0) = \mu$ *and* f *satisfy in* $(0, 1)$

$$f'(x) = -f(x)h(x) + g(x)$$

Proof Making $k = \lfloor nx \rfloor$ in the recursion, we have that

$$\begin{aligned}F_n(\lfloor nx \rfloor) &= G_n(\lfloor nx \rfloor) + H_n(\lfloor nx \rfloor)F_n(\lfloor nx \rfloor - 1)\\ F_n(\lfloor nx \rfloor - 1) - F_n(\lfloor nx \rfloor) &= F_n(\lfloor n(x - 1/n) \rfloor, n) - F_n(\lfloor nx \rfloor)\\ &= f_n(x - 1/n) - f_n(x)\end{aligned}$$

Now, let

$$
\begin{aligned}
W_n(x) &:= \frac{f_n(x-1/n) - f_n(x)}{-1/n} \\
W_n(x) &= \frac{F_n(\lfloor nx \rfloor - 1) - F_n(\lfloor nx \rfloor)}{1/n} \\
&= \frac{F_n(\lfloor nx \rfloor - 1) - G_n(\lfloor nx \rfloor) - H_n(\lfloor nx \rfloor)F_n(\lfloor nx \rfloor - 1)}{-1/n} \\
&= \frac{F_n(\lfloor nx \rfloor - 1)(1 - H_n(\lfloor nx \rfloor)) - G_n(\lfloor nx \rfloor)}{-1/n} \\
&= -F_n(\lfloor nx \rfloor - 1)h_n(x) + g_n(x)
\end{aligned}
$$

Now, taking into account 1 and making $f^*(x) := F_n(\lfloor nx \rfloor - 1)$, we have that W_n converge uniformly on $[\varepsilon, \varepsilon']$ for all $0 < \varepsilon < \varepsilon' < 1$

$$\lim_{n\to\infty} W_n(x) = \lim_{n\to\infty} (f_n^*(x)h_n(x) - g_n(x)) = f(x)h(x) - g(x)$$

In short, from the above lemma, $f'(x) = -f(x)h(x) + g(x)$ and, due to the continuity of f, $f(0) = \mu$. □

We obtain a similar result with a forward recursion.

Theorem 2 *Consider a sequence of functions $F_n : [0, n] \cap \mathbb{Z} \to \mathbb{R}$, each of which are defined recursively by the conditions:*

$$F_n(k) = G_n(k) + H_n(k)F_n(k+1) \text{ and } F_n(n) = \mu$$

Let $f_n(x) := F_n(\lfloor nx \rfloor)$, $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$. If both $h_n(x)$ and $g_n(x)$ converge on $(0, 1)$ and uniformly on $[\varepsilon, \varepsilon']$ for all $0 < \varepsilon < \varepsilon' < 1$ to continuous functions in $(0, 1)$, $h(x)$ and $g(x)$, respectively, and $f_n(x) \to f(x)$ uniformly on $[0, 1]$ with $f \in C[0, 1]$, then $f(1) = \mu$ and f satisfies in $(0, 1)$

$$f'(x) = f(x)h(x) - g(x)$$

Remark In the case of the classic secretary problem, the function of the probability of success using the threshold k, $P_n(k)$, can be expressed with the following recurrence relation:

$$P_n(k) = \frac{1}{n} + \frac{k}{k+1}P_n(k+1);\ P_n(n) = 0$$

Making $p_n(x) = P_n(\lfloor nx \rfloor)$ and assuming the uniform convergence $p_n(x) \to p(x)$ on $[0, 1]$, $p(x)$ must satisfy $p'(x) = \frac{p(x)}{x} - 1$, which, with the condition $p(1) = 0$, we have the well-known function $p(x) = -x\log(x)$, whose maximization in $[0, 1]$ provides the asymptotic value of the optimal threshold, n/e, and the asymptotic probability of success, e^{-1}.

3 The returning secretary problem

Let us assume that there are n secretaries that arrive in an online fashion. Each secretary arrives twice and the order is chosen uniformly at random from the $(2n)!$ possible orders. At all times, it is only possible to discern who is the best secretary of all the secretaries seen so far and whether it is their first or second arrival. Once we accept a secretary, the process ends. We win if we accept (or select) the best secretary. We would like to maximize the probability of winning.

With the aim of offering simpler reasoning that is easier to follow on the part of the reader, we shall consider the following equivalent game.

(1) Let us assume we have an urn with $2n$ objects; n different, rankable objects, each one of which has a clone (twin)
(2) Objects are randomly extracted from the urn and are shown to the player, until the player chooses one of them.
(3) When shown an object, all those that are worse than this object and whose twin has already been shown (inspected) previously are removed from the urn.
(4) To select one of two copies of the best object is considered success.

It should be noted that equivalence is to be understood assuming that, when an object is strictly inferior to the best of the already inspected objects, it is irrelevant for the player if its twin has or has not already been inspected. In other words, the relevant information for the player is the number of different objects shown and whether the best object or an object equal to the previous ones has been inspected once or twice. Having said this, once an object that is not better than or equal to all the previous ones has been inspected, its twin may effectively be considered to have also been inspected; hence point (3).

For the sake of brevity, we will say that an already inspected object is **maximal** if it is better than all the other inspected objects, while a maximal candidate in its second inspection will be called a **nice candidate**.

It should be noted that if k different objects are inspected and one is selected that is maximal, the probability of success is k/n, as in the secretary problem. However, it is obvious that it is always preferable to reject it on its first inspection, as we will always be able to select it on its second inspection and, until then, other better candidates might appear. As in the classic secretary problem, the optimal strategy in this problem is a threshold strategy and it may be formalized as follows.

Theorem 3 *For the returning secretary problem, if n is the number of different objects, there exists* $\mathbf{k}_n$ *such that the following strategy is optimal:*

(1) *Reject the* $\mathbf{k}_n$ *first different inspected objects.*
(2) *After that, accept the first* ***nice candidate***.

The optimal threshold is denominated by $\mathbf{k}_n$. We now introduce the following functions defined on $\{0, 1, ..., n\}$, where n represents the number of different objects (number of twins) and the variable k, the number of different objects already inspected.

- We denote by $\Phi_n(k)$ the probability of success when k different objects have been inspected, the maximal **has only been inspected once** and the aim is to select the next nice candidate to be inspected.

- We denote by $\Psi_n(k)$ the probability of success when k different objects have been inspected, the maximal **has been inspected twice** and the aim is to select the next nice candidate to be inspected.
- We denote by $\Upsilon_n(k)$ the probability that, after having inspected the kth different object for the first time, the maximal has been inspected twice.
- We denote by $\digamma_n(k)$ the probability of success after having interviewed the kth different object for the first time, pending the selection of the first nice candidate to appear. In other words, $\digamma_n(k)$ represents the probability of success using k as the threshold. This probability will depend on whether the kth different inspected object is maximal or not and, if it is not, it will depend on whether the maximal has been explored twice or only once. Having stated this, the following expression is straightforward

$$\digamma_n(k) = \Upsilon_n(k)\Psi_n(k) + (1 - \Upsilon_n(k))\Phi_n(k)$$

3.1 $\Phi_n(k)$: probability of success having inspected k different objects, with the maximal having been inspected only once

We now calculate the probability of success when k different objects have been examined, with the maximal having been inspected only once, and following the strategy of accepting the next nice candidate to appear.

If k different objects have been inspected and the maximal only once, we can effectively consider that there are still $2n-2k+1$ objects in the urn: the twin of the maximal and all the pairs of identical objects that have not been inspected. Two scenarios now exist for the next inspection (with the understanding, as already stated, that the non-inspected objects whose twin has already been inspected and is not maximal may in effect be considered already inspected).

(A) The twin of the maximal is inspected before a new object appears. This occurs with probability

$$\mathfrak{p}_n(k) := \frac{1}{1 - 2\,k + 2\,n}$$

as there is only one favourable case (the twin of the maximal is inspected) and $(1-2\,k+2\,n)$ possible cases. In this case, the object is selected and the probability of success will be k/n.

(B) A new and different object is inspected. It does not matter whether it is a new maximal or not; in both cases, we shall be in the situation of having inspected $k+1$ different objects and of having inspected the maximal once. The probability of this scenario occurring is $1 - \mathfrak{p}_n(k)$ and the probability of success in this case will be that corresponding to the same problem but with $k+1$ different inspected objects.

In short, we have the recursive relationship between the probability of success when k different objects have been inspected and the maximal has been inspected only once, $\Phi_n(k)$, and the probability of success with $k+1$ different objects, $\Phi_n(k+1)$, under the same conditions. The recursion is based on the fact that $\Phi_n(n) = 1$. This is because if n different objects have been seen and the twin of the maximal has not been inspected, success is obviously assured waiting for the twin of the maximal to appear.

$$\begin{aligned}\Phi_n(k) &= \frac{k}{n}\mathfrak{p}_n(k) + (1 - \mathfrak{p}_n(k))\Phi_n(k+1) \\ &= \frac{k + 2\,n\ (-k+n)}{n\ (1 - 2\,k + 2\,n)}\Phi_n(k+1) \\ \Phi_n(n) &= 1\end{aligned}$$

Lemma 4 *For all $n \in \mathbb{N}$ and $0 \le k \le n$, the function Φ_n defined recursively above satisfies*

$$\Phi_n(k) = \frac{2n+k}{3n}$$

Proof By recursion backwards. It suffices to observe that it is true for $k = n$ and that, if it is true for $k+1$, this means that it is also true for k. In effect,

$$\Phi_n(k+1) = \frac{2n + (k+1)}{3n} \Rightarrow \Phi_n(k) = \frac{2n+k}{3n}$$

□

Remark Note that for $k = 1$, we have the case called *The No Waiting Case* in Vardi (2015); i.e. accept the first nice candidate to appear. Note that for all n, the probability of success with this (non-optimal) strategy is at least $2/3$.

Proposition 2 *The sequence of functions $\widehat{\Phi}_n(x) := \Phi_n(\lfloor nx \rfloor)$ converges uniformly on $[0, 1]$ to*

$$\Phi(x) := \frac{2+x}{3}$$

Proof It is straightforward. □

3.2 $\Psi_n(k)$: probability of success having inspected k different objects, with the maximal having been inspected twice

We now calculate the probability of success when k different objects have been examined, with the maximal having been inspected twice, and following the strategy of accepting the next nice candidate to be inspected. If we have inspected k different objects and the maximal has been inspected twice, we can effectively consider that $2n - 2k$ objects remain in the urn: all the pairs of identical objects that have not been inspected. Two scenarios now exist for the next inspection:

(A) The next new inspected object is maximal. This occurs with probability $1/(k+1)$ and places us in the situation where the probability of success is $\Phi_n(k+1)$.

(B) The next new inspected object is not maximal. This occurs with probability $k/(k+1)$ and places us in the same situation but with one more inspected object; i.e. the probability of success in this case will be: $\Psi_n(k+1)$.

We thus have the recursive definition for Ψ_n:

$$\Psi_n(k) = \frac{1}{k+1}\Phi_n(k+1) + \frac{k}{k+1}\Psi_n(k+1)$$

The recursion is based on the fact that $\Psi_n(n) = 0$, which is obviously true, since if all the objects have been inspected and the maximal has already been examined twice, the best object is this maximal and can no longer be selected. In other words, this is effectively equivalent to the fact that there are no more objects in the urn and therefore the player has failed.

Lemma 5 *For all $n \in \mathbb{N}$ and $0 \le k \le n$, the function Ψ_n defined recursively above satisfies*

$$\Psi_n(k) = \frac{-2\,k + 2\,n - k\,\mathrm{H}(k-1) + k\,\mathrm{H}(n-1)}{3\,n}$$

where $\mathrm{H}(m) := \sum_{i=1}^{m} \frac{1}{i}$ *represents the mth harmonic number.*

Proof By recursion backwards. It suffices to observe that it is true for $k = n$ and that, if it is true for $k+1$, this means that it is also true for k. □

Proposition 3 *The sequence $\widehat{\Psi}_n(x) := \Psi_n(\lfloor nx \rfloor)$ converges uniformly on $[0, 1]$ to*

$$\Psi(x) := \frac{2 - 2\,x - x\,\log(x)}{3}$$

Proof This is straightforward, bearing in mind that $\hbar_n(x) := \lfloor nx \rfloor \frac{\mathrm{H}(n-1) - \mathrm{H}(\lfloor nx \rfloor - 1)}{n}$ converges uniformly on $[0, 1]$ to $\hbar(x) := -x \log(x)$ continuously extended to 0 by means of $\hbar(0) := 0$ □

Remark The optimal threshold, $\mathbf{k}_n$, can be determined by simply considering that, when faced with a nice candidate, the decision to select this candidate or not is that the probability of success is greater when doing so ($\mathbf{k}_n/n$) than the probability of success when rejecting this candidate ($\Psi_n(\mathbf{k}_n)$); i.e. the optimal threshold, $\mathbf{k}_n$, will be the first integer k that fulfills the condition $\frac{k}{n} \ge \Psi_n(k)$. That is,

$$\mathbf{k}_n = \min\left\{k : 5 - \frac{2n}{k} \ge \sum_{i=k}^{n-1} \frac{1}{i}\right\}$$

The following will likewise be fulfilled:

$$\Psi_n\left(\frac{\mathbf{k}_n}{n}n\right) = \Psi_n(\mathbf{k}_n) \leq \frac{\mathbf{k}_n}{n}$$
$$\Psi_n\left(\frac{\mathbf{k}_n - 1}{n}n\right) = \Psi_n(\mathbf{k}_n - 1) > \frac{\mathbf{k}_n - 1}{n}$$

Thus, taking limits, we now obtain the asymptotic optimal threshold by solving the equation $\Psi(x) = x$

$$x = \frac{2 - 2x - x\,\log(x)}{3} \Rightarrow x = \frac{2}{W(2e^5)} = 0.4709265...$$

So, we can already affirm that

$$\lim_{n\to\infty} \frac{\mathbf{k}_n}{n} = \frac{2}{W(2e^5)} = 0.4709265...$$

However, the asymptotic probability of success requires more effort.

3.3 $\Upsilon_n(k)$: probability that the maximal has been inspected twice after inspecting the kth different object for the first time

We denote by $\Upsilon_n(k)$ the probability that when we inspect the kth different object for the first time, the maximal has been inspected twice. When we inspect the kth different object for the first time, we are starting out from two possible scenarios:

(A) In the first appearance of the $(k-1)$th different object, the maximal has been inspected twice. This occurs with probability $\Upsilon_n(k-1)$. In this situation, the probability of the maximal being repeated is the probability that the new object is not maximal; i.e. $(k-1)/k$.

(B) In the first appearance of the $(k-1)$th different object, the maximal has been inspected only once. This occurs with probability $1 - \Upsilon_n(k-1)$. In this case, for the kth different object to be inspected for the first time, with the maximal having been inspected twice, the twin of the maximal must be inspected immediately, which occurs with probability $\mathfrak{p}_n(k-1) = \frac{1}{2n-2(k-1)+1}$, and the kth different object is not maximal (this occurs with probability $(k-1)/k$).

This allows $\Upsilon_n(k)$ to be defined recursively. The initial condition is obviously $\Upsilon_n(0) = 0$.

$$\begin{aligned}\Upsilon_n(k) &= \frac{(-1+k)}{k}\,(1 - \Upsilon_n(k-1))\,\mathfrak{p}_n(k-1) + \frac{(-1+k)}{k}\Upsilon_n(k-1)\\ &= \frac{1-k}{k\,(-3+2k-2n)} + \frac{2\,(-1+k)\,(-1+k-n)}{k\,(-3+2k-2n)}\Upsilon_n(k-1)\end{aligned}$$

Although its justification falls outside the scope of this paper, the evolution of its graph, as n increases, means it is reasonable to assume that the sequence $\widehat{\Upsilon}_n(x) :=$

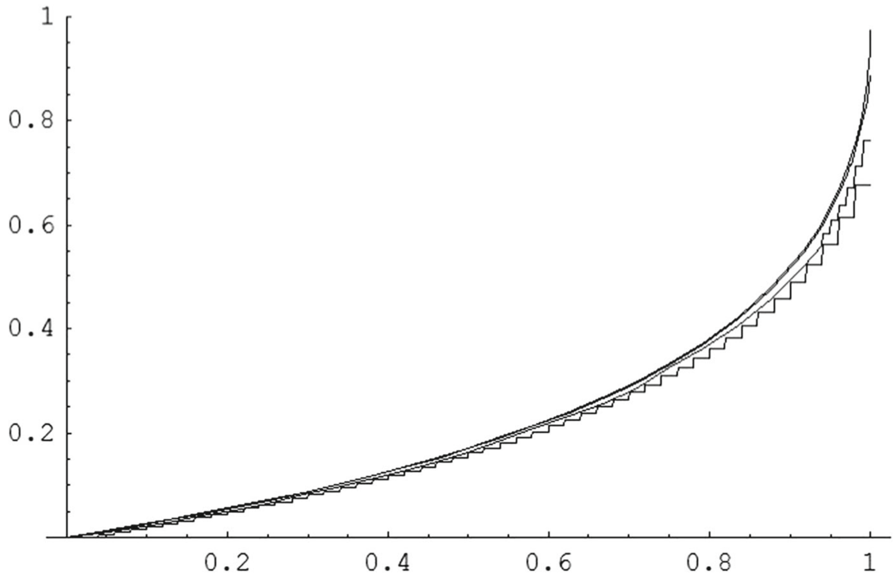


Fig. 1 $\widehat{\Upsilon}_n(x)$ for $n = 50$, 100, 500 and 1000

$\Upsilon_n(\lfloor xn \rfloor)$ converges uniformly on [0, 1] to a continuous and differentiable function Υ on [0, 1) (see Fig. 1 for n = 50, 100, 500 and 1000). The graphical representation also suggests that Υ is not differentiable on 1, but this is not a problem.

Proposition 4 *Assuming that the function* $\widehat{\Upsilon}_n(x) := \Upsilon_n(\lfloor xn \rfloor)$ *converges uniformly on* [0, 1] *to a continuous and derivable function* Υ *on* [0, 1)*, then*

$$\Upsilon(x) := \frac{2 - 2\sqrt{1-x} - x}{x}$$

Proof

$$\Upsilon_n(k) = G_n(k) + H_n(k)\Upsilon_n(k-1)$$

where

$$G_n(k) = \frac{1-k}{k\ (-3 + 2\,k - 2\,n)};\ H_n(k) = \frac{2\ (-1+k)\ (-1+k-n)}{k\ (-3+2\,k-2\,n)}$$

Making

$$h_n(x) := n(1 - H_n(\lfloor nx \rfloor)) \text{ and } g_n(x) := nG_n(\lfloor nx \rfloor)$$

we have that $h_n(x)$ converges on (0, 1) to

$$h(x) := \frac{-2+x}{2\ (-1+x)\ x}$$

and $g_n(x)$ converges on (0, 1) to

$$g(x) := \frac{1}{2-2x}$$

Now, considering 1, Υ satisfies in (0, 1)

$$\Upsilon'(x) = g(x) - h(x)\Upsilon(x) = \frac{1}{2-2x} - \frac{(-2+x)\ \Upsilon(x)}{2\ (-1+x)\ x}$$

Solving this differential equation, we now have that

$$\Upsilon(x) = \frac{2 - 2K\sqrt{1-x} - x}{x}$$

And, given that $\Upsilon(x)$ is continuous in 0 with $\Upsilon(0) = 0$, it follows that $K = 1$ □

3.4 $\digamma_n(k)$: probability of success using the threshold k

Let us now see the probability of success using the threshold k; i.e. accept the first nice candidate to appear after having inspected k different objects. When the kth different object is inspected, two possible scenarios arise: the maximal has been inspected twice or only once, which occurs with respective probabilities $\Upsilon_n(k)$ and $(1-\Upsilon_n(k))$. In the former case, the probability of success is $\Phi_n(k)$ and in the latter, $\Psi_n(k)$. Hence,

$$\digamma_n(k) = \Psi_n(k)\Upsilon_n(k) + (1-\Upsilon_n(k))\Phi_n(k)$$

Proposition 5 *The sequence of functions* $\widehat{\digamma}_n(x) := \digamma_n(\lfloor nx \rfloor)$ *converges uniformly on* $[0, 1]$ *to the function*

$$\digamma(x) := \frac{-4 + 6\sqrt{1-x} + 4x + \left(-2 + 2\sqrt{1-x} + x\right)\log(x)}{3}$$

Proof

$$\digamma_n(\lfloor nx \rfloor) = \Psi_n(\lfloor nx \rfloor)\Upsilon_n(\lfloor nx \rfloor) + (1-\Upsilon_n(\lfloor nx \rfloor))\Phi_n(\lfloor nx \rfloor)$$

and, employing the notations introduced above,

$$\widehat{\digamma}_n(x) = \widehat{\Psi}_n(x)\widehat{\Upsilon}_n(x) + (1-\widehat{\Upsilon}_n(x))\widehat{\Phi}_n(x)$$

where uniform convergence is clear and, passing to the limit, we have that

$$\digamma(x) := \lim_n \widehat{\digamma}_n(x) = \Psi(x)\Upsilon(x) + (1-\Upsilon(x))\Phi(x)$$

where, let us recall,

$$\Phi(x) := \frac{2+x}{3};\ \Psi(x) := \frac{2-2x-x\ \log(x)}{3};\ \Upsilon(x) := \frac{2-2\sqrt{1-x}-x}{x}$$

□

Proposition 6 *If* $\mathbf{k}_n$ *is the optimal threshold; i.e. where* $\mathcal{F}_n$ *reaches its maximum value, then*

$$\lim_n \left(\frac{\mathbf{k}_n}{n}\right) = \vartheta = \frac{2}{W(2e^5)} = 0.4709265..$$

and denoting by $\mathbf{P}_n := \mathcal{F}_n(\mathbf{k}_n)$ *the probability of success using the threshold* $\mathbf{k}_n$

$$\lim_n \mathbf{P}_n = \mathcal{F}(\vartheta) = 0.76797426...$$

Proof We need only consider Proposition 1 and that ϑ is where $\mathcal{F}$ reaches its maximum value on [0, 1]. □

Figure 2 compares the function $\mathcal{F}(x)$ and its counterpart in the classic secretary problem, $-x\log(x)$.

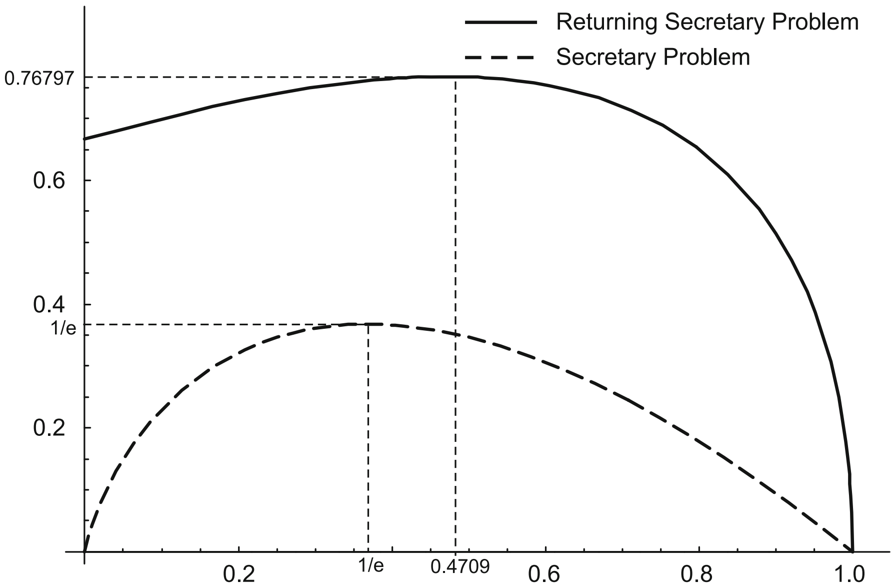


Fig. 2 $\mathcal{F}(x)$ versus $-x\log(x)$

4 Exact calculation of $\mathbf{k}_n$ and $\mathbf{P}_n$

What was set forth in the previous section guarantees that $\mathbf{k}_n \sim \vartheta n$, which means, in virtue of Proposition 1, that for large values of n, this estimation is good enough if one simply wishes for the error in the optimum success probability, $\mathbf{P}_n$, to be small.

An interesting challenge in problems of this kind is to efficiently compute the exact values of $\mathbf{k}_n$ and $\mathbf{P}_n$.

The exact calculation of $\mathbf{k}_n$, using $\mathbf{k}_n = \min\{k : 5 - \frac{2n}{k} \geq \sum_{i=k}^{n-1} \frac{1}{i}\}$, requires a number of operations of linear order, $\mathcal{O}(n)$, whereas $\mathcal{F}_n(\mathbf{k}_n)$, using the formula

$$\mathbf{P}_n = \mathcal{F}_n(\mathbf{k}_n) = \Psi_n(\mathbf{k}_n)\Upsilon_n(\mathbf{k}_n) + (1 - \Upsilon_n(\mathbf{k}_n))\Phi_n(\mathbf{k}_n)$$

requires the calculation of $\Phi_n(\mathbf{k}_n)$ and $\Psi_n(\mathbf{k}_n)$ (closed formulas) and of $\Upsilon_n(\mathbf{k}_n)$, whose recursive calculation also requires requires a number of operations of order $\mathcal{O}(n)$. In short, the calculation of $\mathcal{F}_n(\mathbf{k}_n)$ has computational complexity $\mathcal{O}(n)$.

In contrast, the formula provided by Garrod et al. (2012)

$$\mathbf{P}_n = \frac{1}{3n}\left(2n + \mathbf{k}_n - \left(\mathbf{k}_n - \sum_{s=0}^{\mathbf{k}_n-1}\prod_{r=1}^{s}\frac{2(n-\mathbf{k}_n+r)}{2(n-\mathbf{k}_n+r)+1}\right)\left(3 - \sum_{i=\mathbf{k}_n}^{n-1}\frac{1}{i}\right)\right)$$

requires a number of operations of order $\mathcal{O}(n^2)$; this is due to the presence of the nested sum and product.

Another very interesting challenge is to find closed formulas that yield the exact value of the optimal threshold or that do so with very few exceptions.

In the case of the classic secretary problem, denoting by κ_n the optimal threshold for n secretaries, we have the following expression

$$\kappa_n^* = -\frac{1}{2W\left(-\frac{e^{-(1+1/(2n))}}{2n}\right)}$$

such that $\lceil \kappa_n^* \rceil$ coincides with κ_n for all $n > 3$, with no known exceptions.

This expression can be deduced (see http://mathworld.wolfram.com/SultansDowryProblem.html) from the condition

$$\kappa_n = \min\{k : \mathrm{H}(k) \geq \mathrm{H}(n) - 1\}$$

solving the equation

$$\mathrm{H}(k) = \mathrm{H}(n) - 1$$

Using a series expansion for H about infinity,

$$\mathrm{H}(x) \approx \gamma + \frac{1}{2x} + \log(x) + \ldots$$

where γ is the Euler–Mascheroni constant, and using this approximation in the equation, we have that

$$\frac{1}{2k} + \log(k) = \frac{1}{2n} + \log(n) - 1$$

Although $\lceil \kappa_n^* \rceil$ seems to coincide without exceptions for $n > 3$ with the exact value κ_n, without entering into technical details, it can only really be stated that $\kappa_n \approx \kappa_n^*$, but this does not ensure the coincidence $\kappa_n = \lceil \kappa_n^* \rceil$ *ad infinitum*.

Proceeding similarly in our problem, where the stopping rule is

$$\mathbf{k}_n = \min\left\{k : 5 - \frac{2n}{k} \geq \sum_{i=k}^{n-1} \frac{1}{i}\right\} = \min\left\{k : 5 - \frac{2n}{k} \geq \mathrm{H}(n-1) - \mathrm{H}(k-1)\right\}$$

we obtain the equation

$$5 - \frac{2n}{k} + \mathrm{H}(k) - \frac{1}{k} = \mathrm{H}(n) - \frac{1}{n}$$

Using a series expansion for H about infinity

$$5 - \frac{2n}{k} + \log(k) + \frac{1}{2k} - \frac{1}{k} = \log(n) + \frac{1}{2n} - \frac{1}{n}$$

and solving to k, we have the solution

$$k_n^* := \frac{1 + 4\,n}{2\,W\left(\frac{e^{5+\frac{1}{2\,n}}\left(4+\frac{1}{n}\right)}{2}\right)}$$

which, in fact gives $\lceil k_n^* \rceil = \mathbf{k}_n$ for all $n > 0$ without any known exceptions. We likewise have that, without any known exceptions, $\mathbf{k}_n = \lceil \widehat{k_n} \rceil$, which is obtained from the approximation $k_n^* \approx \widehat{k_n}$,where

$$\widehat{k_n} = n\vartheta + \frac{3}{2\left(1 + \frac{2}{\vartheta}\right)} - \frac{\vartheta}{2} = n \cdot 0.470927... + 0.050417268...$$

5 The same problem with different payoff functions

We will now see that the method proposed in this paper similarly allows the solution of variants of the same problem when considering other payoff functions. For example, it is easy to solve the problem considering interview costs, penalizing failure, etc. It is only necessary to modify the construction of $\Phi_n(k)$, substituting the probability of success, k/n, when a nice candidate is accepted, for the expected payoff, which we shall denote by $\mathbb{E}_n(k)$. Likewise, the final condition for Φ_n must be substituted by $\Phi_n(n) = \mathbb{E}_n(n)$ and, finally, the final condition for Ψ_n must be substituted by $\Psi_n(n) = \Theta$, where Θ represents the payoff when the interviews end without having selected any object. This is what occurs if n different objects are interviewed and the maximal has already been interviewed twice. Accordingly, we will have that:

$$\begin{aligned}\mathfrak{p}_n(k) &:= \frac{1}{1-2k+2n}\\ \Phi_n(k) &= \mathbb{E}_n(k)\mathfrak{p}_n(k) + (1-\mathfrak{p}_n(k))\Phi_n(k+1);\ \Phi_n(n) = \mathbb{E}_n(n)\\ \Psi_n(k) &= \frac{1}{k+1}\Phi_n(k+1) + \frac{k}{k+1}\Psi_n(k+1);\ \Psi_n(n) = \Theta\end{aligned}$$

and the remaining functions, Υ_n and $\digamma_n$, will keep their expression of the standard version intact. Let us see the following three variants.

5.1 The wrong selection is penalized

If we consider that selecting an object without success is penalized in the same way as success is rewarded (a monetary unit) and the lack of selection is not penalized, then we have that the expected payoff when a nice candidate is selected, having inspected k different objects, will be

$$\mathbb{E}_n(k) = \frac{k}{n} - \left(1 - \frac{k}{n}\right) = \frac{2k}{n} - 1.$$

We now have the following recursion for Φ_n,

$$\begin{aligned}\Phi_n(k) &= \mathbb{E}_n(k)\mathfrak{p}_n(k) + (1-\mathfrak{p}_n(k))\Phi_n(k+1)\\ &= G_n(k) + \Phi_n(k+1)H_n(k);\ \Phi_n(n) = 1\end{aligned}$$

where

$$G_n(k) = \frac{\frac{2k}{n} - 1}{1-2k+2n};\ H_n(k) = \frac{-2k+2n}{1-2k+2n}$$

Making

$$h_n(x) := n(1 - H_n(\lfloor nx \rfloor)) \text{ and } g_n(x) := nG_n(\lfloor nx \rfloor),$$

$h_n(x)$ converge on $(0, 1)$ and uniformly on $[a, b]$ for all $0 < a < b < 1$ to

$$h(x) = \frac{1}{2-2x},$$

$g_n(x)$ converge on $(0, 1)$ and uniformly on $[a, b]$ for all $0 < a < b < 1$ to

$$g(x) = \frac{1-2x}{-2+2x}.$$

Assuming the uniform convergence of $\widehat{\Phi}_n(x) := \Phi_n(\lfloor nx \rfloor)$ to $\Phi(x)$ in [0, 1], we are now in the conditions of Theorem 2 and $\Phi(x)$ will be the solution of the differential equation

$$y'(x) = \frac{1 - 2x + y(x)}{2 - 2x}$$

and, together with the final condition, $\Phi(1) = 1$, it may be concluded that

$$\Phi(x) = \frac{1 + 2x}{3}$$

On the other hand,

$$\Psi_n(k) = \frac{1}{k+1}\Phi_n(k+1) + \frac{k}{k+1}\Psi_n(k+1)$$

and, once again making use of Theorem 2 and omitting the details, we have that $\widehat{\Psi}_n(x) := \Psi_n(\lfloor nx \rfloor)$ converge uniformly on [0, 1] to

$$\Psi(x) = \frac{1 - x - 2x\,\log(x)}{3}$$

We also have that

$$F(x) := \Psi(x)\Upsilon(x) + (1 - \Upsilon(x))\Phi(x)$$
$$F(x) = \frac{-5 + 6\sqrt{1-x} + 5x + 2\left(-2 + 2\sqrt{1-x} + x\right)\log(x)}{3}$$

We now have that the maximum value of F in [0, 1] is reached in

$$\alpha := \frac{2}{W(2e^{\frac{7}{2}})} = 0.65123...$$

Such that if k_n is the optimal threshold, $\frac{k_n}{n} \to \alpha$ and the asymptotic expected payoff used as the threshold, $\lfloor n\alpha \rfloor$, is

$$F(\alpha) = 0.64778...$$

5.2 Failure is penalized: incorrect selection or no selection

If we consider that selecting an object without success or not selecting is penalized in the same way as success is rewarded (a monetary unit), then we have the expected payoff when a nice candidate is selected, having inspected k different objects will be

$$\mathbb{E}_n(k) = \frac{n}{k} - \left(1 - \frac{n}{k}\right) = \frac{2k}{n} - 1.$$

In this case, it is necessary to modify the final status of the recursion of Ψ_n: $\Psi_n(n) = -1$. Now, proceeding in a similar way to the previous case, we have that:

$$\Phi(x) = \frac{1 + 2x}{3}; \Psi(x) = \frac{1 - 4x - 2x \log(x)}{3}$$

and, finally, that

$$\digamma(x) = \frac{-11 + 12\sqrt{1-x} + 8x + 2\left(-2 + 2\sqrt{1-x} + x\right)\log(x)}{3}$$

We now have that the maximum value of $\digamma$ in [0, 1] is reached in $\vartheta = \frac{2}{W(2e^5)}$, as in the standard problem. Hence, if k_n is the optimal threshold, $\frac{k_n}{n} \to \vartheta$, the asymptotic expected payoff, using $\lfloor n\vartheta \rfloor$ as the threshold, is

$$\digamma(\vartheta) = 0.535948...$$

5.3 Cost for interviews without any penalty for failure

If we consider that each inspection of a different object has a cost of $1/n$, which detracts from the payment for success (which is unitary) and failure is not penalized, we will have that

$$\mathbb{E}_n(k) = \frac{k}{n}\left(1 - \frac{k}{n}\right); \Phi_n(n) = 0$$

Once again proceeding similarly, we have that

$$\Phi(x) = \frac{(1-x)(2+3x)}{15}; \Psi(x) = \frac{2 - 5x + 3x^2 - x\log(x)}{15}$$

and, finally, that

$$\digamma(x) = \frac{(1-x)\left(-10 + 12\sqrt{1-x} + 9x\right) + \left(-2 + 2\sqrt{1-x} + x\right)\log(x)}{15}$$

We now have that the maximum value of $\digamma$ in [0, 1] is reached in $\zeta = 0.127863637.....$ Hence, if k_n is the optimal threshold, $\frac{k_n}{n} \to \zeta$, the asymptotic expected payoff using the $\lfloor n\zeta \rfloor$ as the threshold, is

$$\digamma(\zeta) = 0.137662...$$

6 Considerations about the work and future perspectives

This paper has presented a novel technique for the Returning Secretary Problem that can be applied to the study of variants of this same problem and which may be used more generally in the study of the asymptotic solution of optimal threshold problems via the use of differential equations derived from the recursion of the problem of underlying dynamic programming. We have also obtained a recursive formula that allows the calculation in linear time, with respect to the number of objects, of the exact probability of success.

A natural continuation of this paper is to consider that there are c copies of n objects (candidates). This study has already been initiated by Garrod (the best c-tuplet) and by Shai Vardi (the $(c-1)$-returning secretary problem), but, so far, not even the asymptotic values are known in the case of $c = 3$. Also of interest is the true returning problem, in which instead of n pairs of twin secretaries we have n secretaries who, after their first interview return to a random place in the queue of candidates (to the urn), which, obviously, is not the same as the problem posed in this paper. Another interesting problem is that of considering that only one candidate, chosen by the player, returns to a random place in the candidate queue after the first interview.

An interesting challenge in problems of this kind is to investigate the decreasing nature of the probability of success with respect to the number of objects. The (non-strict) decrease is quite reasonable: with a larger number of objects, there can be no greater probability of success. However, the strict decrease of the probability is more delicate. In the case of the classic secretary problem, the probability of success with n secretaries, $\mathbf{P}_n$, is strictly decreasing for $n > 3$ and $\mathbf{P}_2 = \mathbf{P}_3 = 1/2$. In the Best-or-Worst variant (Bayón et al. 2017), we have that

$$\mathbf{P}_n = \frac{\lfloor \frac{1+n}{2} \rfloor}{2\lfloor \frac{1+n}{2} \rfloor - 1}$$

so that $\mathbf{P}_n = \mathbf{P}_{n+1}$ for all odd n.

In the variant of the returning secretary problem, the probability of success for the first values of n is shown in the following table: suggesting that, with the exception

n	1	2	3	4	5	6	7	8	9	10
$\mathbf{P}_n =$	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{407}{504}$	$\frac{761}{945}$	$\frac{2837}{3564}$	$\frac{3001}{3780}$	$\frac{1138153}{1441440}$	$\frac{3102431}{3938220}$	$\frac{126193}{160650}$
$\mathbf{P}_n \approx$	1.	0.833	0.833	0.807	0.805	0.796	0.793	0.789	0.787	0.785

of $\mathbf{P}_2 = \mathbf{P}_3 = 5/6$, the decrease in the probability of success is strict. However, this is an open problem.

References

Bayón L, Fortuny P, Grau JM, Oller-Marcén AM, Ruiz MM (2017) The best-or-worst and the postdoc problems. J Comb Optim 35(3):703–723

Babaioff M, Immorlica N, Kleinberg R (2007) Matroids, secretary problems, and online mechanisms. In: Proceedings of SODA, pp 434–443

Bearden JN (2006) A new secretary problem with rank-based selection and cardinal payoffs. J Math Psychol 50:58–59

Dynkin EB (1963) the optimum choice of the instant for stopping a markov process. Sov Math Dokl 4:627–629

Ferguson TS (1989) Who solved the secretary problem? Stat Sci 4(3):282–296

Ferguson TS, Hardwick JP, Tamaki M (1991) Maximizing the duration of owning a relatively best object. In: Ferguson T, Samuels S (eds) Contemporary mathematics: strategies for sequential search and selection in real time, vol 125. American Mathematics Association, Washington, pp 37–58

Freij R, Wastlund J (2010) Partially ordered secretaries. Electron Commun Probab 15:504–507

Garrod B (2011) Problems of optimal choice on posets and generalizations of acyclic colourings. PhD Thesis

Garrod B, Morris R (2012) The secretary problem on an unknown poset. Random Struct Algorithms 43(4):429–451

Garrod B, Kubicki G, Morayne M (2012) How to choose the best twins. Siam J Discrete Math 26(1):384–398

Georgiou N, Kuchta M, Morayne M, Niemiec J (2008) On a universal best choice algorithm for partially ordered sets. Random Struct Algorithms 32:263–273

Gilbert J, Mosteller F (1966) Recognizing the maximum of a sequence. J Am Stat Assoc 61:35–73

Lindley DV (1961) Dynamic programming and decision theory. J R Stat Soc Ser C (Appl Stat) 10(1):39–51

Soto JA (2011) Matroid secretary problem in the random assignment model. In: Proceedings of SODA, pp 1275–1284

Szajowski KA (2009) A rank-based selection with cardinal payoffs and a cost of choice. Sci Math Jpn 69(2):285–293

Vardi S (2014) The secretary returns. arXiv:1404.0614

Vardi S (2015) The returning secretary. In: 32nd International symposium on theoretical aspects of computer science (STACS 2015), Leibniz international proceedings in informatics (LIPIcs), pp 716–729