Definitions of Entwinement

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with Marine De Clerck and Alejandro Vilar López

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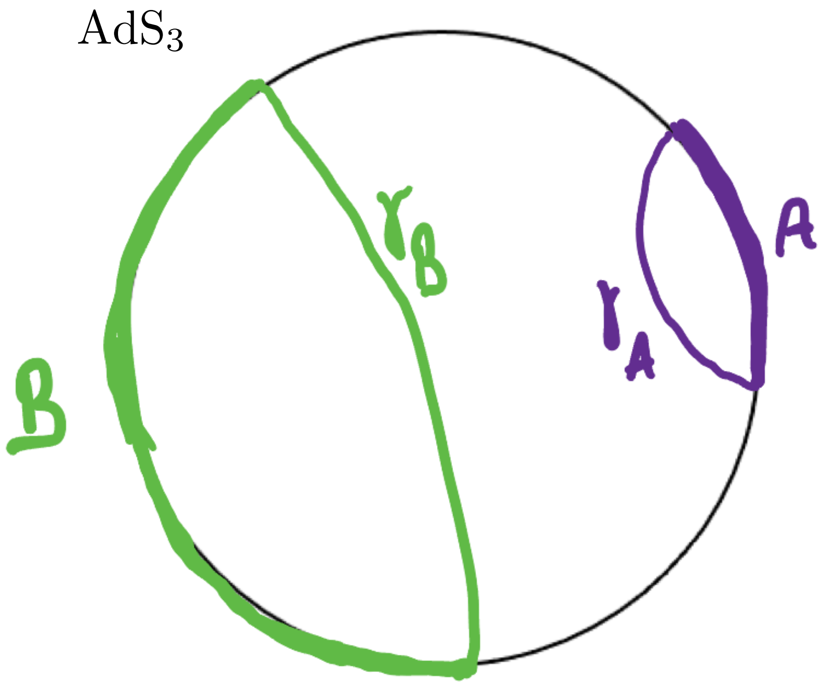
Holographic entanglement entropy in vacuum

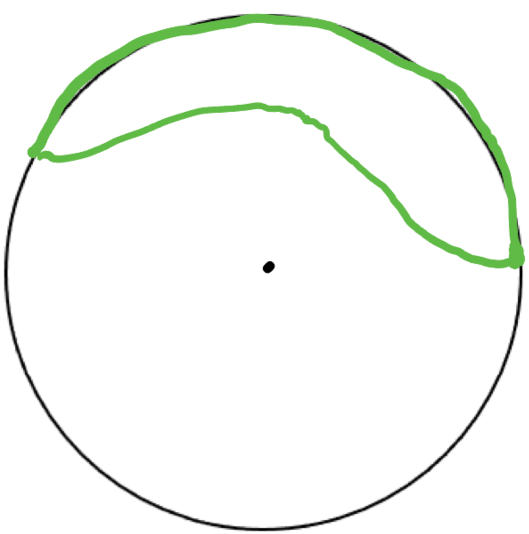


$$S(A) = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

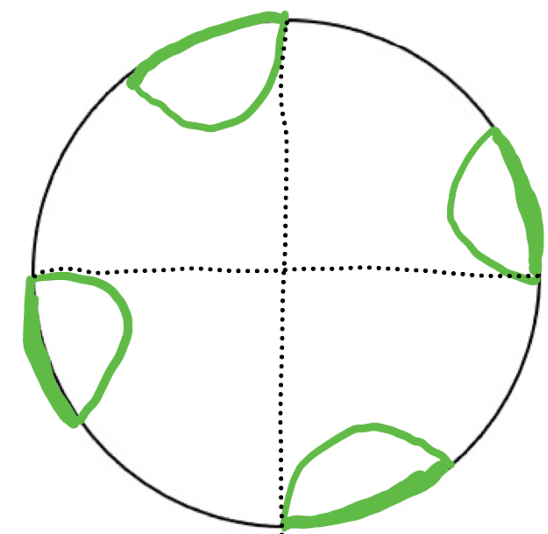
[Ryu, Takayanagi 2006]

Conical defects have entanglement shadows

$\mathrm{AdS}_3/\mathbb{Z}_4$



AdS_3



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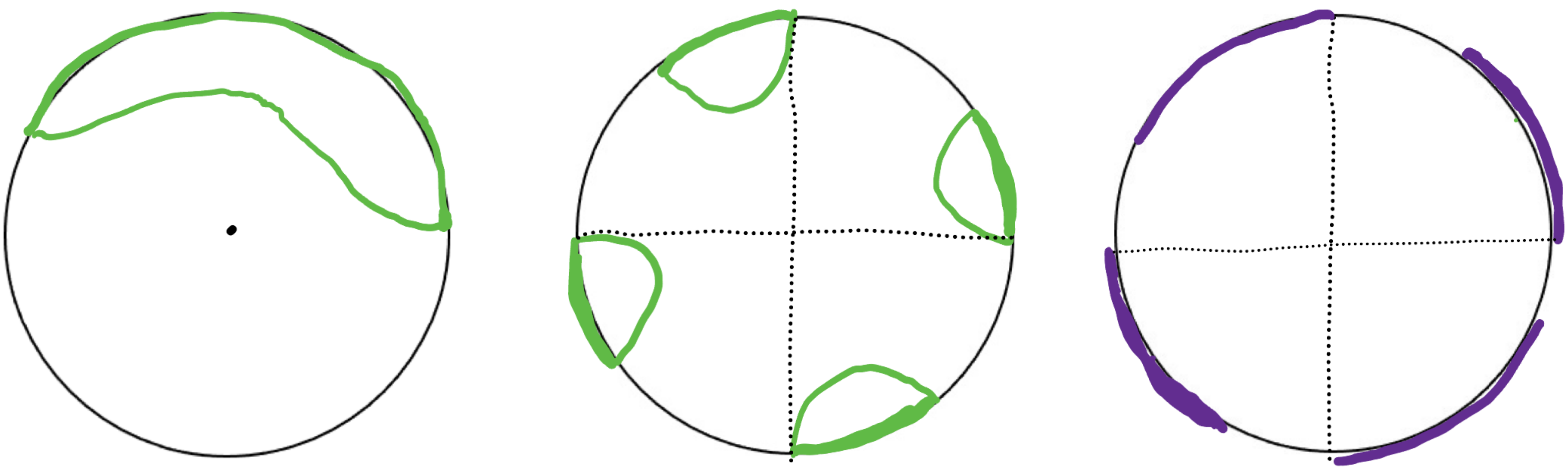
$\mathrm{AdS}_3/\mathbb{Z}_4$ AdS_3



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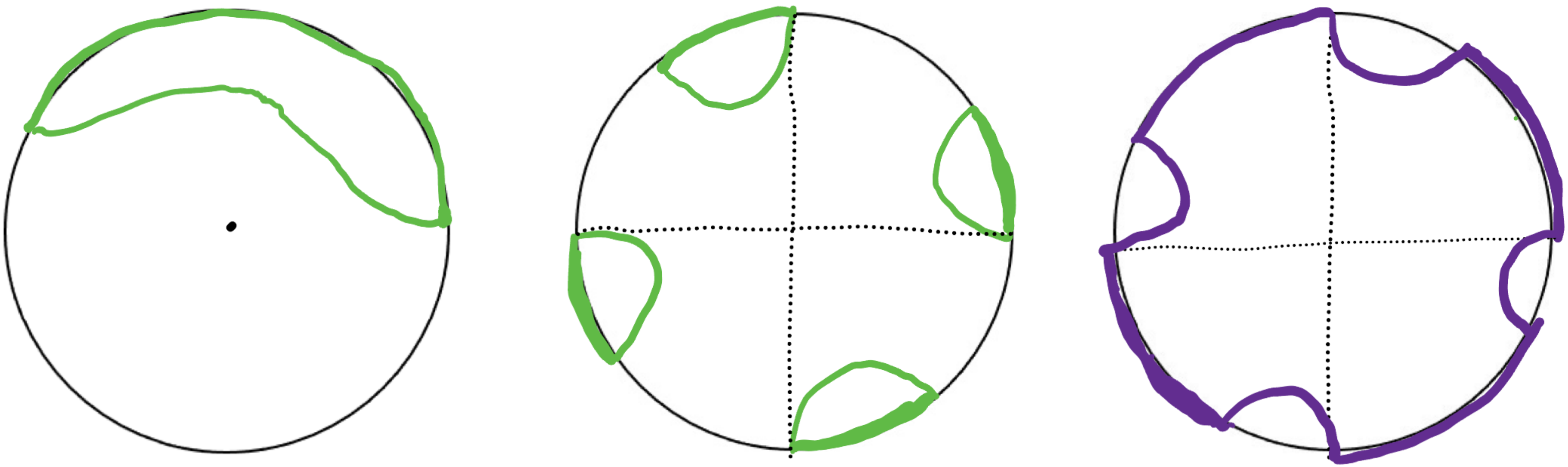

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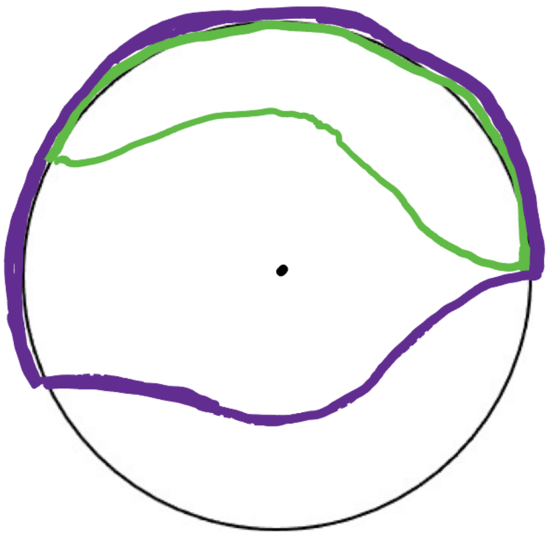

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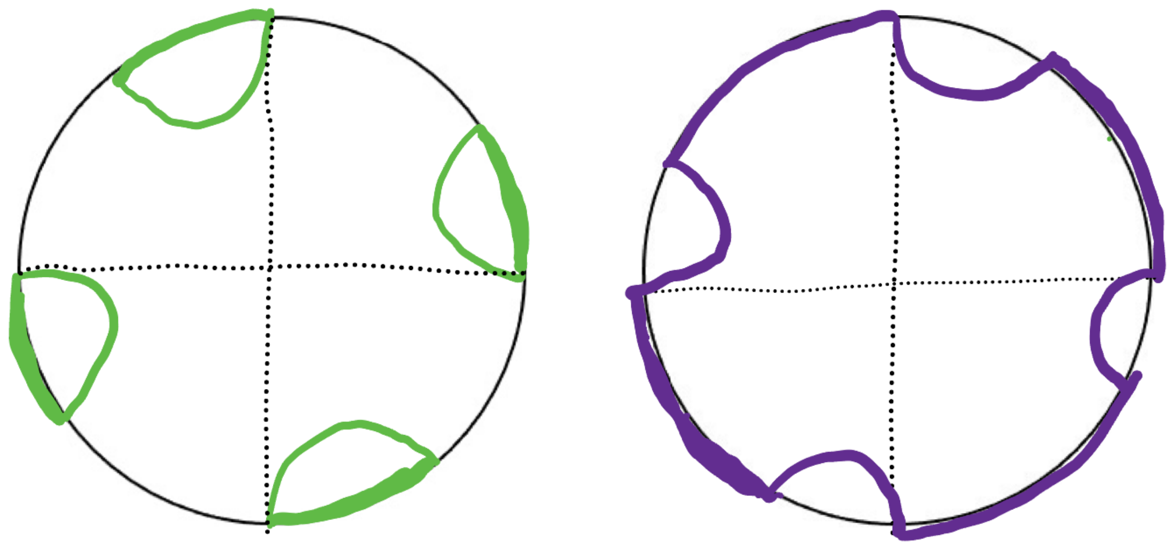

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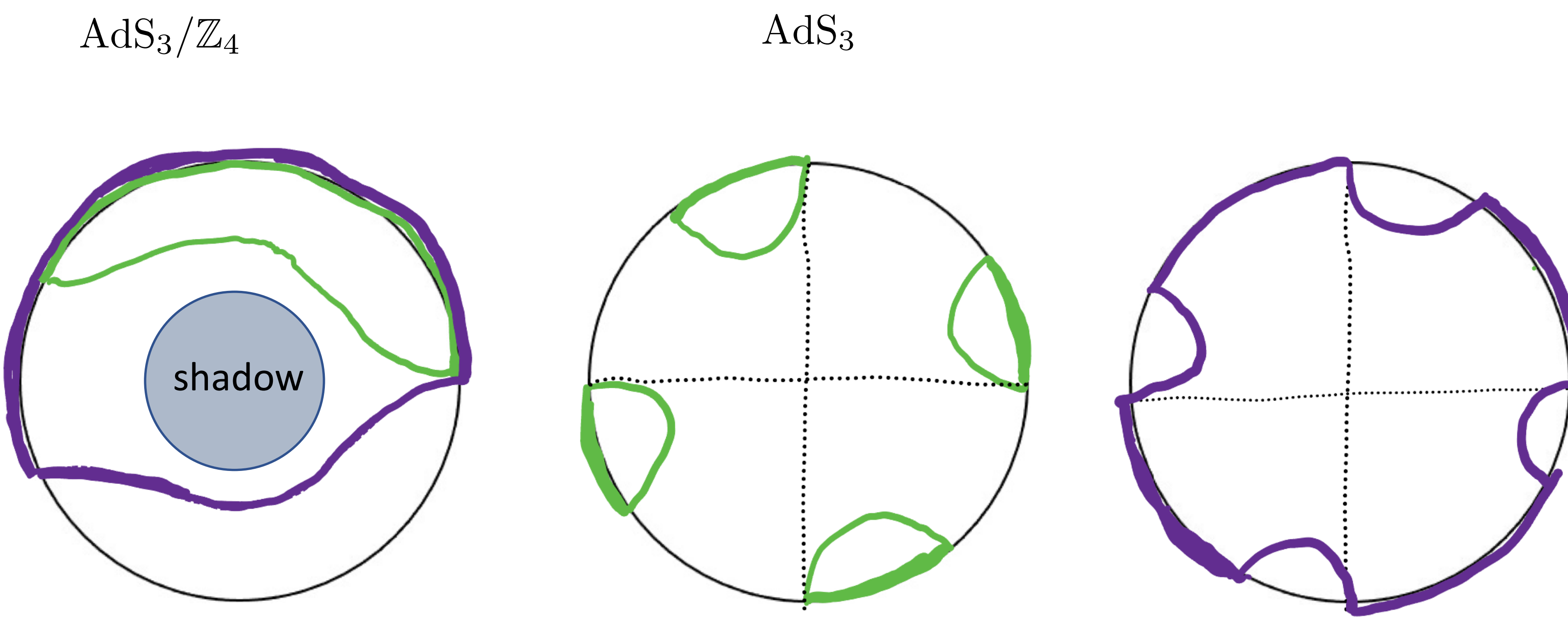

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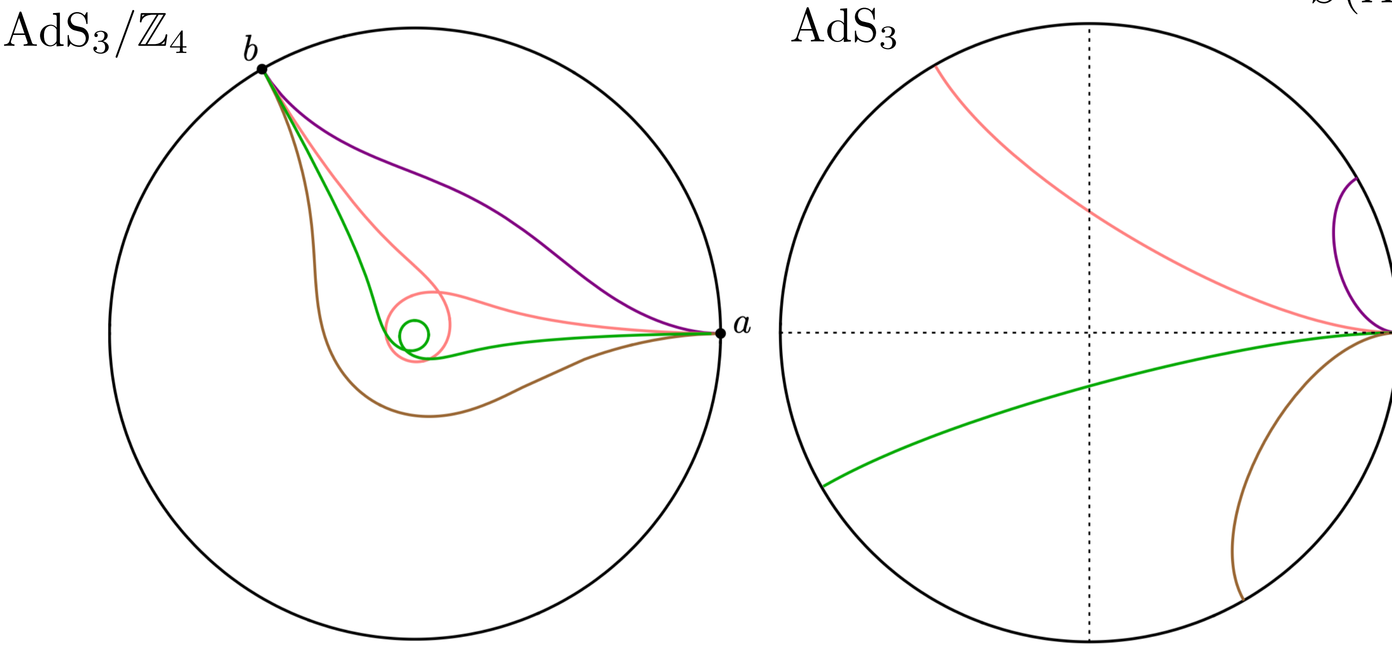
Long geodesics probe entanglement shadow



$$S(A) = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

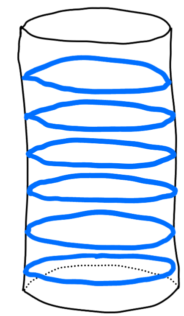
Long geodesics probe entanglement shadow → dual information theoretic quantity?

[Balasubramanian, Chowdhury, Czech, de Boer 2014]

D1/D5 CFT: long string states dual to conical defect

D1/D5 CFT (at specific point in moduli space): symmetric product orbifold $(T^4)^N/S_N$

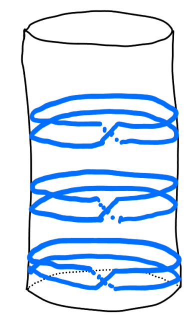
Empty AdS: dual to vacuum in "untwisted sector" describing N "short" closed strings.



"Conical defect" $(\mathrm{AdS}_3 \times \mathrm{S}^3)/\mathbb{Z}_n$: dual to "twisted sector" with N/n "long strings" of length n

$|\Psi_u\rangle = [\sigma_n(0)]^{N/n}\,|0\rangle$ with

$u = (1\,2\ldots n)(n{+}1\ldots 2n)\ldots(N{-}n{+}1\ldots N)$



"Unwrapping" the long strings in CFT $\approx$ going to AdS covering space of $\mathrm{AdS}_3/\mathbb{Z}_n$

[Martinec, McElgin 2002]

Entwinement from covering space

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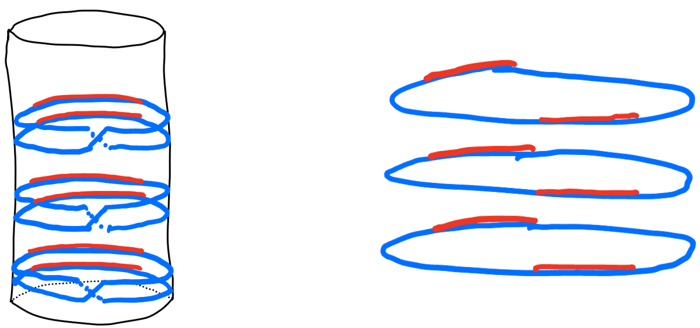
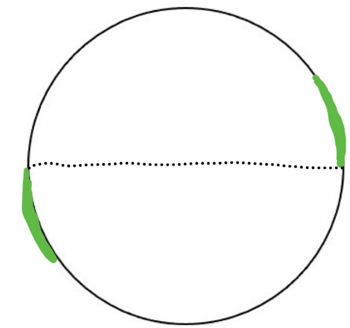
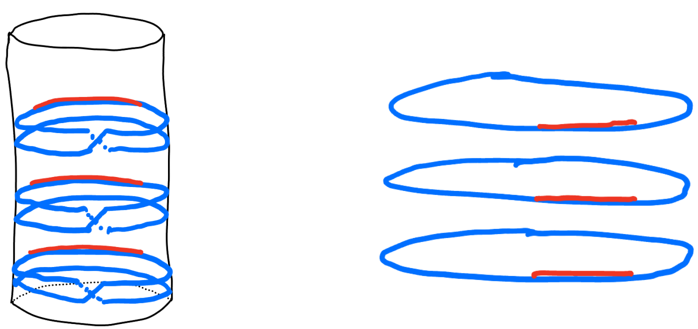
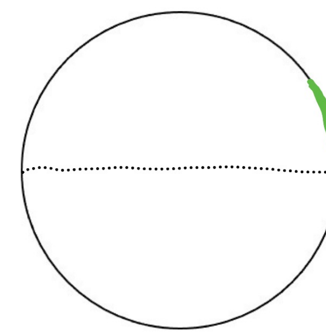


spatial
entanglement





entwinement



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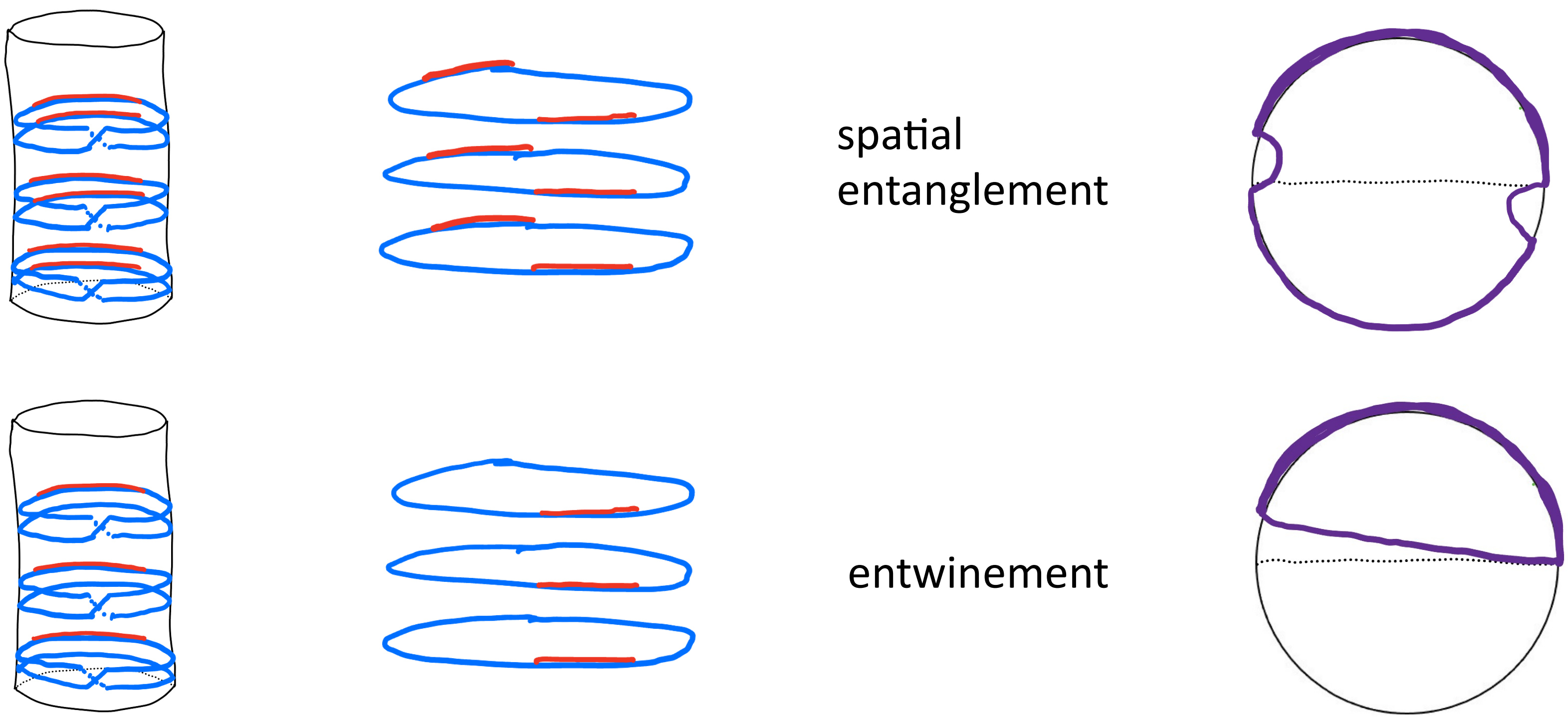

[Balasubramanian, Chowdhury, Czech, de Boer 2014]

Entwinement from extended Hilbert space

Entwinement = entanglement of internal, gauged degrees of freedom

Other example: matrix theory, where spatial entanglement does not exist

Length of nonminimal geodesic is dual to entwinement of segment of long string

Gauge-invariant under $\mathbb{Z}_n \subset S_N$?

Due to $\mathbb{Z}_n$ gauge symmetry, defining entwinement is a priori tricky (Hilbert space does not factorize, no subalgebra of operators associated to segment of long string)

Prescription: ungauge the symmetry, compute entanglement entropy, then gauge again. Involves tracing over non-gauge-invariant states.

[Balasubramanian, Chowdhury, Czech, de Boer 2014]

Candidate-definitions of entwinement

- From extended Hilbert space

  [Balasubramanian, Chowdhury, Czech, de Boer 2014]

- From operator subalgebras

  [Lin 2016]

- From replica method

  [Balasubramanian, Bernamonti, Craps, De Jonckheere, Galli 2016]

- From reduced density matrix

  [Balasubramanian, Craps, De Jonckheere, Sárosi 2018]

- From probability distributions of measurements

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Outline

1. Introduction
2. Entanglement entropy of identical particles
3. Symmetric product orbifolds as lattice gauge theories
4. Entwinement from reduced density matrix
5. Entwinement from replica method
6. Conclusions and outlook

Wavefunctions for identical particles

Identical bosons: symmetric Hilbert space spanned by

$$|x_1, x_2, \dots, x_N\rangle_S = \frac{1}{\sqrt{N!}} \sum_{g \in S_N} |x_{g(1)}, x_{g(2)}, \dots, x_{g(N)}\rangle$$

Gauge-invariant states can be expanded as

$$|\psi_S\rangle = \frac{1}{\sqrt{N!}} \int \left(\prod_{i=1}^{N} \mathrm{d}x_i \right) \psi_S(x_1, x_2, \dots, x_N) |x_1, x_2, \dots, x_N\rangle_S$$

with fully symmetric wavefunction

$$\psi_S(x_1, x_2, \dots, x_N) = \psi_S(x_{g(1)}, x_{g(2)}, \dots, x_{g(N)})$$

Reduced density matrix for identical particles

How are k particles entangled with the remaining N-k?

Consider a generic k-particle operator

$$\mathcal{O}^{(k)} = \frac{1}{N!}\int \mathcal{O}_k(x_1,\ldots,x_k;y_1,\ldots,y_k)|x_1,\ldots,x_k,z_{k+1},\ldots,z_N\rangle_S\,{}_S\langle y_1,\ldots,y_k,z_{k+1},\ldots,z_N|$$

with all variables integrated over. Expectation values can be written in terms of a reduced density matrix:

$$\langle\psi_S|\mathcal{O}^{(k)}|\psi_S\rangle = \int \mathcal{O}_k(x_1,\ldots,x_k;y_1,\ldots,y_k)\rho_S(y_1,\ldots,y_k;x_1,\ldots,x_k)$$

with

$$\rho_S(x_1,\ldots,x_k;x'_1,\ldots,x'_k) = \int\left(\prod_{i=k+1}^{N}\mathrm{d}y_i\right)\psi_S(x_1,\ldots,x_k,y_{k+1},\ldots,y_N)\psi_S^*(x'_1,\ldots,x'_k,y_{k+1},\ldots,y_N)$$

From this reduced density matrix, entanglement entropy can be defined.

No natural subalgebra for subset of particles

If Hilbert space factorizes, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, there exists a natural subalgebra $\mathcal{A}$ associated to $\mathcal{H}_A$. Given a global state ρ , there exists a unique "reduced density matrix" ρ_A in $\mathcal{A}$ s.t.

$$\mathrm{Tr}[\rho\,(\mathcal{O}_A \otimes \mathbf{1}_{\bar{A}})] = \mathrm{Tr}[\rho_A \mathcal{O}_A] \qquad \text{for all } \mathcal{O}_A \in \mathcal{A}$$

The entanglement entropy of $\mathcal{H}_A$ can then be defined as $S_{EE} = -\mathrm{Tr}_{\mathcal{H}_A}[\rho_A \log \rho_A]$

Spatial regions in gauge theories: Hilbert space does not factorize, but one can associate to a region a subalgebra with nontrivial center and a reduced density matrix.

However, for identical particles, the Hilbert space of gauge invariant states does not factorize, nor does there exist a natural subalgebra of gauge-invariant operators associated to a subset of particles. E.g. product of one-particle operators is two-particle operator:

$$\left(\sum_{i=1}^{N} \mathcal{O}_i\right)\left(\sum_{j=1}^{N} \mathcal{Q}_j\right) = \sum_{ij} \mathcal{O}_i \mathcal{Q}_j$$

We used linear subspace rather than subalgebra to define reduced density matrix. [Balasubramanian, Craps, De Jonckheere, Sárosi 2018]

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Standard CFT treatment of orbifolds

Start by projecting on G-invariant states: $P = \frac{1}{|G|} \sum_{g\in G} g$

$$\frac{1}{|G|}\sum_{g\in G} g\,\underset{1}{\square}$$

But this is not modular invariant, e.g. under S modular transform: $g\,\underset{1}{\square} \rightarrow 1\,\underset{g}{\square}$

To remedy this, introduce twisted sectors: $Z_{T/G} \equiv \sum_{h\in G} \frac{1}{|G|} \sum_{g\in G} g\,\underset{h}{\square} = \frac{1}{|G|} \sum_{g,h\in G} g\,\underset{h}{\square}$

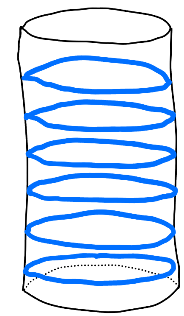
For nonabelian groups, twisted sectors are labeled by conjugacy classes:

$$X(\tau, \sigma + 2\pi) = hX(\tau,\sigma) \Longleftrightarrow gX(\tau,\sigma+2\pi) = (ghg^{-1})gX(\tau,\sigma)$$

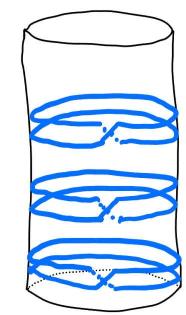
[Ginsparg Les Houches lectures 1988]

Example: symmetric product orbifold

D1/D5 CFT (at specific point in moduli space): symmetric product orbifold $(T^4)^N/S_N$



Untwisted sector describes states of N short strings with wavefunctions invariant under permutations

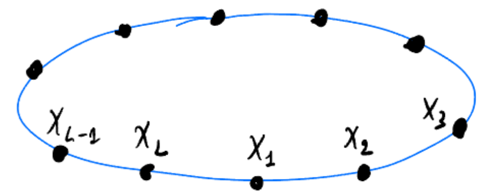


In a twisted sector labeled by a permutation u, strings are periodic up to the action of u. The N strands of string combine into a collection "long strings". More precisely, the twisted sector is labeled by the conjugacy class of u, i.e. the lengths of the long strings.

Twisted sectors are obtained by acting with twist fields on the vacuum of the untwisted sector.

Lattice model: copies of seed theory

Seed theory: 1+1d lattice model of bosons $X \equiv (X^{(\mu)}) = (X^{(0)}, X^{(1)}, \dots, X^{(D-1)})$
We will suppress the index (μ)

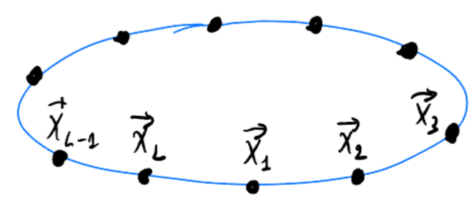


Variables are defined on a circle: $X_i, \; i = 1, \dots, L$

For concreteness: $$H_{\text{seed}} = \sum_{i=1}^{L} \frac{\delta}{2} \left[\Pi_i^2 + \frac{1}{\delta^2}(X_{i+1} - X_i)^2 + \dots \right]$$ with $X_{L+1} = X_1$

Introduce N copies X_i^a, $a = 1, \dots, N$, and permutations $g_i \in S_N$:

$$\hat{g}_i X_i^a \hat{g}_i^{-1} = X_i^{g_i(a)} \; , \qquad \hat{g}_i \Pi_i^a \hat{g}_i^{-1} = \Pi_i^{g_i(a)}$$



Assemble N copies in column vectors:

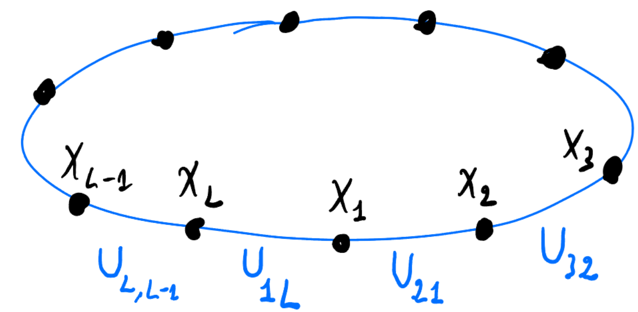
$$\hat{g}_i \mathbf{X}_i \hat{g}_i^{-1} = g_i^{-1} \mathbf{X}_i \; , \qquad \hat{g}_i \mathbf{\Pi}_i \hat{g}_i^{-1} = g_i^{-1} \mathbf{\Pi}_i$$

[Craps, De Clerck, Vilar López 2022]

Covering theory uses link variables

$$H_{\text{seed}} = \sum_{i=1}^{L} \frac{\delta}{2} \left[\Pi_i^2 + \frac{1}{\delta^2}(X_{i+1} - X_i)^2 + \dots \right]$$

$$\hat{g}_i \mathbf{X}_i \hat{g}_i^{-1} = g_i^{-1} \mathbf{X}_i \ , \qquad \hat{g}_i \mathbf{\Pi}_i \hat{g}_i^{-1} = g_i^{-1} \mathbf{\Pi}_i$$



To make N-copied Hamiltonian invariant under local S_N transformations, introduce nondynamical discrete gauge fields U living on links between neighboring lattice points:

$$H_{S_N} = \sum_{i=1}^{L} \frac{\delta}{2} \left[\mathbf{\Pi}_i^T \mathbf{\Pi}_i + \frac{1}{\delta^2}(\mathbf{X}_{i+1} - U_{i+1,i}\mathbf{X}_i)^T (\mathbf{X}_{i+1} - U_{i+1,i}\mathbf{X}_i) + \dots \right]$$

with

$$\hat{g}_{i+1} U_{i+1,i} \hat{g}_{i+1}^{-1} = g_{i+1}^{-1} U_{i+1,i} \ , \qquad \hat{g}_i U_{i+1,i} \hat{g}_i^{-1} = U_{i+1,i} g_i$$

[Craps, De Clerck, Vilar López 2022]

Link variables appear in finite energy condition

$$H_{S_N} = \sum_{i=1}^{L} \frac{\delta}{2} \left[\mathbf{\Pi}_i^T \mathbf{\Pi}_i + \frac{1}{\delta^2} (\mathbf{X}_{i+1} - U_{i+1,i} \mathbf{X}_i)^T (\mathbf{X}_{i+1} - U_{i+1,i} \mathbf{X}_i) + \ldots \right]$$

The Hilbert space of this "covering theory", in which the permutation symmetry is not gauged yet, is spanned by eigenstates of X and U:

$$\mathbf{X}_i | \{\mathbf{x}_j\}, \{u_{j+1,j}\} \rangle = \mathbf{x}_i | \{\mathbf{x}_j\}, \{u_{j+1,j}\} \rangle$$

$$U_{i+1,i} | \{\mathbf{x}_j\}, \{u_{j+1,j}\} \rangle = u_{i+1,i} | \{\mathbf{x}_j\}, \{u_{j+1,j}\} \rangle$$

Action of local permutations:

$$\hat{g}_i | \{\mathbf{x}_j\}, \{u_{j+1,j}\} \rangle = | \{g_i \mathbf{x}_i, \mathbf{x}_j\}_{j \neq i}, \{u_{i+1,i} g_i^{-1}, g_i u_{i,i-1}, u_{j+1,j}\}_{j \neq i, i-1} \rangle$$

Allowed states (finite energy): $\mathbf{x}_{i+1} - u_{i+1,i} \mathbf{x}_i = \mathcal{O}(\delta) \quad \forall i = 1, \ldots, L$

[Craps, De Clerck, Vilar López 2022]

Orbifold: gauging away all link variables but one

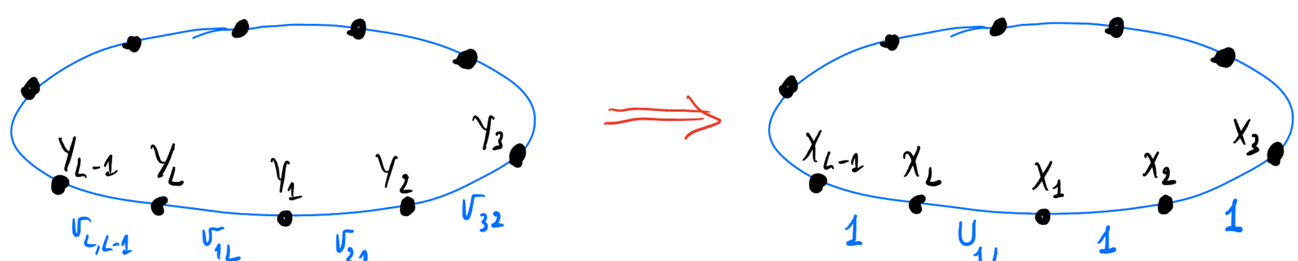
Gauge the S_N symmetry by projecting on permutation invariant states using

$$\hat{P} = \bigotimes_{i=1}^{L} \left(\frac{1}{N!} \sum_{g_i \in S_N} \hat{g}_i \right)$$

Single out global S_N :

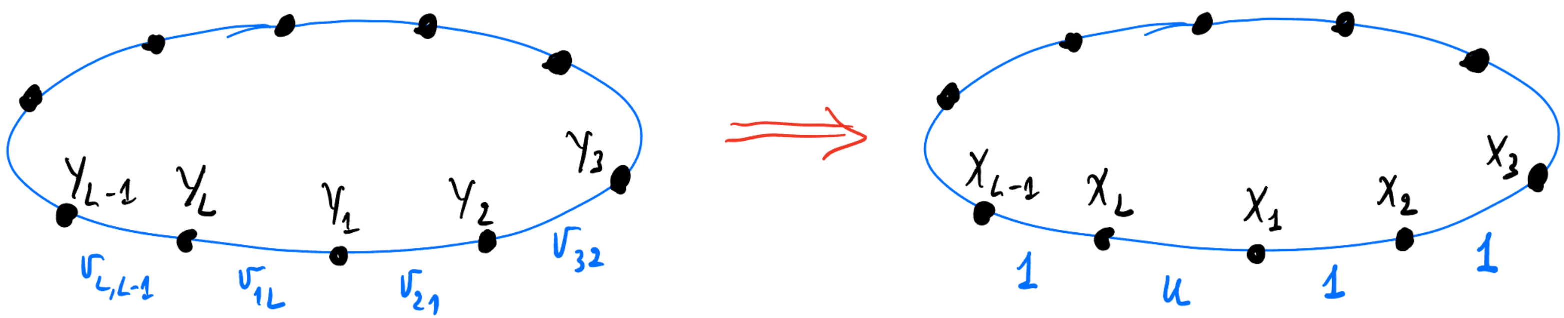
$$\hat{P} = \left[\frac{1}{N!} \sum_{g \in S_N} \hat{g} \otimes \ldots \otimes \hat{g} \right] \left[\bigotimes_{i=1}^{L-1} \left(\frac{1}{N!} \sum_{g_i \in S_N} \hat{g}_i \right) \right] \equiv \hat{P}_{gl} \hat{P}_{L-1}$$

Gauge way first $L-1$ link variables: $\hat{P}_{L-1} |\{\mathbf{y}_i\}, \{v_{i+1,i}\}\rangle = \hat{P}_{L-1} |\{\mathbf{x}_i\}, \{1, \ldots, 1, u_{1,L}\}\rangle$



[Craps, De Clerck, Vilar López 2022]

Gauging away all link variables but one



Define the "global" states $\quad |\{\mathbf{x}_i\}, u\rangle_{gl} \equiv \sqrt{(N!)^{L-1}}\, \hat{P}_{L-1} |\{\mathbf{x}_i\}, \{1, \ldots, 1, u\}\rangle$

with normalization $\quad {}_{gl}\langle\{\mathbf{y}_i\}, v|\{\mathbf{x}_i\}, u\rangle_{gl} = \delta_{u,v} \prod_{i=1}^{L} \delta^{(N)}\left(\mathbf{x}_i - \mathbf{y}_i\right)$

Discrete continuity condition: $\quad \mathbf{x}_{i+1} - \mathbf{x}_i = \mathcal{O}(\delta) \quad \forall i \neq L \;, \qquad \mathbf{x}_1 - u\mathbf{x}_L = \mathcal{O}(\delta)$

Remaining link variable u instructs how N strands should be glued into "long strings".

[Craps, De Clerck, Vilar López 2022]

Twisted sectors are labeled by conjugacy classes

Residual gauge symmetry: global S_N $\quad (\hat{g} \otimes \cdots \otimes \hat{g}) \, |\{\mathbf{x}_i\}, u\rangle_{gl} = |\{g\mathbf{x}_i\}, gug^{-1}\rangle_{gl}$

Gauge-invariant states: $\quad |\{\mathbf{x}_i\}, u\rangle_S \equiv \sqrt{N!}\hat{P}_{gl}|\{\mathbf{x}_i\}, u\rangle_{gl} = \frac{1}{\sqrt{N!}} \sum_{g \in S_N} |\{g\mathbf{x}_i\}, gug^{-1}\rangle_{gl}$

Redundancy: $\quad |\{\mathbf{x}_i\}, u\rangle_S = |\{g\mathbf{x}_i\}, gug^{-1}\rangle_S$ → pick representatives $u \in \mathcal{C}(S_N)$

(one group element per conjugacy class)

$$\mathcal{H}_S = \operatorname{span}\left\{ |\{\mathbf{x}_i\}, u\rangle_S \,|\, \mathbf{x}_i \in \mathcal{M}^N, u \in \mathcal{C}(S_N) \right\}$$

Hilbert space splits into direct sum of twisted sectors labeled by conjugacy classes

Still some redundancy left: $\quad |\{\mathbf{x}_i\}, u\rangle_S = |\{h\mathbf{x}_i\}, u\rangle_S \quad$ with $\quad h \in C_u \quad$ (centralizer of u)

For permutation u with N_i cycles of length i: $\quad C_u = \prod_i \left(S_{N_i} \ltimes \mathbb{Z}_i^{N_i} \right)$

Inner product: $\quad {}_S\langle \{\mathbf{y}_i\}, v | \{\mathbf{x}_i\}, u\rangle_S = \delta_{u,v} \sum_{h \in C_u} \left[\prod_{i=1}^{L} \delta^{(N)} \left(\mathbf{y}_i - h\mathbf{x}_i \right) \right]$

[Craps, De Clerck, Vilar López 2022]

Wavefunctions are symmetric under centralizer

$$|\Psi_u\rangle = \frac{1}{\sqrt{|C_u|}} \int \left(\prod_{i=1}^{L} \prod_{a=1}^{N} \mathrm{d}x_i^a \right) \Psi_u(\{\mathbf{x}_i\}) |\{\mathbf{x}_i\}, u\rangle_S$$

$$|\{\mathbf{x}_i\}, u\rangle_S = |\{h\mathbf{x}_i\}, u\rangle_S$$

$$= \frac{1}{\sqrt{|C_u|}} \int \left(\prod_{i=1}^{L} \prod_{a=1}^{N} \mathrm{d}x_i^a \right) \Psi_u(\{\mathbf{x}_i\}) \frac{1}{|C_u|} \sum_{h \in C_u} |\{h\mathbf{x}_i\}, u\rangle_S$$

$$= \frac{1}{\sqrt{|C_u|}} \int \left(\prod_{i=1}^{L} \prod_{a=1}^{N} \mathrm{d}x_i^a \right) \left(\frac{1}{|C_u|} \sum_{h \in C_u} \Psi_u(\{h^{-1}\mathbf{x}_i\}) \right) |\{\mathbf{x}_i\}, u\rangle_S$$

Wavefunction in twisted sector can be chosen to be symmetric under centralizer:

$$\Psi_u(\{\mathbf{x}_i\}) = \Psi_u(\{h\mathbf{x}_i\})$$

No symmetrization over full S_N. Choosing different representative of conjugacy class would change labelling of strands without changing connectedness.

[Craps, De Clerck, Vilar López 2022]; see also [Erdmenger, Gerbershagen 2019]

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1. Introduction
2. Entanglement entropy of identical particles
3. Symmetric product orbifolds as lattice gauge theories
4. Entwinement from reduced density matrix
5. Entwinement from replica method
6. Conclusions and outlook

Operators acting on arbitrary subset of vertices

Partition vertex variables X_i^a, $i = 1, \ldots, L$, $a = 1, \ldots, N$, in arbitrary subset A and its complement (not necessarily spatially organized):

$$(\mathbf{x}_A, \mathbf{x}_{\bar{A}}) = \mathbf{x}$$

In analogy with identical particles, consider gauge-invariant operators acting only on A:

$$\mathcal{O}^{(A)} = \frac{1}{|C_u|} \int_{\mathbf{x},\mathbf{y}} \mathcal{O}_A(\mathbf{y}_A; \mathbf{x}_A) \delta(\mathbf{y}_{\bar{A}} - \mathbf{x}_{\bar{A}}) |\mathbf{y}, u\rangle_S \, {}_S\langle \mathbf{x}, u|$$

Using C_u redundancy of symmetric states:

$$\mathcal{O}^{(A)} = \frac{1}{|C_u|^2} \int_{\mathbf{x},\mathbf{y}} \left(\sum_{h \in C_u} \mathcal{O}_A((h\mathbf{y})_A; (h\mathbf{x})_A) \delta((h\mathbf{y})_{\bar{A}} - (h\mathbf{x})_{\bar{A}}) \right) |\mathbf{y}, u\rangle_S \, {}_S\langle \mathbf{x}, u|$$

Can specify e.g. that A is a segment consisting of Y connected strands within a long string of length Z, but it would not be gauge-invariant to specify which strands or which long string.

[Craps, De Clerck, Vilar López 2022]

Entwinement from reduced density matrix

$$\mathcal{O}^{(A)} = \frac{1}{|C_u|} \int_{\mathbf{x},\mathbf{y}} \mathcal{O}_A(\mathbf{y}_A; \mathbf{x}_A) \delta(\mathbf{y}_{\bar{A}} - \mathbf{x}_{\bar{A}}) |\mathbf{y}, u\rangle_S \, {}_S\langle \mathbf{x}, u|$$

$$|\Psi_u\rangle = \frac{1}{\sqrt{|C_u|}} \int \left(\prod_{i=1}^{L} \prod_{a=1}^{N} \mathrm{d}x_i^a \right) \Psi_u(\{\mathbf{x}_i\}) |\{\mathbf{x}_i\}, u\rangle_S$$

Expectation value can be written as $\langle \Psi_u | \mathcal{O}^{(A)} | \Psi_u \rangle = \int_{\mathbf{x}_A, \mathbf{y}_A} \mathcal{O}_A(\mathbf{y}_A; \mathbf{x}_A) \rho_S(\mathbf{x}_A; \mathbf{y}_A)$

with reduced density matrix $\rho_S(\mathbf{x}_A, \mathbf{x}'_A) = \int_{\mathbf{y}_{\bar{A}}} \Psi_u(\mathbf{x}_A, \mathbf{y}_{\bar{A}}) \Psi_u^\star(\mathbf{x}'_A, \mathbf{y}_{\bar{A}})$

Entwinement from reduced density matrix: $S_{\mathrm{vN}}(\rho_S) = -\mathrm{Tr}[\rho_S \log \rho_S]$

In contrast to identical particles, reduced density matrix captures some info about location of A within long string configuration specified by u. Can compute entanglement of Y connected strands within long string of length Z.

[Craps, De Clerck, Vilar López 2022]

Linear subspaces versus subalgebras

$$\mathcal{O}^{(A)} = \frac{1}{|C_u|} \int_{\mathbf{x},\mathbf{y}} \mathcal{O}_A(\mathbf{y}_A; \mathbf{x}_A) \delta(\mathbf{y}_{\bar{A}} - \mathbf{x}_{\bar{A}}) |\mathbf{y}, u\rangle_S \, {}_S\langle \mathbf{x}, u|$$

In general, the set of gauge-invariant operators acting only on A does not close under multiplication:

$$\mathcal{O}^{(A)} \mathcal{Q}^{(A)} = \frac{1}{|C_u|^2} \int_{\mathbf{x},\mathbf{y},\mathbf{z}} \sum_{h \in C_u} \mathcal{O}_A(\mathbf{y}_A; (h\mathbf{z})_A) \mathcal{Q}_A(\mathbf{z}_A; \mathbf{x}_A) \delta(\mathbf{y}_{\bar{A}} - (h\mathbf{z})_{\bar{A}}) \delta(\mathbf{z}_{\bar{A}} - \mathbf{x}_{\bar{A}}) |\mathbf{y}, u\rangle_S \, {}_S\langle \mathbf{x}, u|$$

generically does not have the required $\delta(\mathbf{y}_{\bar{A}} - \mathbf{x}_{\bar{A}})$

Spatial partitions are an exception, because then h does not mix A and $\bar{A}$, and one can bring the product to the desired form.

We managed to define a reduced density matrix using linear subspaces rather than subalgebras.

[Balasubramanian, Craps, De Jonckheere, Sárosi 2018] [Craps, De Clerck, Vilar López 2022]

Entwinement in D1/D5 orbifold CFT

$(T^4)^N/S_N$

$|\Psi_u\rangle = [\sigma_n(0)]^{N/n}\,|0\rangle \qquad u = (1\,2\ldots n)(n{+}1\ldots 2n)\ldots(N{-}n{+}1\ldots N)$

In states dual to conical defects, entwinement reproduces lengths of long geodesics. For segment of length L:

$$\mathcal{L}(L) = 2R_{\text{AdS}}\log\left[\frac{2nr_\infty}{R_{\text{AdS}}}\sin\left(\frac{L}{2n}\right)\right]$$

$$E(L) \equiv S_{\text{vN}}(\rho_L) = \frac{N}{n}\frac{c_{ls}}{3}\log\left[\frac{2nR_{\text{AdS}}}{\epsilon_{\text{UV}}}\sin\left(\frac{L}{2n}\right)\right] = \frac{\mathcal{L}(L)}{4G_N}$$

where we related bulk IR and boundary UV cutoffs via $r_\infty = R^2_{\text{AdS}}/\epsilon_{\text{UV}}$

and used $Nc_{ls}/n = c = 3R_{\text{AdS}}/(2G_N)$

[Craps, De Clerck, Vilar López 2022]
[Balasubramanian, Bernamonti, Craps, De Jonckheere, Galli 2016]

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Replica method

$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}(\rho_A^n)$$

Represent state ρ created by operator σ via path integral:

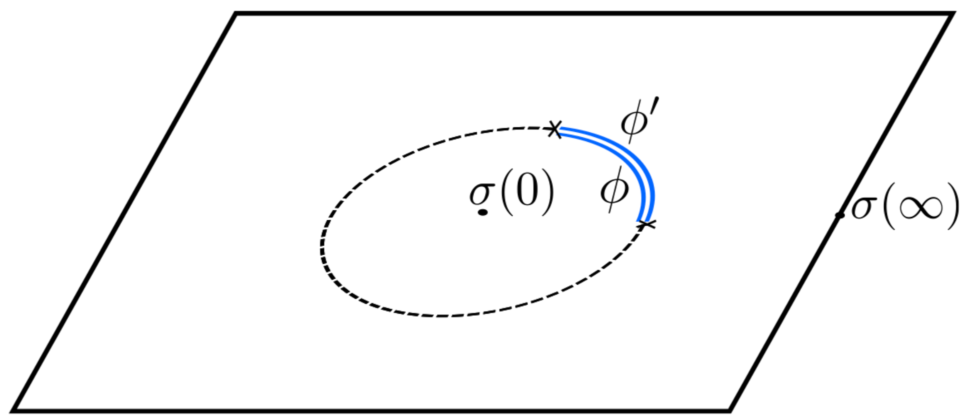
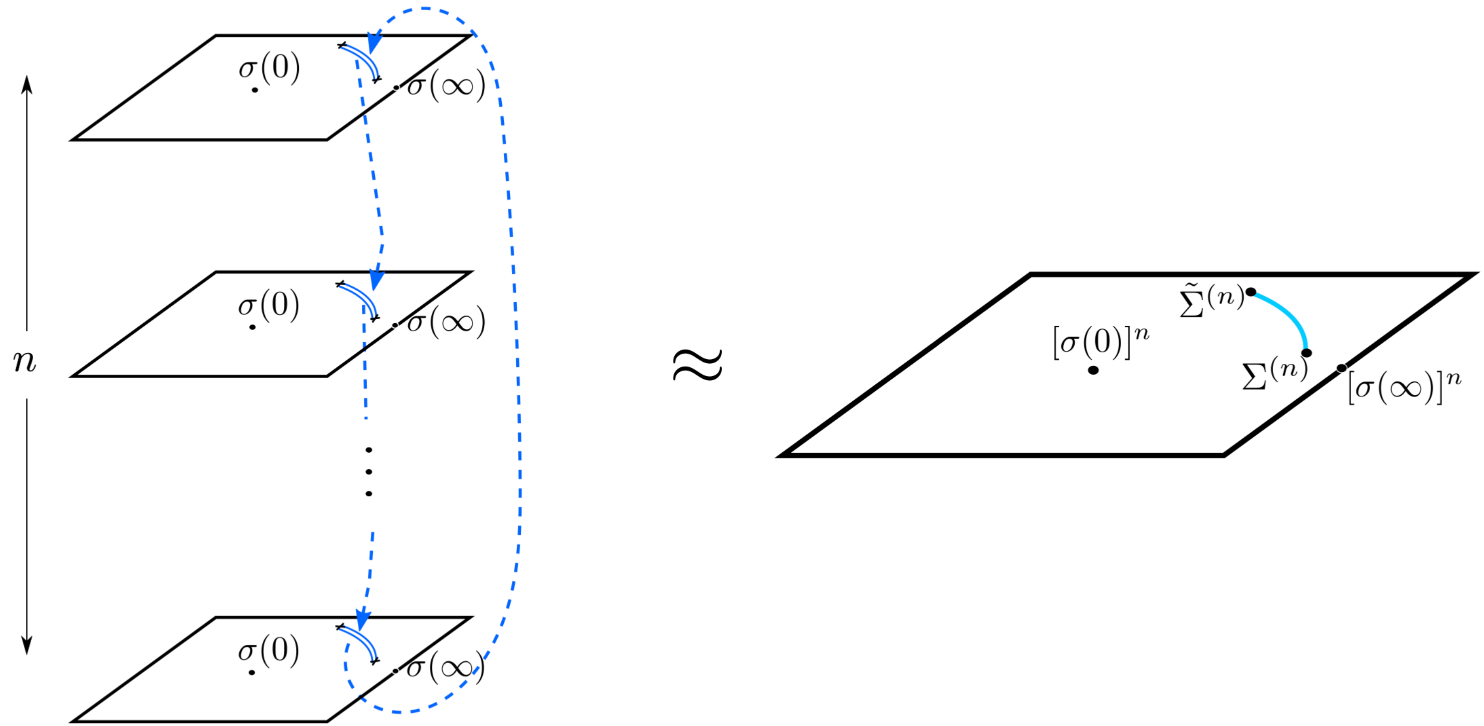


Figure from [Balasubramanian, Bernamonti, Craps, De Jonckheere, Galli 2016]

Replica method

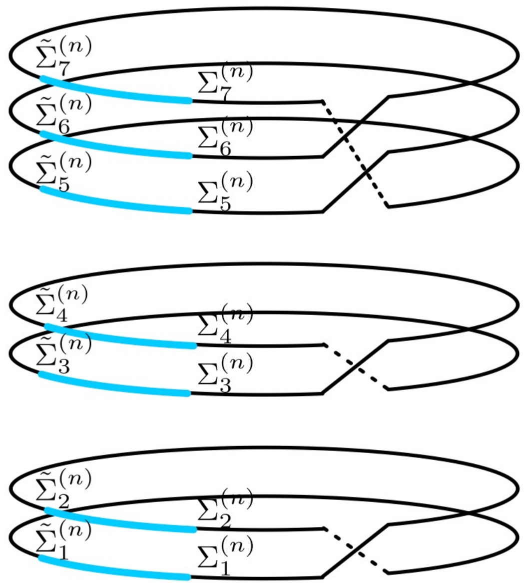
$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}(\rho_A^n)$$

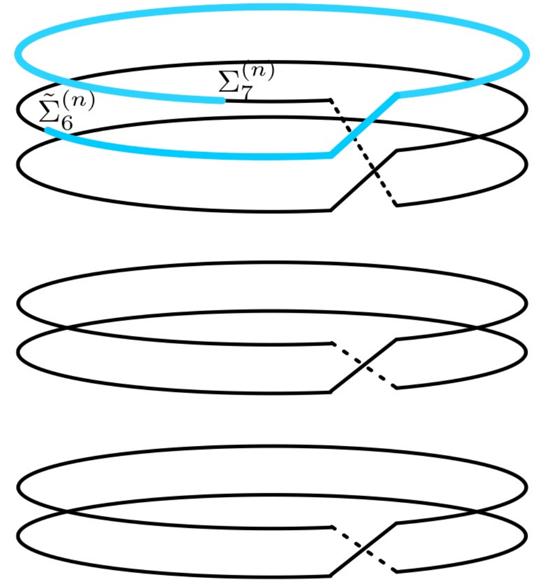


Compute $\mathrm{Tr}(\rho_A^n)$ by introducing n replica copies. Replace by correlator of Rényi twist operators on single sheet of n-fold cover $\mathrm{CFT}^n/\mathbb{Z}_n$

Figure from [Balasubramanian, Bernamonti, Craps, De Jonckheere, Galli 2016]

Spatial entanglement entropy vs entwinement





Spatial entanglement: twist operators act on each strand at same spatial location

$$\Sigma^{(n)} = \Sigma_1^{(n)} \Sigma_2^{(n)} \cdots \Sigma_N^{(n)}$$

Entwinement: twist operators may act on individual strands

[Balasubramanian, Bernamonti, Craps, De Jonckheere, Galli 2016]

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Conclusions and outlook

- Entwinement = entanglement of internal, gauged degrees of freedom
- Motivation from holography: probe inside entanglement shadows
- Original field theory definition involved extended Hilbert space; non-gauge-invariant states involved in intermediate steps
- Lattice model for (symmetric) orbifolds: twisted sectors from link variables
- Entwinement from density matrices for symmetric product orbifolds
- Entwinement from replica method
- Links with other notions of non-spatial entanglement? Entanglement in matrix models?