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A modified backward integration method for optimal control problems with degenerate equilibrium points

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Summary

We present a new numerical indirect method to solve optimal control problems with degenerate equilibrium points. When the optimal control problem is autonomous, the corresponding canonical system becomes an autonomous system of ordinary differential equations. Combining the well known blow-up technique of vector fields with the classic backward integration method, and focusing on a specific type of degenerate singular points, we develop a new method that is both easy to implement and able to deal with complex problems.

KEYWORDS:
Backward Integration, Blow-up Technique, Degenerate Singular Points, Optimal Control Problems

1 | INTRODUCTION

A broad range of numerical algorithms can be used to solve optimal control problems (OCPs). Numerical techniques can often be classified as either direct or indirect. In direct methods, the differential equation and the integral are discretized and the problem is converted into a nonlinear programming (NLP) problem. Conversely indirect methods approximate solutions to OCPs by numerically solving the boundary value problem (BVP) for the differential-algebraic system generated by the Minimum Principle.

In this paper we restrict ourselves to the class of autonomous, infinite time horizon problems. To solve the OCP we first use Pontryagin's Minimum Principle (PMP) to establish the corresponding canonical system, and then we transform the OCP to a two-point BVP. This formulation gives rise, in general, to a system of nonlinear differential equations describing the solution. There exist numerical methods to overcome the problem of nonlinearity: *Multiple Shooting* [1], the *Projection Method* [2], the

Discretization Method of Mercenier and Michel[3], the *method of Time Elimination*[4,5], *Reverse Shooting*[6], the *Backward Integration method*[7], the *Relaxation algorithm*[8], and the *Forward-Backward Sweep method*[9]. Their main features are discussed subsequently.

The Multiple Shooting method[1] subdivides the evaluation interval into N subintervals. Then, it iterates several times the combination of N forward integrations with backward integration once over the whole interval. However, the perfect foresight saddlepoint problem is ill-conditioned for Multiple Shooting. The Projection methods (or Weighted Residual methods)[2] reformulate the problem as a system of differential equations and search an analytic approximation using, frequently, Chebyshev polynomials. Its performance, however, depends on the goodness of the initial guess. Mercenier and Michel[3] transform the original problem into a finite horizon optimal control problem in discrete time with the same stationary solution, and then the solution is obtained by a static optimization procedure. Discretization requires that the system must be derived and linearized at the steady state.

Solving a saddlepoint problem backwards was done first in the method of Time Elimination[4,5], but this can only be applied to low-dimensional saddlepoint systems with monotonous adjustment. This method is also called Reverse Shooting in[6], and like the Backward Integration (BI) method[7], it exploits the backward stability of the stable manifold of a saddle point. The main difference between the two is that Reverse Shooting[6] uses an exogenously guessed terminal time t, whereas BI[7] determines t endogenously. The variant of the Relaxation method presented in[8] to solve a saddlepoint problem, takes first an arbitrary trial solution; then it measures the deviation from the true path using a multidimensional error function and uses the derivative of the error function to improve the trial solution in a Newton-type iteration. The Forward Backward Sweep method[9] solves first the state equation with a Runge-Kutta routine, then the costate equation backwards in time with the same Runge-Kutta solver, and updates the control. In[10] two convergence theorems are proved for this method. More information (and their comparison) about all these procedures can be found in[7] and[8]. All of these are indirect numerical methods. In[11], the limitations of direct and indirect methods for solving OCPs are studied.

Possibly the most relevant limitation of all the above methods is that they are unable to deal with *degenerate singularities*: equilibrium points of the differential equation whose jacobian has zero determinant. Our aim is to introduce the classical blowing-up method for ordinary differential equations[12] and apply it to OCPs.

To this end, we present a modification of the BI method which we use after performing the blow-up of the canonical equation. Blowing-up is a well-known technique in the area of holomorphic and real-analytic vector fields; we point the reader to the main reference[12], its application to plane vector fields[13], and to the quite accessible survey[14]. A deeper but also detailed exposition can be found in[15], where the authors review the main result of[16]. For a complete exposition of singularities of holomorphic differential equations (in which blowing-up plays a significant role), see[17].

It is important to note that we are interested in a type of problem which has not been addressed before: degenerate equilibrium points of what we shall call multiple hyperbolic saddle type, defined below. In [7], and later in [18], the case of backward integration for simple saddlepoint systems is analyzed in detail. They study the case of one state variable avoiding the difficulties posed by multiple state variables [19]. We will introduce a *Modified Backward Integration* (MBI) method for a particular case of degenerate singular points. The method is easy to implement and permits solving complex problems.

Starting with an example of fight against an invasive species (see, for example [20,21,22] for some background), we state an optimal control problem (with parameters in which possible uncertainties and disturbances are included) whose canonical system has, *generically in the parameters*, a hyperbolic saddle as equilibrium point. However, this hyperbolic saddle, for some special family of parameters becomes a degenerate singularity, which upon further inspection is of the type we can solve. Using our technique, we explicitly compute a solution and compare the values we obtain to those found using a different method (which has been proved useful for the non-degenerate case) [23,24,25], which is unable to approach the equilibrium point efficiently.

The paper is organized as follows. In Section 2 we introduce the parametric OCP and briefly describe its main properties in relation to our problem. In Section 3 we summarize (as already known results) the classic backward integration method and the blow-up technique. In Section 4 we formulate the basic ideas of the MBI method for multiple hyperbolic saddles and illustrate it with an example of singular degenerate vector field. In Section 5, we compute the solution approaching the degenerate equilibrium using our technique, and compare it to what other methods can do. In Section 6 we analyze the challenges arising when dealing with discounted functionals. Finally, Section 7 summarizes the main contributions of the paper and gives some pointers for further research .

2 | MOTIVATION: FIGHTING AN INVASIVE SPECIES

Managing invasive species can be a challenging, urgent and complicated problem, as ecologists everywhere known and the general population is starting to notice in large areas. We are thinking, near us (in Spain), of the *camalote* (Eichhornia crassipes) invasion of the Guadiana river [26] or the algae Caulerpa racemosa in the Mediterranean Sea [27], to give a couple of examples.

The problem of total eradication, or of, at least, reaching a reasonable steady state, can be modeled as an optimal control problem in which the optimization is performed on the total cost of the operation. This is our approach in the example that follows.

Consider a species whose population $x(t)$ follows, absent any harvesting, a standard logistic model, and which is subject to harvesting at a rate $u(t)$. Thus, its true evolution is described by the equation:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) - u(t) \tag{1}$$

where r is the *growth rate* and K is the *carrying capacity*. Harvesting, on one hand provides a benefit which we model as proportional to the harvest (we assume the species has some value, say from a chemical point of view or even just as fuel). On the other hand, it incurs some costs, which we model as: linear in the square of the harvest and proportional to the ratio between this square and the actual population (this is due to harvesting being more difficult as the population decreases). Finally, the very existence of the species is also costly (it takes volume from otherwise useful resources), and we model this cost as proportional to the population; see[28] for a detailed explanation of these parameters and[24] for an application to real-world fisheries. Thus, there is a cost function, which after normalization we can assume has three parameters:

$$F(x,u) = u - au^2 - b\frac{u^2}{x} - cx^3 \tag{2}$$

We are going to state the problem of minimizing the costs (which might even become maximizing the benefits if the problem turns out to be economically productive) of harvesting with $t \to \infty$, and we shall see whether this leads to total eradication or to reaching a steady state.

Thus, our optimal control problem is:

$$\begin{aligned} \min_{\mathbf{u}(t)} J &= \int_0^\infty -\left(u(t) - au(t)^2 - b\frac{u(t)^2}{x(t)} - cx(t)^3\right) dt \\ \dot{x}(t) &= rx(t)\,(1 - x(t)/K) - u(t),\ 0 \le t < \infty;\ \ x(0) = x_0 > 0 \end{aligned} \tag{3}$$

with the state $x(t) : \mathbb{R} \to \mathbb{R}$, and the control $u(t) : \mathbb{R} \to \mathbb{R}$, $0 \le t < \infty$.

Let H be the associated Hamiltonian:

$$H(x,u,\lambda) = -\lambda^0\left(u(t) - au(t)^2 - b\frac{u(t)^2}{x(t)} - cx(t)^3\right) + \lambda \cdot \left(rx(t)\,(1 - x(t)/K) - u(t)\right) \tag{4}$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is the co-state variable. From the Pontryagin Minimum Principle, we know that solutions to the problem satisfy the equation

$$\frac{\partial H}{\partial u} = 0 \Leftrightarrow 2au + \frac{2bu}{x} - \lambda - 1 = 0 \tag{5}$$

so that

$$u = \frac{(\lambda+1)x}{2(ax+b)} \tag{6}$$

(the case $\lambda_0 = 0$ has no solution). Substituting this value into $H(x,u,\lambda)$, setting $\lambda_0 = 1$, gives

$$H(x,\lambda) = \frac{x\left(\left(4cx^2(ax+b) + 4\lambda r(ax+b) - \lambda^2 - 2\lambda - 1\right)K - 4\lambda rx(ax+b)\right)}{4K(ax+b)} \tag{7}$$

which is defined for $x \geq 0$ (as long as $b > 0$). The canonical system associated to this optimal control problem is, after substituting u by its value (6):

$$\begin{aligned} \dot{x} &= rx(1 - x/K) - \frac{x(\lambda + 1)}{2(ax + b)} \\ \dot{\lambda} &= \frac{b(\lambda + 1)^2}{4(ax + b)^2} - 3cx^2 + r\lambda(2x/K - 1) \end{aligned} \tag{8}$$

for which the line $x = 0$ is invariant (as should be, from the biological point of view). This system is analytic for $x > 0$, as all the parameters are assumed to be nonzero. Thus, for any initial value (x_0, λ_0) with $x_0 > 0$, it admits a unique solution $(x(t), \lambda(t))$ starting at (x_0, λ_0) defined for all $t > 0$.

The set of equilibria of (8) depends greatly on a, b, c, r and K but there are two special ones:

$$(0, -1 + 2br - 2\sqrt{-br + b^2r^2}),\ (0, -1 + 2br + 2\sqrt{-br + b^2r^2}), \tag{9}$$

which appear from solving a quadratic equation after setting $x = 0$ in (8). These two points coalesce into $(0, 1)$ when $b = 1/r$. Furthermore, there are three extra equilibria, one of which happens to be $(0, 1)$ in some special cases (which involve $b = 1/r$ and $K = 1/b$); this multiple confluence will provide us with the degenerate singularity we shall study.

In the *generic* case, all the equilibria are different and have non-zero jacobian (i.e. are *non-degenerate*). Some of them can be non-real: there are values of the parameters for which only one equilibrium is real, v.gr. $a = 1.25, b = 0.73, c = 0.76, r = 0.63$ and $K = 1.89$, values for which there are three real ones, v.gr. $a = 20, b = 30, c = 2, r = 0.25, K = 1.52$, and values for which there are five, v.gr. $a = 2, b = 3, c = 0.2, r = 0.9, K = 1.25$. These cases degenerate when there are relations between the parameters.

2.1 | Statement of the degenerate problem

Consider Problem (3) with parameters $a = 1, b = 3/2; c = 5/2, r = 2/3, K = 3/2$. Notice how both equilibria of the form (9) correspond to $(x, \lambda) = (0, 1)$. The canonical system becomes:

$$\begin{aligned} \dot{x}(t) &= -\frac{x\left(8x^2 + 9\lambda - 9\right)}{9(2x + 3)} \\ \dot{\lambda}(t) &= -\frac{15x^2}{2} + \frac{3(\lambda + 1)^2}{2(2x + 3)^2} + \frac{2}{9}(4x - 3)\lambda \end{aligned} \tag{10}$$

which we wish to study at the equilibrium point $(x, \lambda) = (0, 1)$. Notice that, if $\lambda(t)$ is bounded on a trajectory $(x(t), \lambda(t))$ with $x(t) \to 0$, then $H(x(t), \lambda(t))$ in (7) tends to 0, which is a necessary condition for PMP to hold in problems with infinite horizon[29]. Thus, we need to study the trajectories converging to this equilibrium point, as they give rise to admissible solutions. As a matter of fact (although we are not really interested in a detailed study of this problem), the limit of $H(x, \lambda)$ on trajectories of (8) converging to $x = 0$ but $\lambda \to \infty$ is ∞, so that the only trajectories we need to study are, indeed, those converging to

$(x, \lambda) = (0, 1)$. After changing variables in order to place this point at $(0, 0)$, that is, setting $x = x$, $\lambda = y + 1$, we obtain:

$$\begin{aligned} \dot{x}(t) &= -\frac{8x^3 + 9xy}{18x + 27} \\ \dot{y}(t) &= \frac{-540x^4 + 64x^3y - 1556x^3 + 144x^2y - 1071x^2 + 27y^2}{18(2x+3)^2} \end{aligned} \tag{11}$$

which has multiplicity 2 at $(0, 0)$, i.e. the origin is a degenerate singularity, which —as explained in the Introduction— is unsuitable for the classical methods.

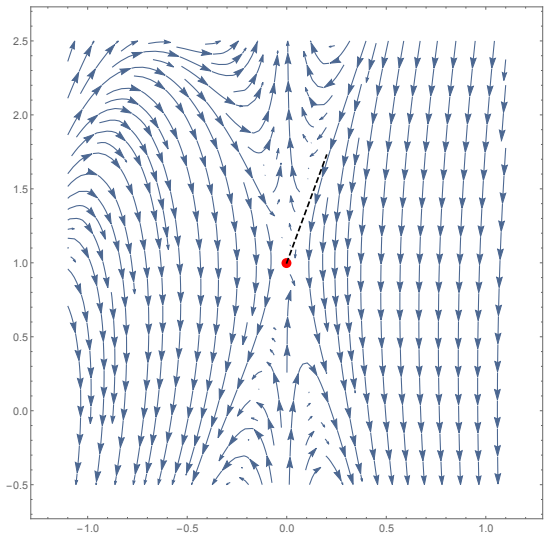

FIGURE 1 The degenerate singularity at $(x, \lambda) = (0, 1)$ (axes not to scale): notice the three "invariant" directions at $(0, 1)$. The dashed line marks the direction on which a trajectory converges to the singularity (as we shall show).

In the following sections we introduce a method which provides a way to compute the admissible trajectories converging to this singularity, by transforming the degenerate point into several new ones amenable to the application of the backward integration method[7]. This transformation is carried out by what is known as the *blow-up* technique. Then, we proceed to perform the computations compare them with the results obtained using two other methods (backward integration and shooting) in a straightforward way, as if the equilibrium were non-degenerate.

3 | PRELIMINARY RESULTS

3.1 | The Backward Integration Method

The idea of Backward integration[7] consists first in the transformation of a BVP ("reaching the steady-state") into an initial value problem (IVP) ("starting near the steady-state"), and then to use the attractive property of the unstable manifold of a hyperbolic saddle: trajectories near the saddle tend to approach it exponentially. This allows the use of standard numerical methods. In order

to use the latter property, Backward Integration reverses the time variable and, as a consequence, swaps the roles of the stable and unstable manifolds. Since the state (x_s, λ_s) of the system is completely known at time $t = \infty$, the problem can be solved recursively walking backwards in time to obtain the (approximate) values $(x(0), \lambda(0))$ at time $t = 0$ of the solution to the OCP, which will remain near the original stable manifold (for reasonable values of t, obviously): the computed solution converges towards the stable manifold of the forward system. One uses the eigenvectors at the steady-state in order to start sufficiently "near" the original stable manifold (one cannot start exactly at the steady state because, by definition, one would stay there indefinitely).

In[7] the canonical system is formulated using the control and the state as variables. Nevertheless, in[18], the canonical system is formulated with the state and the costate as variables. We follow this same approach using the canonical system associated to an OCP (with one state), which we write:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} f(x, \lambda) \\ g(x, \lambda) \end{pmatrix} = \Phi(x, \lambda) \tag{12}$$

Assume, for simplicity, that the system possesses a single steady state (x_s, λ_s), which is (this is the requirement of[7,18] which we are going to relax) a stable saddle point. From the hypotheses follows that the solution approaching the steady state corresponds to the stable manifold W_s of (12), parametrised by the time t. Thus, the initial condition $x(0) = x_0$ and the boundary condition

$$\lim_{t\to\infty}(x(t), \lambda(t)) = (x_s, \lambda_s) \tag{13}$$

are satisfied by that parametrization of the W_s. Consider the effect of a time reversal on the original system (12): that is, perform the change of variables $t = -t$:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = -\Phi(x, \lambda) \tag{14}$$

As the linear part of (12) is reversed in (14), the eigenvalues change sign and, as a consequence, the stable W_s and unstable W_u manifolds are swapped. It is well known (as[7] explains) that trajectories of (12) approach the unstable manifold exponentially. Hence, trajectories of (14) approach W_s exponentially. Let $v_s = (v_x, v_\lambda)$ be an eigenvector associated to the eigenvalue corresponding to W_s. The point $P_0 = (\overline{x}_0, \overline{\lambda}_0) = -\epsilon(v_x, v_\lambda)$ is sufficiently near both W_s and (x_s, λ_s) for $\epsilon \ll 1$. The solution $\overline{\gamma}(t)$ of (14) starting at P_0 approaches W_s exponentially. This solution can be efficiently computed using numerical algorithms, as it corresponds to a standard IVP. Let x_0 be any initial condition of (12): when $\overline{\gamma}(t) = (x_0, \lambda_0)$, the value λ_0 (which is unique, certainly, by the Cauchy-Kowalevski Theorem) is (approximately) the corresponding initial value for the costate variable for the optimal solution of (3). At this point, one only needs to compute the value of the integral in the original problem (3) for that trajectory (with the time properly reversed) in order to obtain the optimal solution.

3.2 | The blow-up technique

We are also going to apply the well-known (in the area of holomorphic and real-analytic vector fields) technique of blowing-up. We point the reader to the standard reference [12], the application of the technique to plane vector fields [13], and the quite accessible survey [14]. A deeper but also detailed exposition can be found in [15], where the authors review the main result of [16]. For a complete exposition of singularities of holomorphic differential equations (in which blow-up plays a significant role), see [17], mainly pages 112-120.

Consider the real-analytic vector field

$$X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} \tag{15}$$

which we shall always write in the differential-equation notation

$$X \equiv \begin{cases} \dot{x} = A(x,y) \\ \dot{y} = B(x,y) \end{cases} \tag{16}$$

and assume that $P = (0,0)$ is a singularity (equilibrium point): that is $A(0,0) = B(0,0) = 0$. Rewrite (16) using the homogeneous expansion of $A(x,y)$ and $B(x,y)$:

$$X \equiv \begin{cases} \dot{x} = A_m(x,y) + A_{m+1}(x,y) + \cdots \\ \dot{y} = B_m(x,y) + B_{m+1}(x,y) + \cdots \end{cases} \tag{17}$$

where at least one of $A_m(x,y)$, $B_m(x,y)$ is not zero. The number m is called the *multiplicity* of the vector field at P. A non-singular vector field at P satisfies $m = 0$. A non-degenerate singularity has $m = 1$ and, if $A_1(x,y) = a_{10}x + a_{01}y$ and $B_1(x,y) = b_{10}x + b_{01}y$, then:

$$|J_1| = \begin{vmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{vmatrix} \neq 0 \tag{18}$$

Where J_1 is called the *linear part* (or Jacobian matrix) of X. As a matter of fact, a hyperbolic saddle corresponds to a nondegenerate singularity such that the eigenvalues of the linear part are both real and of different sign. The stable manifold corresponds to the negative eigenvalue, and the unstable manifold, to the positive one.

We assume, from now on, that $m \geq 1$. If $m > 1$, then J_1 is the zero matrix and its only eigenvalue is, obviously, 0. In this case, instead of the eigenvectors, the directions in which some trajectory may approach P are given by the points in the *tangent cone of X at P* (see [13] and [14]):

Definition 1. The *tangent cone of X at P* is the set of projective points $[a : b]$ (i.e. directions in the plane $\mathbb{R}^2$) such that

$$bA_m(a,b) - aB_m(a,b) = 0 \tag{19}$$

If $T(x,y) := yA_m(x,y) - xB_m(x,y) \equiv 0$ then the singularity is called *dicritical*.

If the singularity is dicritical, then it is known that, for every tangent direction at P except possibly a finite number, there is an analytic trajectory of X adherent to P tangent to that direction (analogue to radial singularities). We do not deal with this type of points. Notice, on the other hand, that there needs not be a trajectory adherent to P tangent to every direction of the tangent cone (this topic is dealt with, in the complex plane, in [16]).

In order to take full advantage of the tangent cone, one performs the *blow-up* of the plane at P (for the last time, we refer the reader to [14], which is the easiest reference):

Definition 2. The *blow-up* of $\mathbb{R}^2$ at P is the analytic manifold M covered by the charts U_1, U_2, both diffeomorphic to $\mathbb{R}^2$, with respective coordinates (x_1, y_1) and (x_2, y_2), with the following glueing on $U_1 \cap U_2$:

$$x_1 = x_2 y_2, \quad y_1 = \frac{1}{x_2} \tag{20}$$

defined on $x_2 \neq 0, y_1 \neq 0$.

It is well known that there is a map $\pi : M \to \mathbb{R}^2$ (also called the blow-up) having the respective expressions:

$$\pi_{|U_1}(x_1, y_1) = (x_1, x_1 y_1), \quad \pi_{|U_2}(x_2, y_2) = (x_2 y_2, y_2). \tag{21}$$

(usually, and informally, U_1 is called the "first chart" of π and U_2, the "second chart"). Notice how the preimage of the line $y = \alpha x$ is $\pi^{-1}(x, \alpha x) \equiv (x_1, \alpha)$: this gives the main geometric idea of the blow-up: the lines passing through P become parallel when taking their preimages. As a consequence, the preimage or P, $E = \pi^{-1}(P)$ is a whole projective line (a closed circumference): one point for each of the directions through P. This set (the preimage of P by π) is called the *exceptional divisor* of π. This is why the map π is called the blow-up: the point P "blows-up" to become a circumference and, as a side effect, the lines through P part. Moreover, the vector field X can be "pulled-back" to M:

Theorem 1. Assume $m \geq 1$. There exists a vector field $\overline{X}$ on M such that $\pi_*(\overline{X}) = d\pi(\overline{X}) = X$. Moreover, in local coordinates in U_1, this vector field has the expression

$$\overline{X} \equiv \begin{cases} \dot{x}_1 = A(x_1, x_1 y_1) \\ \dot{y}_1 = \dfrac{1}{x_1}(B(x_1, x_1 y_1) - y_1 A(x_1, x_1 y_1)) \end{cases} \tag{22}$$

(notice that the division by x_1 makes sense because $m \geq 1$). Even more, there exists a vector field X_1 on U_1 and a vector field X_2 on U_2 (notice that they are not defined globally on M) such that

$$\overline{X} = x_1^{m-1} X_1, \quad \overline{X} = y_2^{m-1} X_2 \tag{23}$$

If X is non-dicritical, then both X_1 and X_2 have only a finite number of singular points in the exceptional divisor E.

We include a proof for the sake of completeness.

Proof. The manifold M is covered by the two charts U_1 and U_2 above. In order to have a vector field defined on M, we need to show that there are two vector fields X_1 and X_2, defined on U_1 and U_2 respectively, which coincide on $U_1 \cap U_2$.

Let $X_1 = \overline{X}$ be given by (23) on U_1; this vector field is well-defined on U_1 because $U_1 = \pi^{-1}(x \neq 0)$ and $x_1 = x$ on U_1. Let

$$X_2 = \begin{cases} \dot{x}_2 = \dfrac{1}{y_2}(A(x_2y_2, y_2) - x_2B(x_2y_2, y_2)) \\ \dot{y}_2 = B(x_2y_2, y_2) \end{cases} \tag{24}$$

which is defined on U_2 for the same reason: $U_2 = \pi^{-1}(y \neq 0)$ and $y_2 = y$ in U_2. We claim that $X_1 = X_2$ in $U_1 \cap U_2$, so that the vector field $\overline{X}$ of the statement exists. In order to prove this equality, recall that by (22), $x_1 = x_2y_2$ and $y_1 = 1/x_2$ in $U_1 \cap U_2$, so that:

$$X_1(x_2) = X_1(1/y_1) = -\frac{1}{y_1^2}X_1(y_1) = -\frac{1}{y_1^2}\frac{1}{x_1}\left(B(x_1, x_1y_1) - y_1A(x_1, x_1y_1)\right) = \star \tag{25}$$

which, taking again into account (22), is:

$$\star = \frac{x_2}{y_2}\left(-B(x_2y_2, y_2) + \frac{1}{x_2}A(x_2y_2, y_2)\right) = \frac{1}{y_2} = \frac{1}{y_2}\left(A(x_2y_2, y_2) - x_2B(x_2, y_2)\right) = X_2(x_2). \tag{26}$$

On the other hand,

$$X_1(y_2) = X_1(x_1y_1) = x_1\frac{1}{x_1}\left(B(x_1, x_1y_1) - y_1A(x_1, x_1y_1)\right) + A(x_1, x_1y_1)y_1 = B(x_1, x_1y_1) = B(x_2y_2, y_2) = X_2(y_2) \tag{27}$$

showing that X_1 and X_2 coincide in $U_1 \cap U_2$, so that they glue together on M to give the global vector field $\overline{X}$. □

The vector fields X_1 and X_2, each of which is only defined, respectively, in U_1 and U_2, are called, the *strict transforms* of X in each chart. Notice, from (22) and (23), that E is always invariant for both X_1 and X_2 if X is non-dicritical. The importance of the tangent cone comes from the following property:

Theorem 2 (See[17]). Assume X has a singular point at $P = (0, 0)$ which is not dicritical. Let $[a : b]$ denote a direction through $P = (0, 0)$. Then: if $a \neq 0$, the point $x_1 = 0, y_1 = b/a$ is a singular point of X_1. If $a = 0$, then $x_2 = 0, y_2 = 0$ is a singular point for X_2. The converse is also true.

Hence, if $P = (0, 0)$ is not a dicritical singularity of X, singular points of X_1 or X_2 correspond to "singular directions" of X. As a matter of fact (ibid.):

Theorem 3 (ibid.). Let $\gamma : (0, \infty) \to \mathbb{R}^2$ be a trajectory of X with $\gamma(t) \to (0, 0)$ as $t \to \infty$ and $\gamma(t) \neq (0, 0)$. Let $\overline{\gamma}$ be the lift of γ to M (which is a trajectory of $\overline{X}$). Assume X is not dicritical at $P = (0, 0)$. If $\overline{\gamma}$ has a single accumulation point Q in $E = \pi^{-1}(P)$, then Q belongs to the tangent cone.

This means that the tangent cone describes —essentially— the directions on which trajectories approach the singular point P.

In this first approach, we are going to restrict ourselves to degenerate singularities which only give rise to new saddle points.

Definition 3. A *multiple hyperbolic saddle* is a degenerate singularity P of a vector field X such that each of the singular points of X_1 and X_2 in Theorem 1 is a hyperbolic saddle.

An easy result whose proof follows from the discussion in[15] is:

Lemma 1. If P is a multiple hyperbolic saddle of X of multiplicity m, then there are exactly $m+1$ singularities in E, for X_1 and X_2 (counting both). Each of these corresponds to a direction in the tangent cone.

A consequence of the Cauchy-Kowalevski Theorem is the following:

Lemma 2. Let $\gamma : (0,\infty)$ be a trajectory of X with $\gamma \to P = (0,0)$ as $t \to \infty$, and assume P is a multiple hyperbolic saddle. Let $\overline{\gamma}$ be the lift of γ to M (which is a trajectory of $\overline{X}$). Then there is $Q \in E = \pi^{-1}(0,0)$ such that $\overline{\gamma} \to Q$, and Q corresponds to one of the hyperbolic saddles of either X_1 or X_2.

Hence, if P is a multiple hyperbolic saddle of X, the directions in which there is a trajectory adherent to P correspond to the points of the tangent cone. Thus, *the tangent cone plays the role of the eigenvalues* for a multiple hyperbolic saddle. In fact, if one performs the blow-up of a non-degenerate hyperbolic saddle, the tangent cone is, exactly, the set of invariant directions of the linear part. However, depending on the sign of the coordinates and the multiplicity of X at P, the trajectory may or may not approach P when the corresponding eigenvalue is positive or negative. We study this in the next section.

4 | MODIFIED BACKWARD INTEGRATION FOR MULTIPLE HYPERBOLIC SADDLES

From the previous section, one deduces that, if $P = (0,0)$ is a multiple hyperbolic saddle of multiplicity m, one can blow-up P in order to compute the tangent cone (i.e. the singular points of the strict transform of X), which will contain exactly $m+1$ directions: $Q_1, \dots, Q_{m+1}$. On each one of these, say Q, one must verify the existence of approaching trajectories, and perform the backward integration method in order to compute (an approximation) to the trajectory accumulating at P in the direction given by Q. This will provide the set of trajectories of X accumulating at P. As the strict transform requires division by a power of the equation of the exceptional divisor, one has to take it into account. Specifically:

Lemma 3. Let m be the multiplicity of X at a multiple hyperbolic saddle P and let $\pi : M \to \mathbb{R}^2$ be the blow-up at P. Let $Q \in E = \pi^{-1}(P)$ be a singular point of X_1 and λ, μ be the eigenvalues of X_1 at Q, such that μ corresponds to E (which is invariant for X_1). Then:

1. If $\lambda < 0$, then there is a trajectory $\gamma \equiv (x(t), y(t))$ of X approaching P in the direction Q with $x(t) > 0$.
2. If $m-1$ is even and $\lambda < 0$, then there is a trajectory $\gamma \equiv (x(t), y(t))$ of X approaching P in the direction Q with $x(t) < 0$.
3. If $m-1$ is odd and $\lambda > 0$, then there is a trajectory $\gamma \equiv (x(t), y(t))$ of X approaching P in the direction Q with $x(t) < 0$.

Otherwise, there is no trajectory of γ approaching P in the direction Q. (An analogous statement holds for X_2 and $y(t) <> 0$).

The proof is a direct application of the definition of hyperbolic saddle and strict transform of a vector field.

Thus, the only possible trajectories of X approaching P correspond to the saddles of X_1 or X_2 in E whose eigenvalues satisfy the properties in the lemma.

Example. By way of illustration, consider the vector field

$$X \equiv \begin{cases} \dot{x} = 3y^2 - x^2 \\ \dot{y} = 2xy \end{cases} \tag{28}$$

which has a degenerate singularity at $P = (0, 0)$. It has multiplicity $m = 2$. The tangent cone is the set of directions $[a : b]$ such that

$$3b^3 - a^2b - 2a^2b = 0 \tag{29}$$

that is:

$$3b(b^2 - a^2) = 0 \Leftrightarrow b = 0, b = a, b = -a \tag{30}$$

Thus, the tangent directions correspond to the lines $y = 0$, $x = y$, $x = -y$: the points $[1 : 0], [1 : 1]$ and $[1 : -1]$ in E. All of them belong to the chart U_1, so that we can just look at the strict transform X_1. The pull-back of $\overline{X}$ of X in M is (in coordinates (x_1, y_1) in U_1):

$$\overline{X} \equiv \begin{cases} \dot{x}_1 = x_1(-x_1 + 3y_1^2x_1) \\ \dot{y}_1 = x_1(3y_1 - 3y_1^3) \end{cases} \tag{31}$$

The strict transform X_1 is

$$X_1 \equiv \begin{cases} \dot{x}_1 = -x_1 + 3y_1^2x_1 \\ \dot{y}_1 = 3y_1 - 3y_1^3 \end{cases} \tag{32}$$

and the singular points with $x_1 = 0$ are given, as they should, by $y_1 = 0$, $y_1 = 1$, $y_1 = -1$. Notice how $E \equiv (x_1 = 0)$ is invariant for X_1, as X is non-dicritical at $(0, 0)$. At $Q = [1 : 0]$ (which corresponds to $x_1 = y_1 = 0$), the eigenvalues are

$$\lambda = -1, \mu = 3 \tag{33}$$

where μ corresponds to the direction of the axis $x_1 = 0$ (i.e. the exceptional divisor). The eigenvalue $\lambda = -1$ provides an approaching trajectory in the direction $y = 0$ (the OX axis) for $x > 0$ but, as $m - 1$ is odd, there is no *approaching* trajectory for $x < 0$ in that direction.

At $Q = [1 : 1]$, the vector field X_1 has equations

$$X_1 \equiv \begin{cases} \dot{x}_1 = 2x_1 + 6x_1y_1 + 3x_1y_1^2 \\ \dot{y}_1 = -6y_1 - 9x_1y_1 - 3y_1^3 \end{cases} \tag{34}$$

which gives $\lambda = 2$, $\mu = -6$. This implies that there is an approaching trajectory in the direction $x = -y$ for $x < 0$ and a diverging one for $x > 0$.

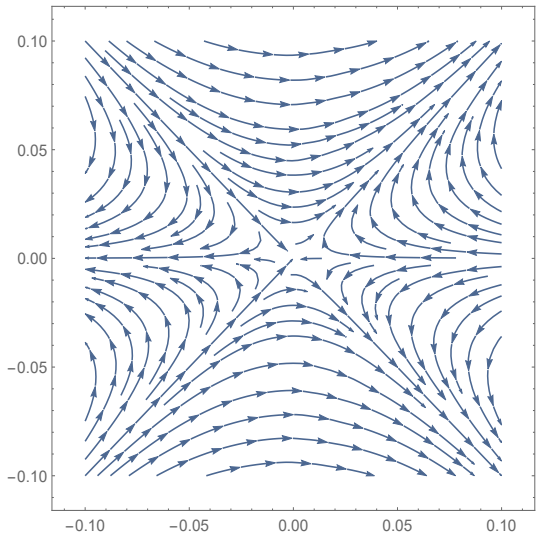

FIGURE 2 The flow corresponding to the vector field (28). Notice the approaching trajectories in the directions of the angles $0, 3\pi/4$ and $5\pi/4$.

At $Q = [1 : -1]$, the vector field X_1 has equations

$$X_1 \equiv \begin{cases} \dot{x}_1 = 2x_1 - 6x_1y_1 + 3x_1y_1^2 \\ \dot{y}_1 = -6y_1 + 9x_1y_1 - 3y_1^3 \end{cases} \tag{35}$$

and the situation is the same as for $[1 : 1]$. Figure 2 shows the local dynamics of X at $P = (0, 0)$, where the approaching curves $(T - t, T - t)$, $(T - t, t - T)$ and $(0, T - t)$ for $t \in (0, T)$ can be clearly seen.

4.1 | Stability of multiple hyperbolic saddles

The local stability of a multiple hyperbolic saddle cannot be studied by means of the linear part of the vector field, as it is always the zero matrix. However, precisely because they have a specific structure, we can provide a description of the local behavior of trajectories near the such an equilibrium point.

Proposition 1. Let P be a multiple hyperbolic saddle for the plane analytic vector field X, of multiplicity m. Then: there exist trajectories $\gamma_1(t), \eta_1(t) \ldots, \gamma_{m+1}(t), \eta_{m+1}(t)$ of X and a neighborhood U of P such that:

1. For $i = 1, \ldots, m + 1$, the trajectory $\gamma_i(t)$ approaches P and $\eta_i(t)$ moves away from P, that is: $P = \lim_{t\to\infty} \gamma_i(t)$, and $P = \lim_{t\to 0} \eta_i(t)$.

2. If $P_0 \in U$ is different from P, then either $P_0 = \gamma_i(t)$ for some i, t or the trajectory of X passing trough P_0 leaves U eventually.

Proof. After a linear change of coordinates, we may assume that the tangent cone of X at P does not include the point $[0 : 1]$, corresponding to the axis $x = 0$. If π is the blow-up of P, we know that, as P is a multiple hyperbolic saddle:

- The exceptional divisor is invariant for the strict transform X_1 of X in U_1.
- All the singularities of the strict transform X_1 on the exceptional divisor are hyperbolic saddles, and there are exactly $m + 1$ of them.

This implies that the structure of the trajectories of X_1, the strict transform of X in the first chart is, schematically, as in Figure 3 (a finite sequence of $m + 1$ alternating saddles all of them included in an algebraic set: the exceptional divisor).

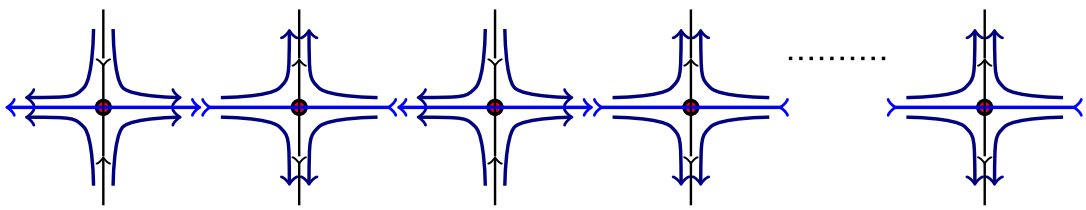

FIGURE 3 Schematic structure of the dynamics of the strict transform X_1 of a multiple hyperbolic saddle. In blue, the exceptional divisor. We assume, for simplicity, that all the singularities are "visible" in U_1.

Denote by E the exceptional divisor, which is a compact 1-dimensional manifold on which the $m+1$ hyperbolic singularities, say $P_1, \dots P_{m+1}$, are. Let $\overline{X}$ denote the pull-back of X. By compacity of E, and because all the singularities of X_1 are hyperbolic, there is an open set $\overline{U} \supset E$ such if $Q \in \overline{U}$ and $Q \notin E$, then: either Q belongs to the unique trajectory of $\overline{X}$ converging to some P_i or the trajectory of X starting at Q eventually leaves $\overline{U}$. Taking $U = \pi(\overline{U})$, γ_i the projection of the trajectories of $\overline{X}$ converging to E and η_i the projection of the trajectories of $-\overline{X}$ converging to E, we are done. □

Notice that in Figure 3, we do not claim that X_1 is a vector field *on the whole of a neighborhood of* E: the strict transform X_1 is defined only on the open set U_1. However, the hyperbolicity of its singularities holds by definition.

As a consequence, we have proved that a multiple hyperbolic saddle at P is always unstable and there are just a finite number of trajectories converging to P (exactly $m + 1$ of them).

5 | SOLUTION TO THE INVASIVE SPECIES PROBLEM

Recall that we had to compute the trajectories (if there are any) of the vector field X given by (11):

$$X = \begin{cases} \dot{x}(t) = -\dfrac{8x^3 + 9x\lambda}{18x + 27} \\ \dot{\lambda}(t) = \dfrac{-540x^4 + 64x^3\lambda - 1556x^3 + 144x^2\lambda - 1071x^2 + 27\lambda^2}{18(2x+3)^2} \end{cases} \tag{36}$$

approaching $(0,0)$ (we had already performed the change of variables sending $(0,1)$ to $(0,0)$). We do not need to worry about the denominators being 0 (i.e. $2x + 3 = 0$, that is $x = -3/2$), as the trajectories starting with $x(0) > 0$ (the only ones we study because $x(t)$ denotes a population) remain positive (that is, $x(t) > 0$) because the line $x = 0$ is invariant. From now on, we use y instead of λ in order to follow the standard convention when dealing with vector fields. As the origin is a degenerate singularity of X, we first compute the tangent cone (19). Taking into account that the multiplicity of X is 2 at $(0,0)$, the tangent cone C is the set of projective zeroes of the homogeneous polynomial $T(x,y)$:

$$T(x,y) = yA_2(x,y) - xB_2(x,y) = -\frac{xy^2}{3} - x\left(\frac{y^2}{6} - \frac{119}{18}x^2\right) \tag{37}$$

that is:

$$C = \left\{ \left[1 : -\frac{\sqrt{119}}{3}\right], \left[1 : \frac{\sqrt{119}}{3}\right], [0 : 1] \right\} \tag{38}$$

which means that the point "at infinity" $[0 : 1]$ is also a singular point of the blow-up $\overline{X}$ of X. Hence, we need to study three points in the exceptional divisor E of the blow-up M of $\mathbb{R}^2$ at $(0,0)$: two in the first chart U_1 and one in the second chart U_2. We shall set $Q_1 = [1 : -\sqrt{119}/3]$, $Q_2 = [1 : \sqrt{119}/3]$ (these two are in U_1) and $Q_3 = [0 : 1]$, which belongs to U_2 .

5.1 | The invariant curve $x = 0$

We first study $Q_3 = [0 : 1]$. The strict transform of X in the second chart U_2 is a vector field which, in local coordinates (x_2, y_2) centered at Q_3, has the form:

$$X_2(x_2, y_2) = \begin{cases} \dot{x}_2 = x_2\left(-\dfrac{1}{2} + \cdots\right) \\ \dot{y}_2 = y_2\left(\dfrac{1}{6} + \cdots\right) \end{cases} \tag{39}$$

where the ellipses indicate terms of order at least 1. Thus, Q_3 is a hyperbolic saddle whose stable manifold is $x_2 = 0$ (notice that both the exceptional divisor $y_2 = 0$ and the curve $x_2 = 0$ are invariant). As the multiplicity of X at $(0,0)$ is 2, Lemma 3 tells us that that the trajectories of X included in $x = x_2 = 0$ approach $(0,0)$ both when $y < 0$ and when $y > 0$. However, the set $x = 0$ describes the equilibrium with no population, which is irrelevant.

5.2 | Trajectories in the direction Q_1 diverge

The strict transform of X in the first chart is:

$$X_1 = \begin{cases} \dot{x}_1 = -\dfrac{x_1(8x_1 + 9y_1)}{9(2x_1 + 3)} \\ \dot{y}_1 = -\dfrac{12x_1^2(8y_1 - 45) + 4x_1\left(9y_1^2 + 48y_1 - 389\right) + 9\left(9y_1^2 - 119\right)}{18(2x_1 + 3)^2} \end{cases} \tag{40}$$

Consider $Q_1 = [1 : -\sqrt{119}/3]$. The linear part of X_1 at $x_1 = 0$, $y_1 = -\sqrt{119}/3$ (the point Q_1) is:

$$X_1(x_1, y_1) = \begin{cases} \dot{x}_1 = \dfrac{\sqrt{119}}{9} x_1 \\ \dot{y}_1 = -\dfrac{540 + 32\sqrt{119}}{81} x_1 - \dfrac{\sqrt{119}}{3} y_1 \end{cases} \tag{41}$$

which is a hyperbolic saddle. The eigenvalue corresponding to the exceptional divisor is $-\sqrt{119}/3$ and the eigenvalue corresponding to the other invariant manifold is $\sqrt{119}/3$. As a consequence, the only trajectory of $\overline{X}$ included in $x_1 > 0$ approaching Q_1 is the exceptional divisor (we do not need to consider the case $x_1 < 0$ because these are not physical solutions, as $x_1 = x$; Lemma 3 is not useful for this specific problem in this case).

5.3 | Convergent trajectory in the direction Q_2

The linear part of X_1 (given by (40)) at $Q_2 = [1 : \sqrt{119}/3]$ is, in coordinates centered at Q_2:

$$X_1(x_1, y_1) = \begin{cases} \dot{x}_1 = -\dfrac{\sqrt{119}}{9} x_1 \\ \dot{y}_1 = -\dfrac{540 - 32\sqrt{119}}{81} x_1 + \dfrac{\sqrt{119}}{3} y_1 \end{cases} \tag{42}$$

which is again a hyperbolic saddle. In this case, the sign of the eigenvalues is transposed from that of Q_1, so that there is a unique trajectory converging to Q_2 in M, included in $x = x_2 > 0$. This is the one we need to study.

5.4 | Blow-up and backward integration

As Q_2 has homogeneous coordinates $[1 : \sqrt{119}/3]$, we might just try to use backward integration straightaway using, as tentative direction, the line $y = \frac{\sqrt{119}}{3}x$. However, our method provides a better initial condition, as blowing-up gives an extra order of approximation.

Consider the linear part of $\overline{X}$, given by (42) at the point Q_2 in M. The unstable manifold of that linear singularity is $y_2 \simeq 0.486x_2$. Therefore, a convenient initial point for the differential equation corresponding to $-\overline{X}$ at Q_2 might be $\overline{P}_0 = (x_1, y_1) = (10^{-4}, 3.63629)$. Notice that this provides, better accuracy than taking $P_0 = (x, y) = (10^{-4}, 0.00036)$, which is what one would

choose in order to use the backward integration method with the direction given by Q_3 without blowing-up (disregarding the fact that the backward integration method is not straightforwardly applicable at $(0,0)$ as the singularity is degenerate).

As $x > 0$, we do not need to consider the sign conditions in Lemma 3 and we just integrate $-\overline{X}$ (i.e. X "backwards") from $\overline{P}_0$ until we get to the desired value $x_1 = 0.25$ along a trajectory $(x_1(u), y_1(u))$. Using Mathematica™, we obtain $x_2 = 0.25$, $y_2 \simeq 3.74913$ (the value of u giving these is useless, as $\overline{X}$ is a modification of X). Therefore, the initial condition for the optimal trajectory $(x^*(t), \lambda^*(t))$ of the canonical system converging to $(0,1)$ is $(x(0), \lambda(0)) = (0.25, 1.937281)$.

We now integrate X forward from this initial condition, in order to compute the trajectory *depending on time* t. As explained above, the points $(x(u), \lambda(u)) = (x_2(u), x_2(u)y_2(u))$ computed from the trajectory of $\overline{X}$ above belong to the set $\{(x^*(t), \lambda^*(t)) : t \in (0, \infty)\}$ but the variable u has no direct relation to the true time t, hence the necessity of this forward integration. This last step provides the optimal solution $(x^*(t), \lambda^*(t))$ which, using (6) gives the solution $(x^*(t), u^*(t))$ to (3) with $x(0) = 0.25$, for the specific values of the parameters. Figure 4 contains the plots of $x^*(t)$ and $u^*(t)$ as a function of time for the solution to the optimal control problem converging to $(x, \lambda) = (0,1)$ with $x(0) = 0.25$.

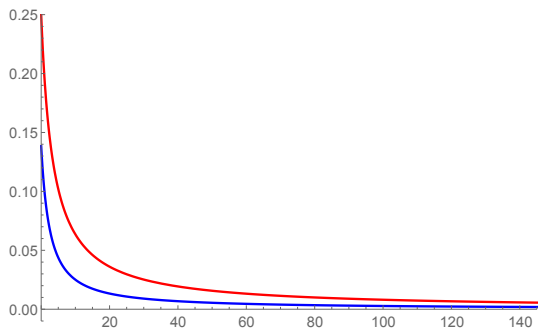

FIGURE 4 Population $x^*(t)$ (red) and harvest $u^*(t)$ (blue) as a function of time for the solution converging to $(x, \lambda) = (0,1)$.

The cost of that trajectory in the control problem is $C = -0.152741$ (computed using a discretization of $t = .01$), so that it is actually profitable. We have seen that this is the only trajectory of the canonical system converging to the steady state and that the hamiltonian only converges to 0 on these. Hence, this value is the cost of the optimal solution to our OCP.

5.5 | Comparison with other methods

We have considered two possible alternative methods: a naive use of the backward integration method[7] and a version of the shooting method which we have successfully used in several other contexts[23,24,25].

We point out in passing that the degenerate singularity we are studying is of the "simplest" type, as there are only three invariant curves passing through it. The drawbacks of these methods will only increase in more complicated cases.

5.5.1 | Backward integration

One might try and use the Backward Integration Method[7] naively, but there is a great problem: what initial condition *near the singularity* does one take in order to —going backwards— approach a trajectory converging to it? The jacobian matrix (i.e. the linear part of the vector field) at the singularity provides no information, as it is the zero matrix, so that it provides no information.

For this specific problem, as long as one chooses an initial value (for backward integration) $\lambda(0) > -\frac{\sqrt{119}}{9}x(0)$, the backward integration method will give a solution truly approaching the trajectory converging to the equilibrium. However, if that condition does not hold, the backward integration method will give a solution approaching the invariant set $x = 0$, which is useless. For example, if one chooses $x(0) = 0.001$ and $y(0) = -0.01$, the backward trajectory is very soon asymptotic to $x = 0$.

One might use the tangent cone for choosing the initial value for the backwards integration method but one also needs —in order for this to provide useful information— to study which points in the tangent cone are points of convergence of trajectories of the blown-up vector field.

5.5.2 | Modified shooting method

Using a version of the shooting method adapted to Optimal Control Problems, we can easily solve the generic case (i.e. when the parameters give rise to an ordinary saddle), with a short convergence time and very good approximation, as in our previous works; v.gr., for $t = 50$, we obtain $x(t) \simeq 0.2209$, the equilibrium being $x \simeq 0.2165$, an error of less than 5 per thousand.

However, when trying to use it for the degenerate case, we get, for $t = 50$, the value 0.06, whereas the true value at $t = 50$ is 0.015 (notice the large relative error). The reason (as we see it) is related to the great curvature of the trajectories near the singular point, as Figure 1 shows. This modified shooting method is not only unable to quickly find the steady state but any increase in the desired approximation requires a delicate and troublesome choice of discretization and bounding parameters, which makes it unfeasible for the degenerate case.

6 | COMMENTS ON DISCOUNTED FUNCTIONALS

In economic models, one usually encounters the special case of a discounted control problem with infinite horizon:

$$\min_{\mathbf{u}(t)} J = \int_0^\infty G(x(t), \mathbf{u}(t))e^{-rt}dt \tag{43}$$

$$\dot{x}(t) = f\left(x(t), \mathbf{u}(t)\right), \; 0 \le t < \infty; \;\; x(0) = x_0$$

where $x(t) : \mathbb{R} \to \mathbb{R}$, $\mathbf{u}(t) : \mathbb{R} \to \mathbb{R}^n$, $0 \le t < \infty$, $r \ge 0$ is the positive rate of discount, and G is a function bounded from above.

Equation (4) defines H, which is called, in the presence of a discount factor, the *present-value Hamiltonian*. Nevertheless, it is usually desirable to define a new Hamiltonian that is free of the discount factor: the *current-value Hamiltonian*:

$$H_c(x, \mathbf{u}, \lambda) = H(x, \mathbf{u}, \lambda) \cdot e^{rt} = G(x, \mathbf{u}) + m \cdot f(x, \mathbf{u}) \tag{44}$$

where a new (current-value) Lagrange multiplier m is defined:

$$m = \lambda \cdot e^{rt}$$

When working with H_c instead of H, all the conditions of the PMP must be reexamined (see[30] and[29]). The next conditions are equivalent to the necessary conditions imposed by the PMP:

$$\dot{x} = \frac{\partial H_c}{\partial m} = f; \quad x(0) = x_0 \tag{45}$$

$$\dot{m} = rm - \frac{\partial H_c}{\partial x}; \quad \lim_{t\to\infty} H_c(t)e^{-rt} = 0 \tag{46}$$

$$H_c(x, \mathbf{u}^*, m) = \min_{\mathbf{u}(t)} H_c(x, \mathbf{u}, m) \tag{47}$$

And the current-value Hamiltonian of an autonomous problem has an additional property: its value along the optimal paths must be constant over time, i.e: $H_c^* = cst$.

Let $\mathbf{u}^*(t)$ be an admissible control of the OCP satisfying the Hamiltonian minimizing condition (47). Then the ODE:

$$\dot{x}(t) = \frac{\partial H_c}{\partial m}(x(t), \mathbf{u}^*(t), m(t)) = f(x(t), \mathbf{u}^*(t)) \tag{48}$$

$$\dot{m}(t) = rm(t) - \frac{\partial H_c}{\partial x}(x(t), \mathbf{u}^*(t), m(t)) \tag{49}$$

is also called the *canonical system* of the OCP. The current-value notation has the advantage that this canonical system becomes an autonomous ODE. The corresponding Jacobian is:

$$J(x_s, m_s) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 H_c^*}{\partial m \partial x} & \dfrac{\partial^2 H_c^*}{\partial m^2} \\ -\dfrac{\partial^2 H_c^*}{\partial x^2} & -r - \dfrac{\partial^2 H_c^*}{\partial x \partial m} \end{pmatrix} \tag{50}$$

where $H_c^* = H_c(x_s, \mathbf{u}^*(x_s, m_s), m_s)$ is the maximized Hamiltonian.

The elements in the diagonal of the Jacobian (50) differ by a constant (the discount factor): this implies that, as long as u^* is an analytic function of x and m, the Jacobian cannot be the zero matrix. As a consequence (see, for instance,[18]), the only possible singularities with monotone approaching trajectories are hyperbolic saddles or saddle-nodes (this have a zero eigenvalue and a nonzero one). The first case is well-know, whereas the second may give rise to an infinite number of trajectories approaching the steady state. If in our OCP (3) we add a discount factor to the functional, say e^{-rt} and compute the current-value canonical system for $\rho = 2/3$ (which is certainly non-real but this is just by way of example), we obtain a vector field whose jacobian at

$(0, -1)$ is:

$$J(0,1) = \begin{pmatrix} 2/3 & 0 \\ -8/9 & 0 \end{pmatrix} \tag{51}$$

which means that the singularity at $(0, 0)$ is degenerate of saddle-node type. Unfortunately, these singularities have the undesirable property of having infinite number of trajectories of the vector field accumulating (as sets) at the singularity. Saddle-nodes are also quite unstable numerically, as they tend to have non-convergent power series solutions, despite having multiplicity 1. We have not carried out the complete study of our example in this case as we reckon it requires a deeper study, for which the tools in this paper are unsuitable.

7 | CONCLUSIONS AND FURTHER RESEARCH

Optimal control theory has become a key discipline in many branches of economics, social sciences and biology. Due to the complexity of the recent applications, OCPs are most often solved numerically. However, qualitative study of steady states is relevant because the nature of the problem may change drastically near one. For degenerate steady-states (whose Jacobian determinant is 0), there is no available technique in the literature.

We show how to modify the Backward Integration method, using the blow-up technique, in order to deal with solutions approaching a specific class of degenerate steady states (multiple hyperbolic saddles). The required modification turns out to be simple and effective, due to the nature of hyperbolic saddles.

Using this technique we can completely analyze multiple hyperbolic saddles with any number of admissible trajectories approaching the steady state, whereas the classical approach can only deal with a single trajectory.

Two examples are presented illustrating the techniques and their feasibility.

The case of discount factor is addressed and shown to require a deeper study and more developed strategies.

The blow-up technique and the qualitative study of degenerate singularities of vector fields may provide greater insight into the nature of steady-states in Optimal Control Problems. We state, as future research paths: the study of equilibria of saddle-node type; singularities in which trajectories become tangent (which may require iterating the blow-up process); bounding the number of possible trajectories approaching an equilibrium…

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