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A NOTE ON LAST-SUCCESS-PROBLEM

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ABSTRACT. We consider the Last-Success-Problem with n independent Bernoulli random variables with parameters $p_i > 0$. We improve the lower bound provided by F.T. Bruss for the probability of winning and provide an alternative proof to the one given in [3] for the lower bound $(1/e)$ when $R := \sum_{i=1}^{n}(p_i/(1-p_i)) \geq 1$. We also consider a modification of the game which consists in not considering it a failure when all the random variables take the value of 0 and the game is repeated as many times as necessary until a "1" appears. We prove that the probability of winning in this game when $R \leq 1$ is lower-bounded by $0.5819... = \frac{1}{e-1}$. Finally, we consider the variant in which the player can choose between participating in the game in its standard version or predict that all the random variables will take the value 0.

Keywords: Last-Success-Problem; Lower bounds; Odds-Theorem; Optimal stopping; Optimal threshold
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1. INTRODUCCION

The Last-Success-Problem is the problem of maximizing the probability of stopping on the last success in a finite sequence of Bernoulli trials. The framework is as follows. There are n Bernoulli random variables which are observed sequentially. The problem is to find a stopping rule to maximize the probability of stopping at the last "1". We restrict ourselves here to the case in which the random variables are independent. This problem has been studied by Hill and Krengel [9], Hsiau and Yang [10] and was simply and elegantly solved by F.T. Bruss in [2] with the following famous result. Other recent papers related to this problem are [5], [4], [8] and [7].

Theorem 1. *(Odds-Theorem, F.T. Bruss 2000). Let $I_1, I_2, ..., I_n$ be n independent Bernoulli random variables with known n. We denote by $(i = 1, ..., n)$ p_i, the parameter of I_i; i.e. $(p_i = P(I_i = 1))$. Let $q_i = 1 - p_i$ and $r_i = p_i/q_i$. We define the index*

$$\mathbf{s} = \begin{cases} \max\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \geq 1\}, & \text{if } \sum_{i=1}^{n} r_i \geq 1; \\ 1, & \text{otherwise.} \end{cases} \tag{1.1}$$

To maximize the probability of stopping on the last "1" of the sequence, it is optimal to stop on the first "1" we encounter among the variables $I_{\mathbf{s}}, I_{\mathbf{s}+1}, ..., I_n$. The optimal win

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probability is given by

$$\mathcal{V}(p_1,...,p_n) := \begin{cases} \left(\prod_{j=\mathbf{s}}^{n} q_j\right)\left(\sum_{i=\mathbf{s}}^{n} r_i\right), & \text{if } p_{\mathbf{s}} < 1; \\ \prod_{j=\mathbf{s+1}}^{n} q_j, & \text{if } p_{\mathbf{s}} = 1. \end{cases} \tag{1.2}$$

Henceforth, we will denote by $\mathcal{G}(p_1,...,p_n)$ the game consisting of pointing to the last 1 of the sequence $\{I_1,...,I_n\}$, where $0 < p_i = P(I_i = 1)$ for all $i = 1,...,n$. We denote $R_k := \sum_{i=k}^{n} \frac{p_i}{1-p_i}$ and $Q_k := \prod_{i=k}^{n}(1-p_i)$. The index $\mathbf{s}$ in Theorem 1 will be called the *optimal threshold* and the probability of winning, using the optimal strategy, will be denoted by $\mathcal{V}(p_1,...,p_n)$.

Bruss also presented in [2] the following bounds for the probability of winning.

Theorem 2. *Let* $\mathbf{s}$ *be the optimal threshold for the game* $\mathcal{G}(p_1,...,p_n)$*, then*

$$\mathcal{V}(p_1,...,p_n) > R_{\mathbf{s}} e^{-R_{\mathbf{s}}}.$$

He subsequently presented an addendum [3] with the following result for the case in which $R_1 \geq 1$.

Theorem 3. *If* $R_1 \geq 1$ *then*

$$\mathcal{V}(p_1,...,p_n) > \frac{1}{e}. \tag{1.3}$$

Very recently this result has been improved in [1] as follows.

Theorem 4. *If* $R_1 \geq 1$ *then*

$$\mathcal{V}(p_1,...,p_n) \geq \left(1 - \frac{1}{n+1}\right)^n > \frac{1}{e}.$$

This bound is, in fact, the same one that is proposed as Exercise 18 in [6] (Chapter 5, Section 5) :

$$\mathcal{V}(p_1,...,p_n) \geq \left(1 + \frac{1}{n}\right)^{-n} > \frac{1}{e}.$$

However, Theorem 4 does not improve (1.3) significantly for large n .

In the present paper, sharper lower bounds are established for the probability of winning than those presented above. In passing, we provide a very different proof of Theorem 3 from that of Bruss.

In those cases where $p_i < 1$ for all i, if all the random variables are zero, then the player fails. This suggests a variant (Variant I) of the standard game in which this is not considered a failure and the game is repeated as many times as necessary until a 1 appears. We study this variant in Section 3, where we will see that the typical value of $1/e$ for the lower bound of the probability of winning is replaced by $\frac{1}{e-1} = 0.5819....$

We also consider the possibility that the player can choose between participating in the game in its standard version or predict that all the random variables will have a value of 0. The study of this variant (Variant II) is very straightforward, but it is pleasing to discover that $1/e$ is the lower bound for the probability of winning in all cases.

The final section summarizes the results obtained with respect to the lower bounds for the probability of winning and establishes that the game with the greatest probability of winning is Variant I.

2. Lower bound for the case in which $R_{\mathbf{s}} = \infty$

Theorem 5. *If* $\mathbf{s}$ *is the optimal threshold and* $R_{\mathbf{s}} = \infty$*, then*

$$\mathcal{V}(p_1, ..., p_n) \geq \left(\frac{n-\mathbf{s}}{n-\mathbf{s}+R_{\mathbf{s}+1}}\right)^{n-\mathbf{s}} > \frac{1}{e^{R_{\mathbf{s}+1}}} > \frac{1}{e}.$$

Proof. If $\mathbf{s}$ is the optimal threshold and $R_{\mathbf{s}} = \infty$, this means that $p_{\mathbf{s}} = 1$ and $R_{\mathbf{s}+1} < 1$. In this case, the probability of winning is

$$\prod_{i=\mathbf{s}+1}^{n} (1-p_i).$$

Minimizing $\prod_{i=\mathbf{s}+1}^{n}(1-x_i)$ with respect to x_i subject to the constraint

$$\sum_{i=\mathbf{s}+1}^{n} x_i/(1-x_i) = R_{\mathbf{s}+1}$$

shows (using Lagrange multiplier technique) that this minimum is obtained by

$$x_{\mathbf{s}+1} = ... = x_n = \frac{R_{\mathbf{s}+1}}{R_{\mathbf{s}+1}+n-\mathbf{s}}$$

and its value

$$\left(\frac{n-\mathbf{s}}{n-\mathbf{s}+R_{\mathbf{s}+1}}\right)^{n-\mathbf{s}}.$$

This is decreasing with n always above its limit, which is $e^{-R_{\mathbf{s}+1}} > e^{-1}$. $\square$

3. Lower bound for the case in which $1 \leq R_{\mathbf{s}} \leq \infty$

The proof presented here is very different from the Bruss's proof and is based on the construction of a problem with a lower probability of winning (always $> 1/e$), adding a sufficiently large number of Bernoulli random variables with the same parameter. Previously, however, let us see several preparatory lemmata.

Lemma 1. *If* $p \in (0,1)$ *and* $\mathbf{x} \geq 1$*, then*

$$(1-p)\frac{\frac{p}{1-p}+\mathbf{x}}{\mathbf{x}} \leq 1.$$

Proof. It should be borne in mind that

$$(1-p)\frac{\frac{p}{1-p}+\mathbf{x}}{\mathbf{x}} = p\left(\frac{1}{\mathbf{x}}-1\right)+1.$$

$\square$

Lemma 2. *If* $\{p, P\} \subset (0,1)$ *with* $p \leq P$ *and* $1 \leq \mathbf{x} \leq \frac{1}{1-P}$*, then*

$$\frac{(1-p)}{1-P}\frac{\frac{p}{1-p}+\mathbf{x}-\frac{P}{1-P}}{\mathbf{x}} \leq 1.$$

Proof.

$$\frac{(1-p)}{1-P}\frac{\frac{p}{1-p}+\mathbf{x}-\frac{P}{1-P}}{\mathbf{x}} - 1 = \frac{(p-P)((P-1)\mathbf{x}+1)}{(P-1)^2\mathbf{x}}$$

$$\frac{(1-p)}{1-P}\frac{\frac{p}{1-p}+\mathbf{x}-\frac{P}{1-P}}{\mathbf{x}} \leq 1 \iff \frac{(p-P)((P-1)\mathbf{x}+1)}{(P-1)^2\mathbf{x}} \leq 0.$$

Now, the last inequality is true, seeing as

$$p \leq P \text{ and } 1 \leq \mathbf{x} \leq \frac{1}{1-P}.$$

$\square$

Lemma 3. *If* $\{p, \mathbf{x}\} \subset (0,1)$, *then*

$$(1-p)\left(\frac{p}{1-p} + \mathbf{x}\right) \leq 1.$$

Proof.

$$(1-p)\left(\frac{p}{1-p} + \mathbf{x}\right) = p + \mathbf{x} - p\mathbf{x}.$$

If we now assume that $p + \mathbf{x} - p\mathbf{x} = 1 + \epsilon > 1$, then we have the following contradiction

$$p = \frac{1 + \epsilon - \mathbf{x}}{1 - \mathbf{x}} > 1.$$

$\square$

We will denote by $\lceil \alpha \rceil$ the least integer greater than or equal to α.

Lemma 4. *Let* $\mathfrak{f} : [0,1] \longrightarrow \mathbb{R}$ *be the following function*

$$\mathfrak{f}(x) := \begin{cases} x \cdot (\lceil 1/x \rceil - 1) \cdot (1-x)^{\lceil 1/x \rceil - 2}, & \textit{if } 0 < x < 1; \\ 1/e, & \textit{if } x = 0; \\ 1, & \textit{if } x = 1. \end{cases}$$

Then, we have that $\mathfrak{f}$ *is continuous and strictly increasing in* $[0,1]$ *and, consequently, that* $1/e < \mathfrak{f}(x)$ *for all* $x \in (0,1]$.

Proof. The function $\mathfrak{f}$ can be rewritten as follows

$$\mathfrak{f}(x) = \begin{cases} x \cdot n \cdot (1-x)^{n-1}, & \text{if } \frac{1}{n+1} \leq x < \frac{1}{n} \text{ and } n \in \mathbb{N}; \\ 1/e, & \text{if } x = 0; \\ 1, & \text{if } x = 1. \end{cases}$$

so it is clear that the function $\mathfrak{f} \in C^1\left(\frac{1}{n+1}, \frac{1}{n}\right)$. Since $\mathfrak{f}(x) = x$ for all $x \in [1/2, 1)$, it follows that

$$\lim_{x \to 1^-} \mathfrak{f}(x) = 1.$$

Moreover,

$$\lim_{x \to \frac{1}{n}^-} \mathfrak{f}(x) = \left(1 - \frac{1}{n}\right)^{n-1} = \frac{1}{n}(n-1)\left(1 - \frac{1}{n}\right)^{n-2} = \lim_{x \to \frac{1}{n}^+} \mathfrak{f}(x)$$

so it follows that $\mathfrak{f}$ is also continuous at $\frac{1}{n}$ for all $n \in \mathbb{N}$. On the other hand, for every $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ we have that

$$\mathfrak{f}'(x) = -n(1-x)^{n-2}(nx-1) > 0.$$

Thus, $\mathfrak{f}$ is strictly increasing in $(0,1]$ and it only remains to prove that $\mathfrak{f}$ is continuous at 0. To do so, note that

$$x \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \Longrightarrow \left(\frac{n}{n+1}\right)^n = \mathfrak{f}\left(\frac{1}{n+1}\right) < \mathfrak{f}(x) < \mathfrak{f}\left(\frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^{n-1},$$

so, taking limits:

$$\frac{1}{e} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \leq \lim_{x\to 0^+}\mathfrak{f}(x) \leq \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-1} = \frac{1}{e} \Longrightarrow \lim_{x\to 0^+}\mathfrak{f}(x) = \frac{1}{e}.$$

and the proof is complete. □

Lemma 5. *Let the game $\mathcal{G}(p_1, ..., p_n)$ have*

$$p_i = p < 1 \text{ for } n - \lceil 1/p\rceil + 2 \leq i \leq n \text{ and } 1 \leq n - \lceil 1/p\rceil + 2.$$

Hence, the optimal threshold is $\mathbf{s} = n - \lceil 1/p\rceil + 2$ and the probability of winning is

$$\mathcal{V}(p_1, ..., p_n) = p\cdot(\lceil 1/p\rceil - 1)\cdot(1-p)^{\lceil 1/p\rceil - 2} = \mathfrak{f}(p) > 1/e.$$

Proof. If $n - \lceil 1/p\rceil + 2 \geq 1$, then $\mathbf{s} = n - \lceil 1/p\rceil + 2 \geq 1$ is the optimal threshold, given that

$$R_{\mathbf{s}} = (n - \mathbf{s} + 1)\frac{p}{1-p} = (\lceil 1/p\rceil - 1)\frac{p}{1-p} \geq 1$$

$$R_{\mathbf{s}+1} = (n - \mathbf{s})\frac{p}{1-p} = (\lceil 1/p\rceil - 2)\frac{p}{1-p} < 1$$

On the other hand,

$$Q_{\mathbf{s}} = (1-p)^{n-\mathbf{s}+1} = (1-p)^{\lceil 1/p\rceil - 1}.$$

And finally, using Lemma 4, we have

$$\mathcal{V}(p_1, ..., p_n) = R_{\mathbf{s}}Q_{\mathbf{s}} = p\cdot(\lceil 1/p\rceil - 1)\cdot(1-p)^{\lceil 1/p\rceil - 2} = \mathfrak{f}(p) > \frac{1}{e}.$$

□

In the particular case that p is the inverse of a natural number, we have the following corollary with a very sharp bound.

Corollary 1. *Let $1 < m \in \mathbb{N}$ and the game $\mathcal{G}(p_1, ..., p_n)$ with $n-m+2 \geq 1$ and $p_i = 1/m$ for $n-m+2 \leq i \leq n$. Hence, the optimal threshold is $s = n-m+2$ and the probability of winning is*

$$\mathcal{V}(p_1, ..., p_n) = \left(\frac{-1+m}{m}\right)^{-1+m} > 1/e.$$

Lemma 6. *Let $\mathbf{s}$ be the optimal threshold and ϑ the probability of winning for the game $\mathcal{G}(p_1, ..., p_n)$. Let $\mathbf{p} \leq \min\{p_i : i = \mathbf{s}, ..., n\}$. Let us now consider the auxiliary game $\mathcal{G}(p_1, ..., p_n, p_{n+1})$, with $p_{n+1} = \mathbf{p}$, and let us denote by ϑ^* the probability of winning for this game. Then:*

$$\vartheta \geq \vartheta^*.$$

Proof. We denote by $V(t)$ the probability of winning when t is the threshold used and by $V^*(t)$ the same probability for the auxiliary problem. Let us denominate by $\mathbf{s}^*$ the optimal threshold for the problem $\mathcal{G}(p_1, ..., p_n, p_{n+1})$.

We will assume, first, that $R_1 \geq 1$.

Given that $\mathbf{s}$ is the optimal threshold for the problem $\mathcal{G}(p_1, ..., p_n)$, we have that:

- If $p_{\mathbf{s}} < 1$:

$$\mathbf{s} = \max\{k : \sum_{j=k}^{n} r_j \geq 1\} \text{ and } V(\mathbf{s}) := \left(\prod_{j=\mathbf{s}}^{n} q_j\right)\left(\sum_{i=\mathbf{s}}^{n} r_i\right) = \vartheta.$$

Let us denominate by $\mathbf{s}^*$ the optimal threshold for the problem $\mathcal{G}(p_1, ..., p_n, p_{n+1})$. Thus,

$$\mathbf{s}^* = \max\{k : r_{n+1} + \sum_{j=k}^{n} r_j \geq 1\} \text{ and } V^*(\mathbf{s}^*) := \left(q_{n+1} \prod_{j=\mathbf{s}^*}^{n} q_j\right)\left(r_{n+1} + \sum_{i=\mathbf{s}^*}^{n} r_i\right) = \vartheta^*.$$

We will see what $\mathbf{s}^* \in \{\mathbf{s}, \mathbf{s}+1\}$. It's clear that $\mathbf{s} \leq \mathbf{s}^*$. Let us assume $\mathbf{s}+k = \mathbf{s}^* > \mathbf{s}+1$. Now, considering that $r_{n+1} = \frac{\mathbf{p}}{1-\mathbf{p}} \leq r_i$, for all $i = \mathbf{s}, ..., n$, it concluded that

$$\sum_{i=\mathbf{s}+1}^{n} r_i \geq \sum_{i=\mathbf{s}+2}^{n} r_i + \frac{\mathbf{p}}{1-\mathbf{p}} \geq \sum_{i=\mathbf{s}+k}^{n} r_i + \frac{\mathbf{p}}{1-\mathbf{p}} \geq 1$$

which contradicts the fact that $\mathbf{s}$ is optimal. Thus, $\mathbf{s}^* \in \{\mathbf{s}, \mathbf{s}+1\}$, and it suffices to prove that

$$V^*(\mathbf{s}) \leq V(\mathbf{s}) \text{ and } V^*(\mathbf{s}+1) \leq V(\mathbf{s}).$$

Now, making $\mathbf{x} := \sum_{i=\mathbf{s}}^{n} r_i \geq 1$, from Lemma 1, we have that

$$\frac{V^*(\mathbf{s})}{V(\mathbf{s})} = q_{n+1} \cdot \frac{r_{n+1} + \mathbf{x}}{\mathbf{x}} \leq 1.$$

On the other hand, since $\mathbf{x} - r_{\mathbf{s}} < 1$ and therefore $\mathbf{x} < \frac{1}{1-p_{\mathbf{s}}}$, from Lemma 2, we have that

$$\frac{V^*(\mathbf{s}+1)}{V(\mathbf{s})} = \frac{q_{n+1}}{q_{\mathbf{s}}} \cdot \frac{r_{n+1} + \mathbf{x} - r_{\mathbf{s}}}{\mathbf{x}} \leq 1.$$

- If $p_{\mathbf{s}} = 1$:

$$\sum_{j=\mathbf{s}+1}^{n} r_j < 1 \text{ and } V(\mathbf{s}) := \prod_{j=\mathbf{s}+1}^{n} q_j = \vartheta.$$

Thus, reasoning as above, we have that $\mathbf{s}^* \in \{\mathbf{s}, \mathbf{s}+1\}$ and then we prove that

$$V^*(\mathbf{s}) \leq V(\mathbf{s}) \text{ and } V^*(\mathbf{s}+1) \leq V(\mathbf{s}).$$

$$V^*(\mathbf{s}) = q_{n+1}\left(\prod_{j=\mathbf{s}+1}^{n} q_j\right) = (1-p)\left(\prod_{j=\mathbf{s}+1}^{n} q_j\right) = (1-p)V(\mathbf{s}) < V(\mathbf{s})$$

$$V^*(\mathbf{s}+1) = \left(q_{n+1} \prod_{j=\mathbf{s}+1}^{n} q_j\right)\left(r_{n+1} + \sum_{i=\mathbf{s}+1}^{n} r_i\right) = (1-p)V(\mathbf{s})\left(\frac{p}{1-p} + \sum_{i=\mathbf{s}+1}^{n} r_i\right)$$

Now, making $\mathbf{x} := \sum_{i=\mathbf{s}+1}^{n} r_i < 1$, from Lemma 3, we have that $V(\mathbf{s}) \geq V^*(\mathbf{s}+1)$.

Finally, if $R_1 < 1$, then $\mathbf{s} = 1$ and, reasoning as in the case $p_s < 1$, we also have that $\vartheta \geq \vartheta^*$.

□

Theorem 6. *Let us consider the game $\mathcal{G}(p_1, ..., p_n)$ and let $\mathbf{s}$ be the optimal threshold with $1 \leq R_{\mathbf{s}} \leq \infty$. Let $p := \min\{p_i : \mathbf{s} \leq i \leq n\}$, then*

$$\mathcal{V}(p_1, ..., p_n) > 1/e.$$

Proof. Using Lemma 6 repeatedly, we can build a sequence of games with a non-increasing probability of winning by attaching successive independent Bernoulli random variables with parameter p. When the attachment process has been carried out as many times as is necessary, we shall be able to use Lemma 5 and shall have

$$\mathcal{V}(p_1, ..., p_n) \geq \mathcal{V}(p_1, ..., p_n, p, ..., p) \geq p \cdot (\lceil 1/p \rceil - 1) \cdot (1-p)^{\lceil 1/p \rceil - 2} = \mathfrak{f}(p) > \frac{1}{e}.$$

□

This result improves the lower bound, $1/e$, quite ostensibly when all the parameters p_i are moderately far from 0.

3.1. ANNEX: The case $p_i = p$ for all i. We end this section with a result for the case in which all the Bernoulli random variables have the same parameter p. This was treated, with a certain degree of imprecision, in [11].

Proposition 1. *Let us consider the game $\mathcal{G}(p_1, ..., p_n)$, with $p_i = p < 1$. Thus,*

- *If $n \geq (\lceil 1/p \rceil - 1)$, then $\mathbf{s} = n - \lceil 1/p \rceil + 2$ is the optimal threshold and*

$$\mathcal{V}(p, ..., p) = p \cdot (\lceil 1/p \rceil - 1) \cdot (1 - p)^{\lceil 1/p \rceil - 2}.$$

- *If $n < (\lceil 1/p \rceil - 1)$, then $\mathbf{s} = 1$ is the optimal threshold and*

$$\mathcal{V}(p, ..., p) = n \cdot p \cdot (1 - p)^{n-1}.$$

Proof. • If $n \geq (\lceil 1/p \rceil - 1)$, then the conditions of the Lemma 5 are met.

• If $n < (\lceil 1/p \rceil - 1)$, then $R_1 = n \frac{p}{1-p} < 1$ and hence $\mathbf{s} = 1$ is the optimal threshold and, moreover, $Q_1 = (1 - p)^n$. Thus

$$\mathcal{V}(p, ..., p) = R_1 Q_1 = n \cdot p \cdot (1 - p)^{n-1}.$$

□

[11] addressed the problem differently, concluding that $1/e$ is a lower bound for the probability of winning. However, the optimal threshold that is considered in the aforementioned paper

$$s^* = \left\lfloor n + 1 + \frac{1}{\log(1 - p)} + \frac{1}{2} \right\rfloor$$

is not correct, as s^* does not always coincide with $s = n - \lceil 1/p \rceil + 2$, obtained in the previous proposition. Although it must be said that it is a very good estimate.

4. Lower bound for the case in which $1 > R_1$

Theorem 7. *If $R_1 < 1$, then*

$$\mathcal{V}(p_1, ..., p_n) > R_1 \left(\frac{n}{n + R_1} \right)^n > R_1 e^{-R_1}.$$

Proof.

$$\mathcal{V}(p_1, ..., p_n) = R_1 Q_1 = \left(\sum_{i=1}^{n} \frac{p_i}{1 - p_i} \right) \prod_{i=1}^{n} (1 - p_i)$$

Considering $f(x_1, ..., x_n) = \sum_{i=1}^{n} \frac{x_i}{1-x_i} \prod_{i=1}^{n}(1 - x_i)$ and using Lagrange multiplier technique we have that the minimum value of f with the constraint $R_1 = \sum_{i=1}^{n} \frac{x_i}{1-x_i}$ is reached in $x_1 = ... = x_n = \frac{R_1}{R_1+n}$ and the minimum value of f is

$$R_1 \left(\frac{n}{n + R_1} \right)^n > R_1 e^{-R_1}.$$

□

5. Variant I: If there have been no $1's$, the game is repeated

In those cases in which $p_i < 1$ for all i the player may fail because he has no chance to point to any *"last 1"*, as all the variables are 0. This suggests a variant of the original game in which the game is repeated as many times as necessary until a 1 appears. Of course, if $p_i = 1$ for some i, then it will never be necessary to repeat the game.

Proposition 2. *If* $\mathbf{s}$ *is the optimal threshold for the game* $\mathcal{G}(p_1, ..., p_n)$*, with* $p_i < 1$ *for all* i*, then the probability of winning with the new rule is*

$$\mathcal{V}^*(p_1, ..., p_n) = \frac{\left(\sum_{i=\mathbf{s}}^n \frac{p_i}{1-p_i}\right) \prod_{i=\mathbf{s}}^n (1-p_i)}{1 - \prod_{i=1}^n (1-p_i)}. \tag{5.1}$$

Proof. Obviously, the optimal strategy, with this rule, is the same as in the game in its original version. The difference lies only in the probability of winning, which is conditioned by $\sum_{i=1}^n I_i > 0$. Thus, bearing in mind that

$$P\left(\sum_{i=1}^n I_i > 0\right) = 1 - P\left(I_1 = I_2 = ... = I_n = 0\right) = 1 - \prod_{i=1}^n (1-p_i),$$

we have that

$$\mathcal{V}^*(p_1, ..., p_n) = P\left(\mathsf{WIN} | \sum_{i=0}^n I_i > 0\right) = \frac{\mathcal{V}(p_1, ..., p_n)}{1 - \prod_{i=1}^n (1-p_i)}.$$

$\square$

The cases in which $1 \geq R_1$ and $1 < R_1$ require a different treatment.

5.1. The case in which $1 \geq R_1$.

Theorem 8. *If* $1 \geq R_1$ *for the game* $\mathcal{G}(p_1, ..., p_n)$*, then*

$$\mathcal{V}^*(p_1, ..., p_n) > \frac{R_1}{-1 + e^{R_1}} \geq \frac{1}{-1 + e} = 0.5819...$$

Proof. Taking into account (5.1), let us now consider

$$\mathcal{V}^*(x_1, ..., x_n) = \frac{\left(\sum_{i=1}^n \frac{x_i}{1-x_i}\right) \prod_{i=1}^n (1-x_i)}{1 - \prod_{i=1}^n (1-x_i)}.$$

Minimizing $\mathcal{V}^*$ with respect to x_i subject to the constraint

$$\sum_{i=1}^n \frac{x_i}{1-x_i} = R_1,$$

shows (using Lagrange multiplier technique) that this minimum is obtained by

$$x_1 = x_2 = ... = x_n = \frac{R_1}{n + R_1}.$$

The minimum value is

$$\frac{R_1 \left(\frac{n}{n+R_1}\right)^n}{1 - \left(\frac{n}{n+R_1}\right)^n} > \frac{R_1}{-1 + e^{R_1}} \geq \frac{1}{-1 + e} = 0.5819...$$

$\square$

5.2. **The case in which $1 < R_1 < \infty$.**

Theorem 9. *If $1 < R_1 < \infty$ for the game $\mathcal{G}(p_1, ..., p_n)$, then*

$$\mathcal{V}^*(p_1, ..., p_n) > \frac{e^{-1}}{1 - \prod_{i=1}^n (1 - p_i)}.$$

Proof. Simply consider in (5.1) the bound (1.3) established in Theorem 3. □

Here is a case where, even though $R_1 > 1$, $\mathcal{V}^*$ is lower-bounded by $\frac{1}{e-1}$.

Theorem 10. *If $p_i = 1/n$ for $i = 1, ..., n$, then*

$$\mathcal{V}^*(p_1, ..., p_n) > \frac{1}{e-1} = 0.5819...$$

Proof. Obviusly, for $n = 1$, $\mathcal{V}^*(1) = 1 > \frac{1}{e-1}$.

For $n \geq 2$, the optimal threshold is $\mathbf{s} = 2$, seeing as

$$R_2 = (n-1)\frac{1/n}{1 - 1/n} = 1 \text{ and } R_3 = (n-2)\frac{1/n}{1 - 1/n} < 1$$

and hence

$$\mathcal{V}^*(p_1, ..., p_n) = \frac{R_2 Q_2}{1 - \prod_{i=1}^n (1 - p_i)} = \frac{\prod_{i=2}^n (1 - p_i)}{1 - \prod_{i=1}^n (1 - p_i)} = \frac{(1 - 1/n)^{n-1}}{1 - (1 - 1/n)^n} > \frac{1}{e-1}.$$

□

6. Variant II: The player can predict that there will be no $1's$

In the game in its original version, one of the ways the player loses is that all the variables are zero. We speculate what will occur if, as an initial possible move, we allow the player to predict that all the random variables will have the value 0 (i.e. there will not be $1's$). The answer is simple: the new game is equivalent to the original game adding a first random variable, I_0 with parameter $p_0 = 1$. Effectively, stopping at stage 0 in the standard game is equivalent to predicting that there will be no $1's$. In this variant, if the player predicts that all the random variables will have the value 0, the probability of winning is $\prod_{i=1}^n (1 - p_i) = Q_1$ and the probability of winning in the standard game, when the optimal threshold is $\mathbf{s}$, is $\mathcal{V}(p_1, ..., p_n) = R_{\mathbf{s}} Q_{\mathbf{s}}$. The probability of winning, let us call it $\mathcal{V}^{**}(p_1, ..., p_n)$, is hence

$$\mathcal{V}^{**}(p_1, ..., p_n) = \max\{Q_1, Q_{\mathbf{s}} R_{\mathbf{s}}\}.$$

Thus, the optimal strategy for **Variant II** is as follows:

(1) If $Q_1 = Q_{\mathbf{s}} R_{\mathbf{s}}$, it makes no difference whether the player predicts that there will be no $1's$ or plays the standard game.
(2) If $Q_1 < Q_{\mathbf{s}} R_{\mathbf{s}}$, then play the standard game.
(3) If $Q_1 > Q_{\mathbf{s}} R_{\mathbf{s}}$, then predict that there will be no $1's$.

Having said so, we now have the following result.

Theorem 11. *Taking $R_1 = \sum_{i=1}^n \frac{p_i}{1-p_i}$, the optimal strategy for **Variant II** is as follows:*

(1) *If $R_1 = 1$, it is indifferent to predict that there will be no $1's$ or play the standard game.*
(2) *If $R_1 > 1$, then play the standard game.*
(3) *If $R_1 < 1$, then predict that there will be no $1's$.*

In any case, the probability of winning, $\mathcal{V}^{**}(p_1, ..., p_n)$, is greater than $1/e$.

Proof. It suffices to bear in mind that the probability of winning when betting that there will be no $1's$ is $\prod_{i=1}^{n}(1-p_i)=Q_1$ and the probability of winning the standard game, when the optimal threshold is $\mathbf{s}$, is $\mathcal{V}(p_1,...,p_n)=R_{\mathbf{s}}Q_{\mathbf{s}}$.

(1) If $R_1=1$, the optimal threshold is $\mathbf{s}=1$ and we have

$$R_{\mathbf{s}}Q_{\mathbf{s}}=R_1Q_1=Q_1.$$

(2) If $R_1>1$, the optimal threshold is $\mathbf{s}\geq 1$ with $R_{\mathbf{s}}\geq 1$ and we have

$$\text{if } \mathbf{s}=1, \text{ then } R_{\mathbf{s}}Q_{\mathbf{s}}=R_1Q_1>Q_1.$$

$$\text{if } \mathbf{s}>1, R_{\mathbf{s}}Q_{\mathbf{s}}\geq Q_{\mathbf{s}}>Q_1.$$

(3) If $R_1<1$, the optimal threshold is $\mathbf{s}=1$ and we have

$$R_{\mathbf{s}}Q_{\mathbf{s}}=R_1Q_1<Q_1.$$

The probability of winning for this variant is greater than $1/e$ because, as has been stated, it is actually equivalent to a standard game with $p_0=1$. □

In summary, if the probability of winning for the standard game is not greater than $1/e$, then the probability that all the variables are zero is greater than $1/e$. And vice versa: if the probability that all the variables are zero is not greater than $1/e$, then the probability of winning for the standard game is greater than $1/e$. Of course it can also occur that the probability is greater than $1/e$ in both cases, but it can never be the case that both probabilities are simultaneously less than $1/e$.

7. Summary and conclusions

We first present the results related to the bounds for the probability of winning. Finally, we find that Variant I always has a higher probability of winning than Variant II, except in the case of $p_i=1$ for some i, in which case the three games are in fact equivalent.

Theorem 12. *Let* $\mathbf{s}$ *be the optimal threshold for the game* $\mathcal{G}(p_1,...,p_n)$ *and* $p:=\min\{p_i : i=\mathbf{s},...,n\}$*, then*

$$\mathcal{V}(p_1,...,p_n)\geq\begin{cases}(1-p)^{-1+\frac{1}{p}}>\frac{1}{e}, & \text{if } 1\leq R_1<\infty\ ;\\ R_1\left(\frac{n}{n+R_1}\right)^n, & \text{if } R_1<1;\\ \max\{e^{-R_{\mathbf{s}+1}},(1-p)^{-1+\frac{1}{p}}\}, & \text{if } R_1=\infty.\end{cases}$$

Proof. For the case $1\leq R_1<\infty$, see Theorem 6.

For the case $R_1<1$, see Theorem 7.

For the case $R_1=R_{\mathbf{s}}=\infty$, see Theorems 6 and 5.

□

Theorem 13. *Let* $\mathbf{s}$ *be the optimal threshold for the game* $\mathcal{G}(p_1,...,p_n)$*, then*

$$\mathcal{V}^*(p_1,...,p_n)\begin{cases}\geq\frac{R_1\left(\frac{n}{n+R_1}\right)^n}{1-\left(\frac{n}{n+R_1}\right)^n}>\frac{R_1}{e^{R_1}-1}\geq\frac{1}{e-1}, & \text{if } R_1\leq 1\ ;\\ \geq\frac{e^{-1}}{1-\prod_{i=1}^{n}(1-p_i)}, & \text{if } 1<R_1<\infty;\\ =\mathcal{V}(p_1,...,p_n), & \text{if } R_1=\infty.\end{cases}$$

Proof. For the case $R_1\leq 1$, see Theorem 8.

For the case $1<R_1<\infty$, see Theorem 9.

For the case $R_1=\infty$, $p_{\mathbf{s}}=1$, and, consequently, $\mathcal{V}^*(p_1,...,p_n)=\mathcal{V}(p_1,...,p_n)$. □

Theorem 14. *If $p_i = 1$ for some $i = 1, ..., n$, then all three games are equivalent by construction. Otherwise,*

$$\mathcal{V}(p_1, ..., p_n) \leq \mathcal{V}^{**}(p_1, ..., p_n) < \mathcal{V}^*(p_1, ..., p_n),$$

Proof. If $R_{\mathbf{s}} \geq 1$, then $R_1 \geq 1$ and, by Theorem 11, we have $\mathcal{V}(p_1, ..., p_n) = \mathcal{V}^{**}(p_1, ..., p_n)$. Moreover,

$$\mathcal{V}^*(p_1, ..., p_n) = \frac{\mathcal{V}(p_1, ..., p_n)}{1 - \prod_{i=1}^{n}(1 - p_i)} > \mathcal{V}(p_1, ..., p_n).$$

If $R_s < 1$, then $\mathcal{V}(p_1, ..., p_n) < \mathcal{V}^{**}(p_1, ..., p_n)$. Moreover, in this case $R_1 < 1$, and so, by Theorem 13,

$$\mathcal{V}^*(p_1, ..., p_n) > \frac{R_1}{-1 + e^{R_1}} \text{ and } \mathcal{V}^{**}(p_1, ..., p_n) = \prod_{i=1}^{n}(1 - p_i).$$

Maximizing $\prod_{i=1}^{n}(1 - x_i)$ with respect to x_i subject to the constraint $R_1 = \sum_{i=1}^{n} \frac{x_i}{1 - x_i}$ shows that this maximum is reached when all the x_i are 0 except for one, which takes the value $\frac{R_1}{1+R_1}$. So that

$$\mathcal{V}^{**}(p_1, ..., p_n) \leq 1 - \frac{R_1}{1 + R_1} = \frac{1}{1 + R_1} < \frac{R_1}{-1 + e^{R_1}} < \mathcal{V}^*(p_1, ..., p_n).$$

□

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