Worldsheet from worldline

Eurostrings `23, 25.4.2023

Umut Gürsoy

ITP, Utrecht

Based on arXiv:2211.16514 w/ D. Gallegos and N. Zinnato

Is holography inherent in perturbative QFT?

Can we see gravity emerging directly from QFT amplitudes?

How to organize QFT to reproduce bulk propagators?

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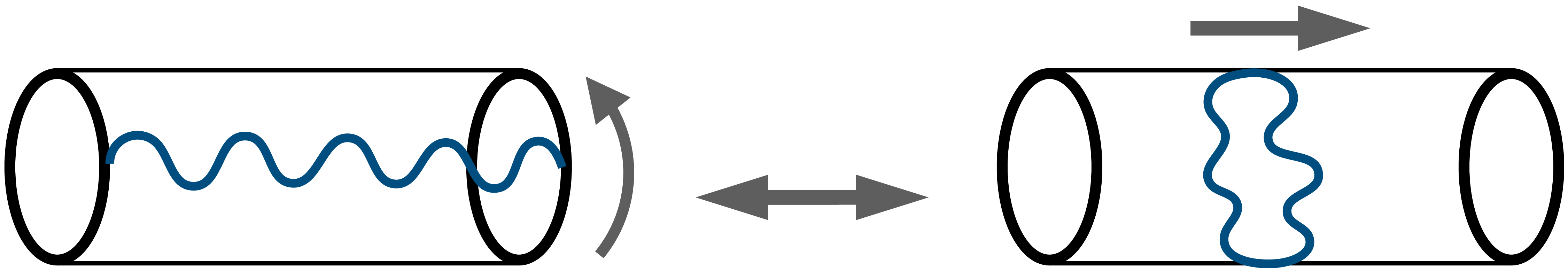

Open-closed duality of D-branes

Maldacena `97 Polyakov `96

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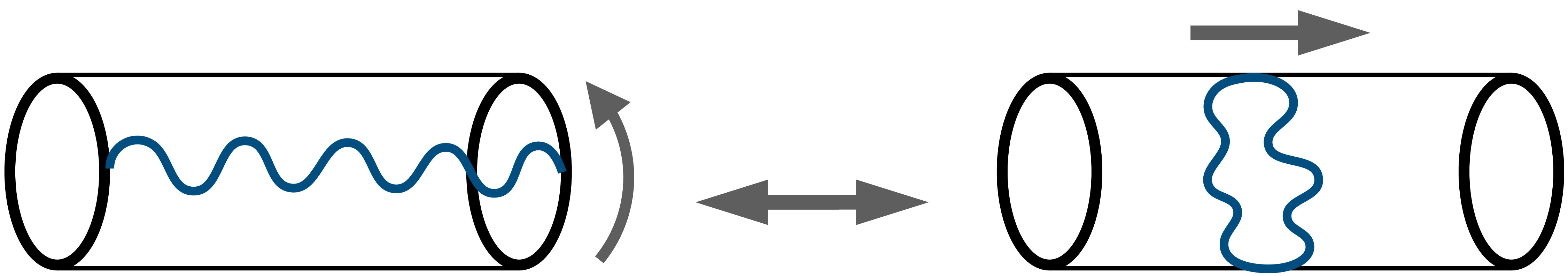
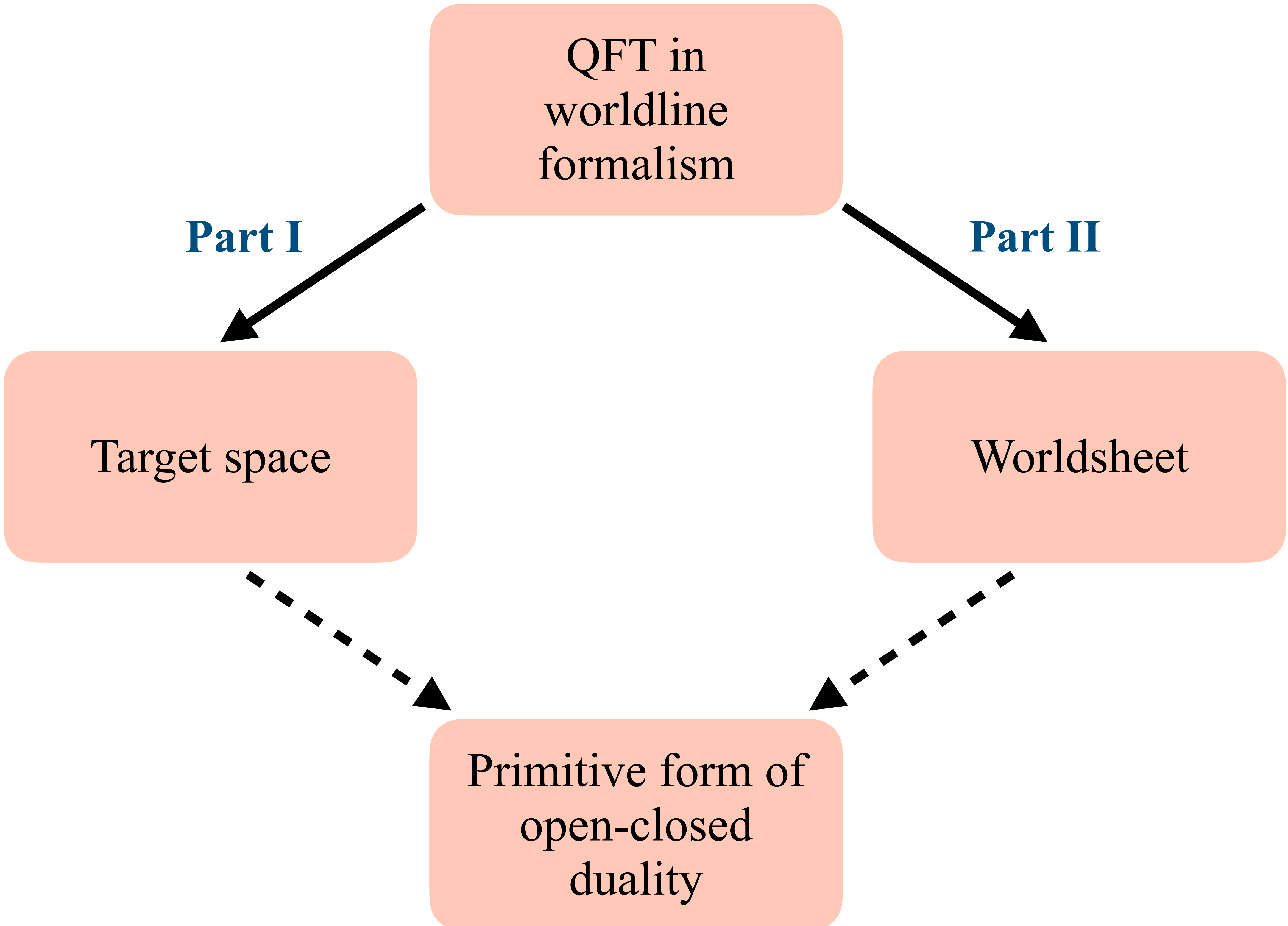


Open-closed duality of D-branes

Maldacena `97  Polyakov `96

$$\langle \mathcal{O}_1(k_1)\cdots\mathcal{O}_n(k_n)\rangle_g = \int_{M_{g,n}} \langle \mathcal{V}_1(k_1,z_1)\cdots\mathcal{V}_n(k_n,z_n)\rangle_{w.s.}$$

Outline


QFT in worldline formalism
Part I
Part II
Target space
Worldsheet
Primitive form of open-closed duality

Worldline formalism of QFT

$$\mathcal{S} = N \int d^d x \, \mathrm{Tr} \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{m^2}{2}\Phi^2 + \sum_h \frac{\lambda_h}{h!}\Phi^h \right)$$

Renormalizable for

$h \leq 2d/(d-2)$

Consider m $\to$ 0 and assume regularization both in the UV and the IR

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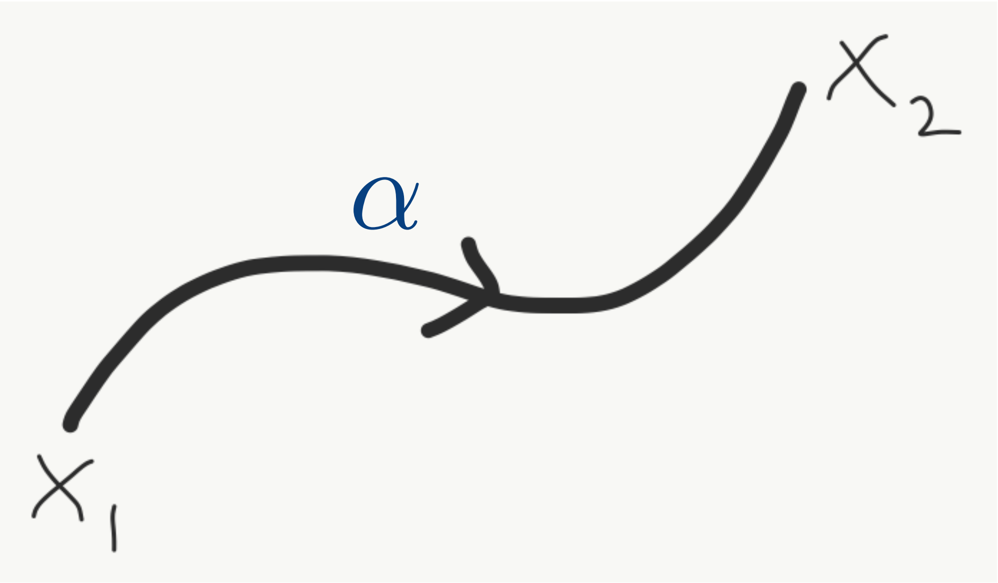


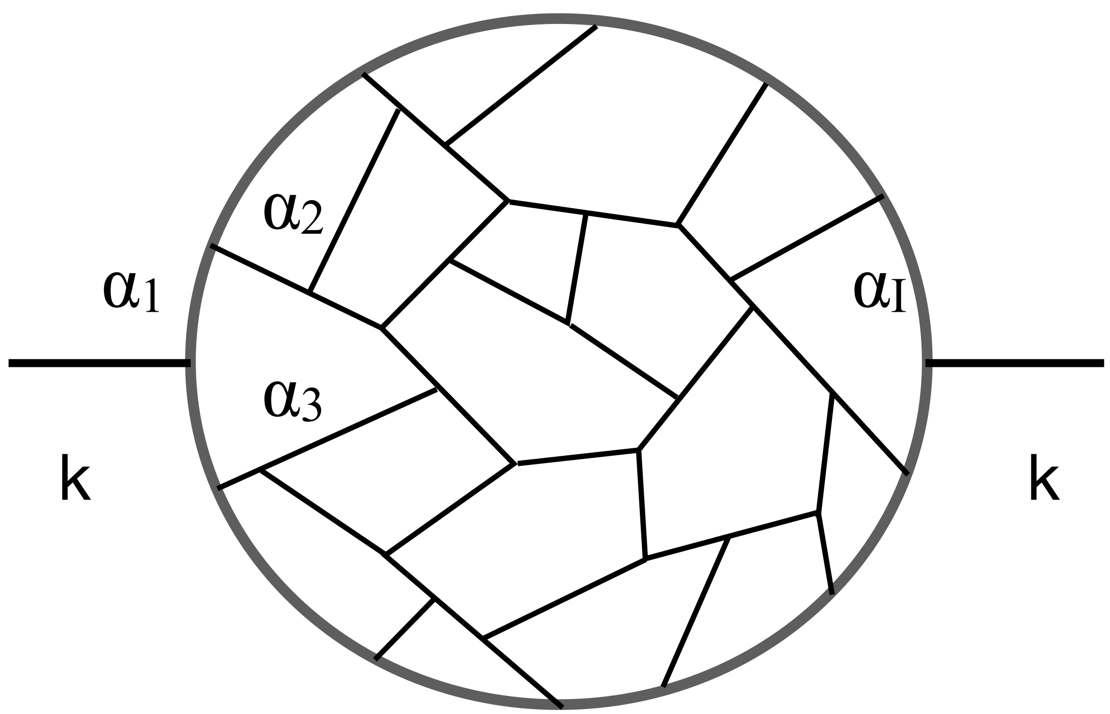
Schwinger `51

$$\langle \Phi(x_1)\Phi(x_2)\rangle = \int d^4 k \, \frac{e^{ik(x_1 - x_2)}}{k^2 + m^2} = \int_0^\infty d\alpha \, \langle x_1 | e^{-\alpha(\hat{k}^2 + m^2)} | x_2 \rangle$$

Worldline formalism for interacting QFT

Itzykson, Zuber `80, Weinzerl `22

Connected graph ℓ-loops, I-lines



Symanzik polynomials

$$\mathcal{U} = \sum_{\mathcal{T}_1} \prod_{r=1}^{l} \alpha_r \qquad \mathcal{A} = \sum_{\mathcal{T}_2} \prod_{r=1}^{\ell+1} \alpha_r$$

"trees" "2-trees"

$$\Gamma(k) = \int_0^\infty \prod_{r=1}^{I} d\alpha_r \, \mathcal{U}^{-d/2}(\alpha_r) \exp[-\mathcal{U}(\alpha_r)^{-1} \mathcal{A}(\alpha_r) k^2]$$

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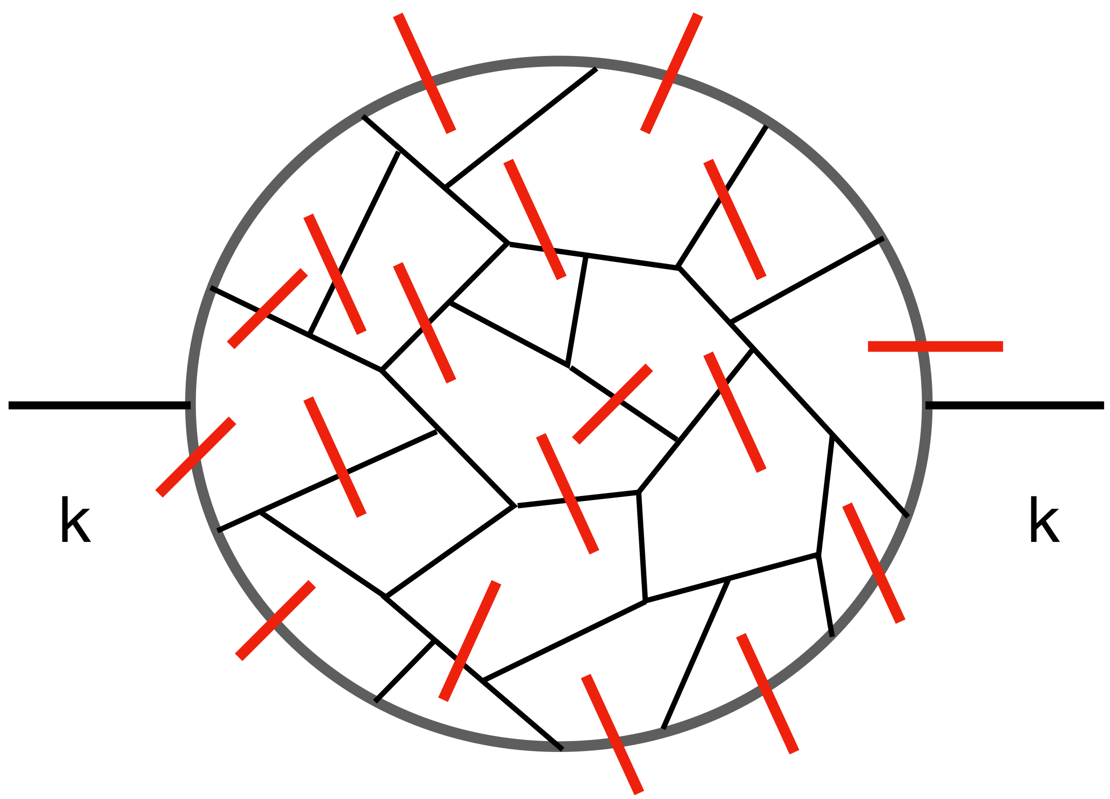

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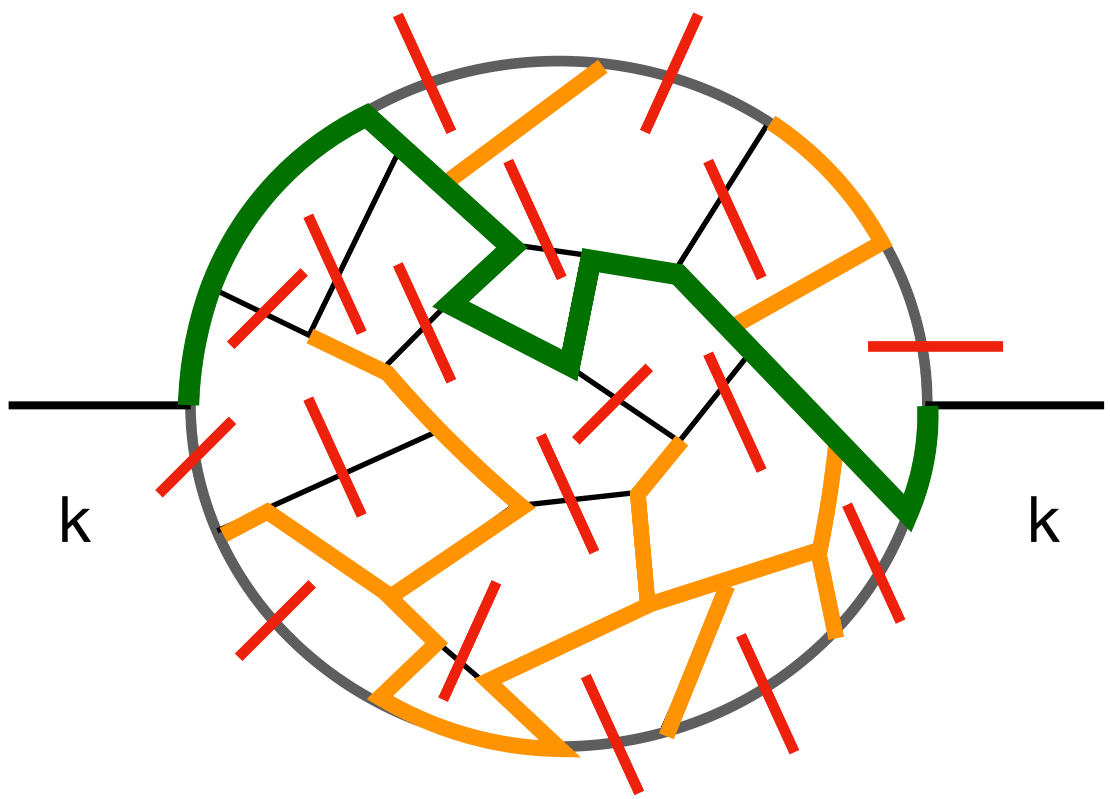

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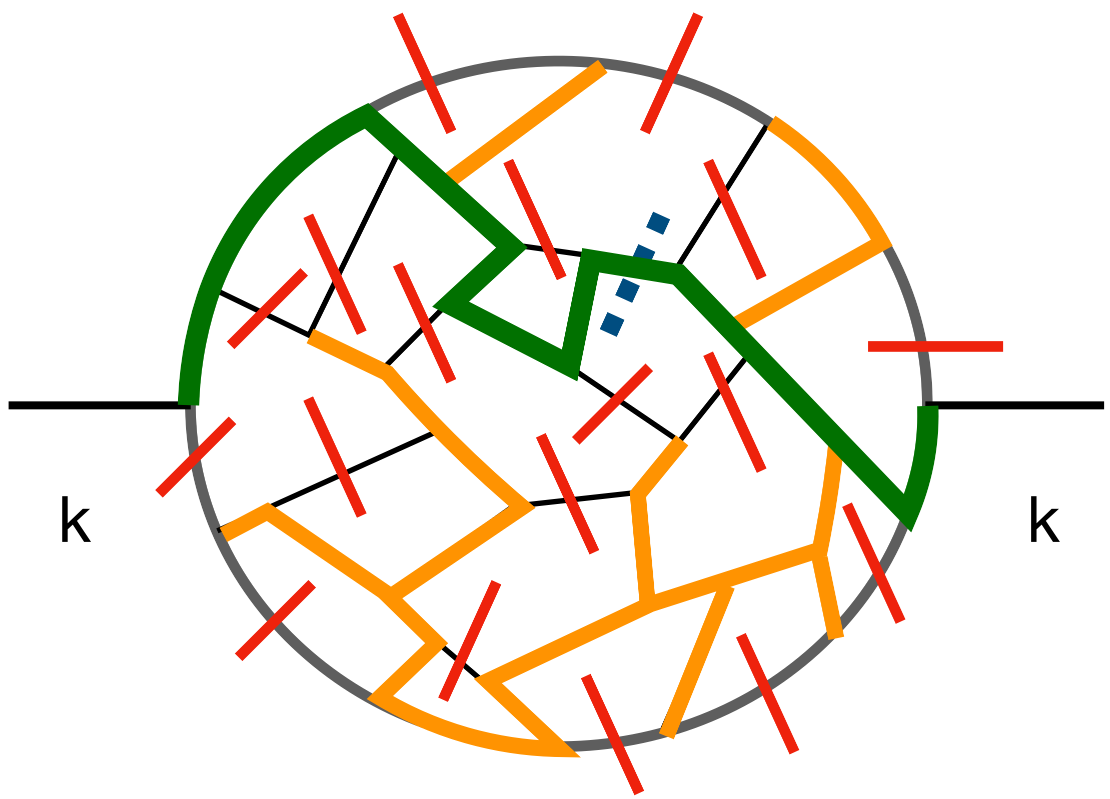

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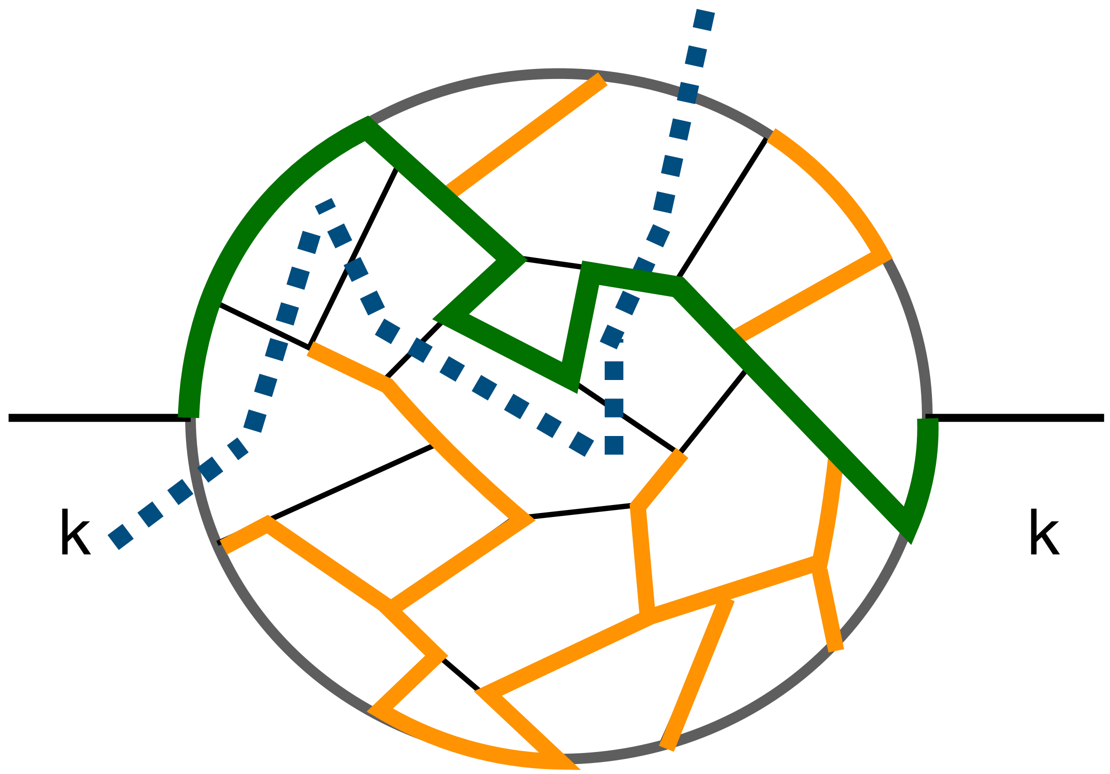

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UG, Gallegos, Zinnato `22

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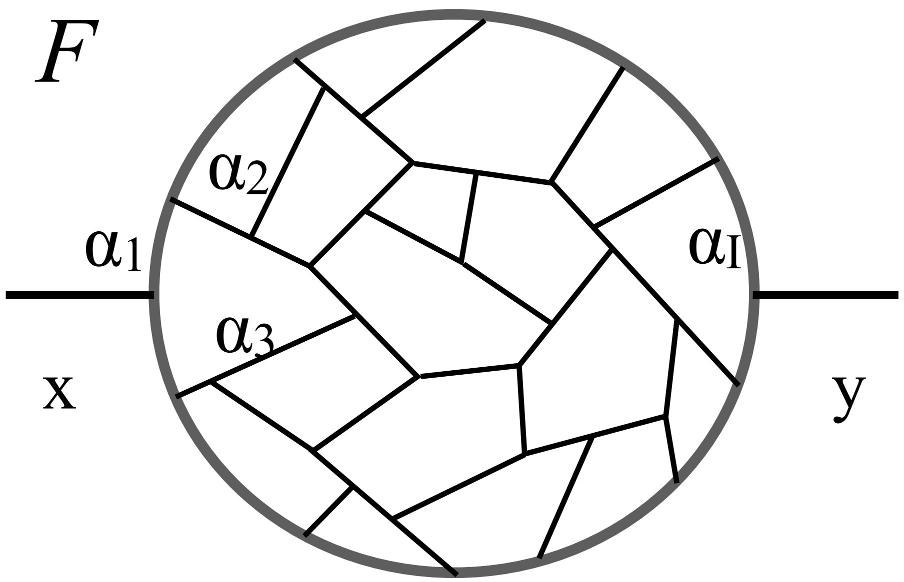
Part I: Target space

Two-point function

ℓ independent momenta and genus g



Define "scale dimension of graph":

$$\Delta = \left(\frac{d}{2} - \frac{h}{h-2}\right)\ell + \frac{d}{2} - 1$$

$$\Gamma_F(x,y) = \mathcal{V}_{\mathcal{F}} \int_0^\infty \frac{d\tau}{\tau}\,\tau^{-2-\Delta}\,e^{-\frac{|x-y|^2}{4\tau}}$$

$$\mathcal{V}_F(\Delta) \equiv \ell \int_0^\infty \left(\prod_{r=1}^{I} db_r\right) \delta\left(1-\mathcal{U}_F\right) \mathcal{A}_F^{\Delta+2-d/2} \qquad \alpha_r = \tau\ b_r$$

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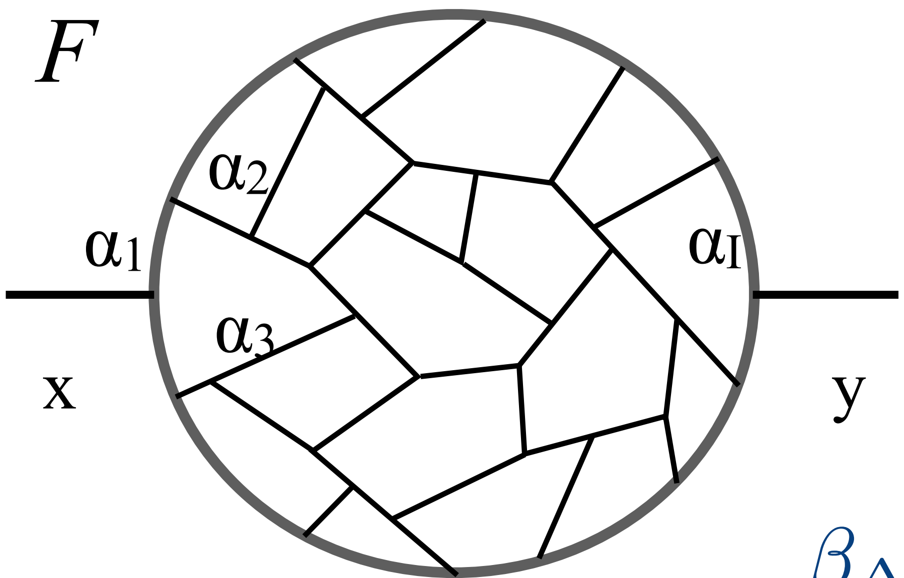

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$\beta_{\Delta+2}(x,y)$:bulk scalar with mass $M^2 = \Delta(d-\Delta)$ in AdS_{d+1}

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Overall proper time $\Longleftrightarrow$ holographic dimension

Gopakumar \`04

Two-point function

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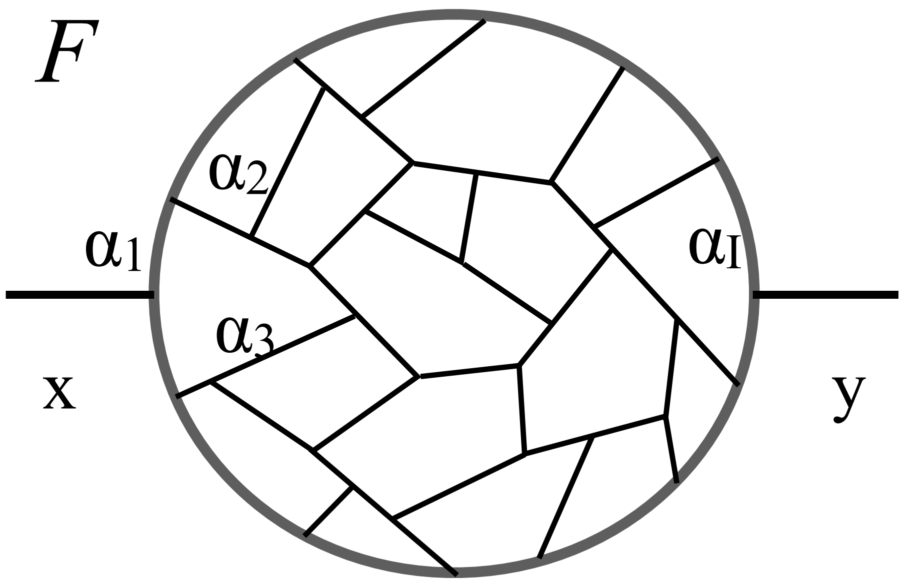

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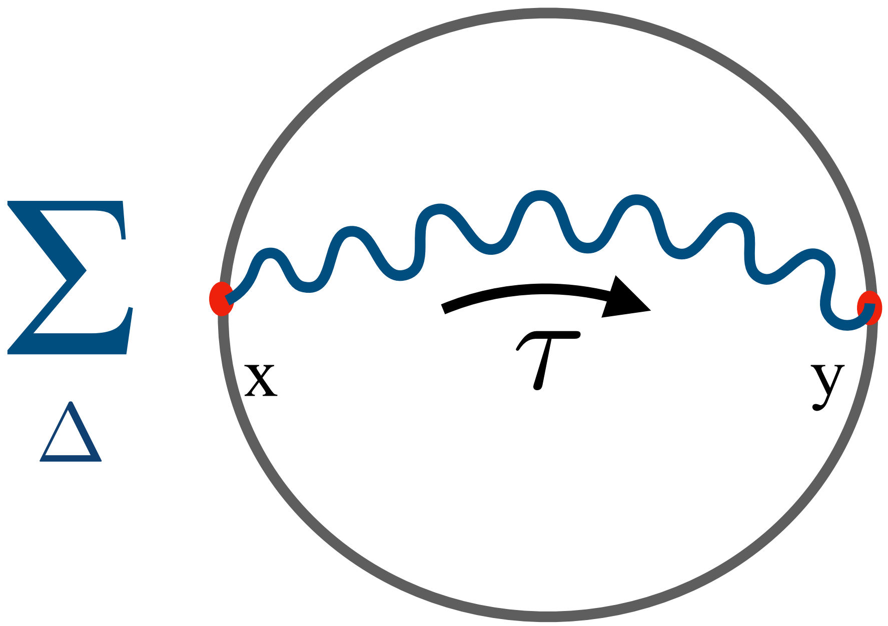

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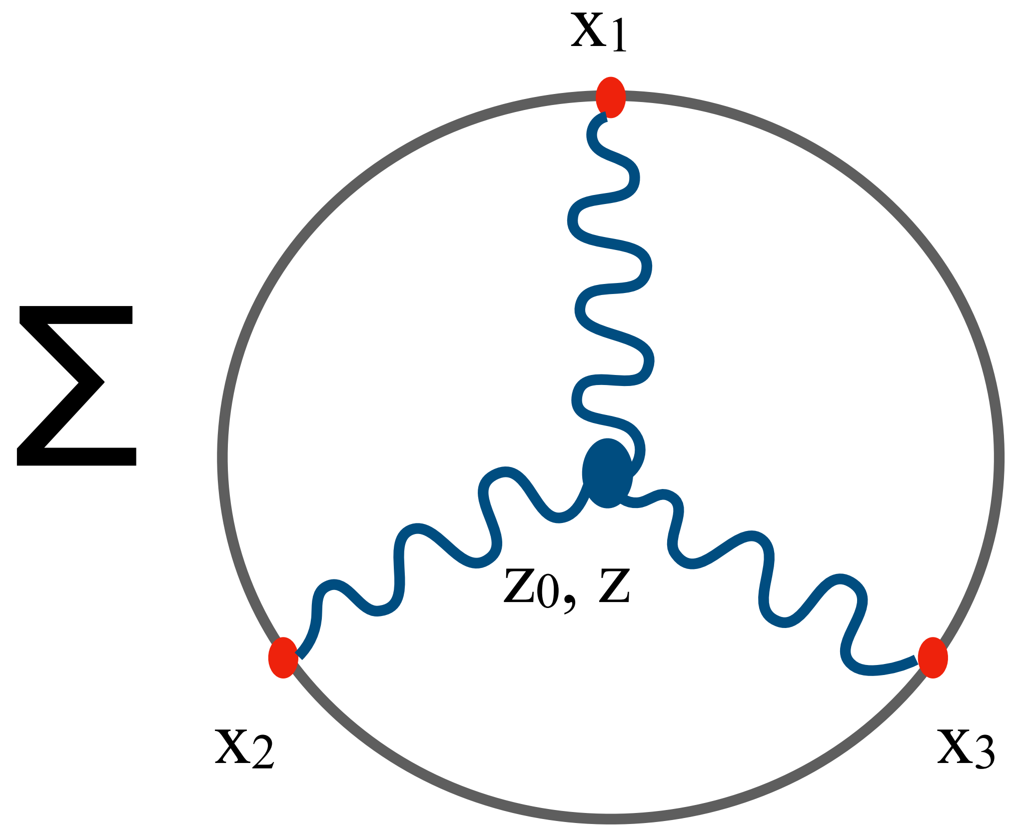
A novel form of Källen-Lehmann representation

Three point function

$$\Gamma_F(x_1,x_2,x_3) = \sum_{j_1,j_2,j_3=0}^{\infty} v^F_{j_1,j_2,j_3} \int \frac{d^{1+d}z}{\tau^{1+d}} \prod_{i=1}^{3} K_{j_i+\Delta+2}(\tau,z,x_i)$$

$$\Delta = \frac{2}{3}\left(\frac{d}{2} - \frac{h}{h-2}\right)\ell \\ + \frac{2}{3}\left(d + \frac{h-3}{h-2}\right)$$

Sum over Witten diagrams



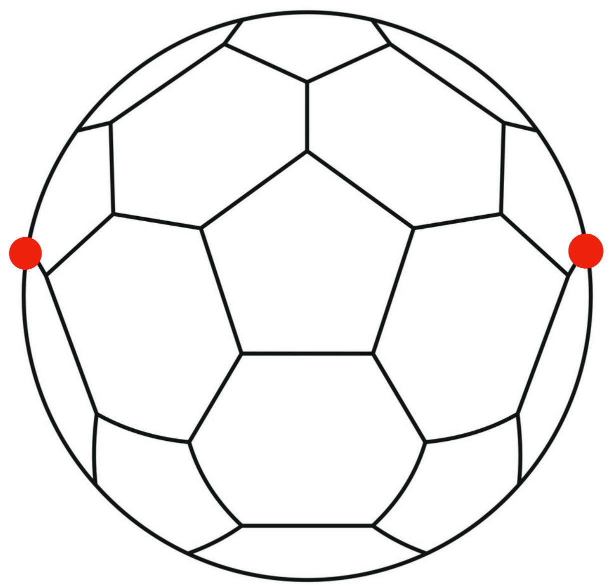
generalizing

Gopakumar `04

Part II: Worldsheet

Mapping QFT to string amplitudes

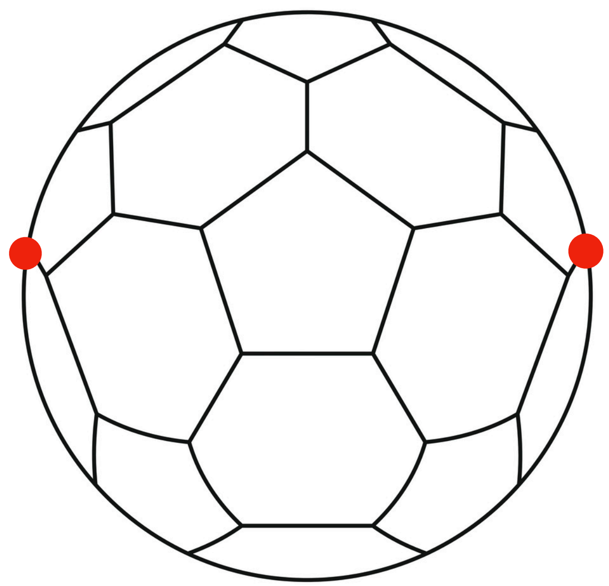
Weinzerl `22



$$\Gamma_F^{QFT}(k) \propto \prod_{r=1}^{I} \int_0^\infty d\alpha_r \; \det M(\alpha)^{-d/2} e^{k^2 \mathcal{G}(1,2)} \times e^{-\frac{1}{2}\sum_{i=1,2}(\mathcal{G}(i,i)+\Delta(i,i))}$$

$$\mathcal{G}(i,j) = -\bar{\sigma}_i \cdot M^{-1} \cdot \bar{\sigma}_j \,, \qquad M_{mn} = \sum_{r=1}^{I} \alpha_r \lambda_{rm} \lambda_{rn} \,, \qquad (\bar{\sigma}_i)_m = \sum_r \lambda_{rm} \sigma_{ri} \alpha_r \,, \qquad \Delta(i,j) = 2\sum_r \sigma_{ri}\sigma_{rj}\alpha_r$$

Mapping QFT to string amplitudes



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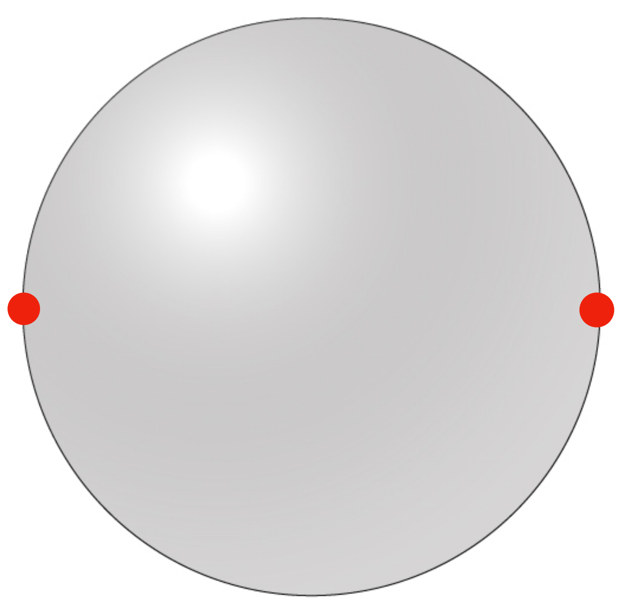

$\mathbf{R}^{\mathbf{d}}$

$$\Gamma_{S_2}^{\text{string}}(k,\sigma_i) \propto \left(\det \nabla^2\right)_{S_2}^{-\frac{d}{2}} e^{k^2 G(\sigma_1,\sigma_2)} \times e^{-\frac{1}{2}\sum_{i=1,2}\left(G(\sigma_i,\sigma_i)+\frac{\alpha'}{2}\log d^2(\sigma_i,\sigma_i)\right)}$$

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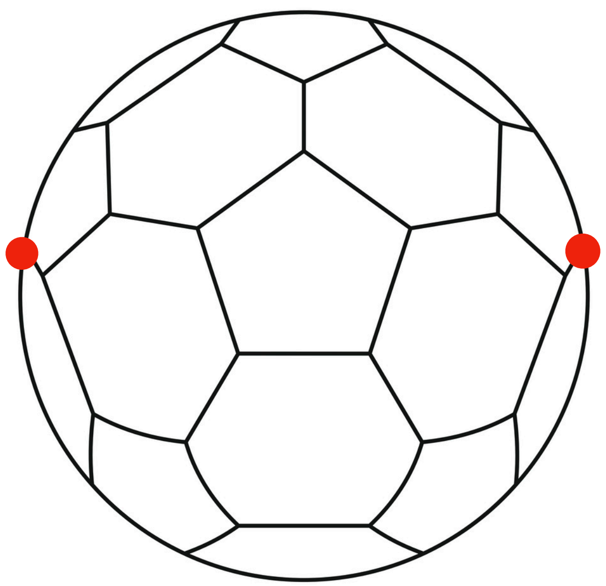

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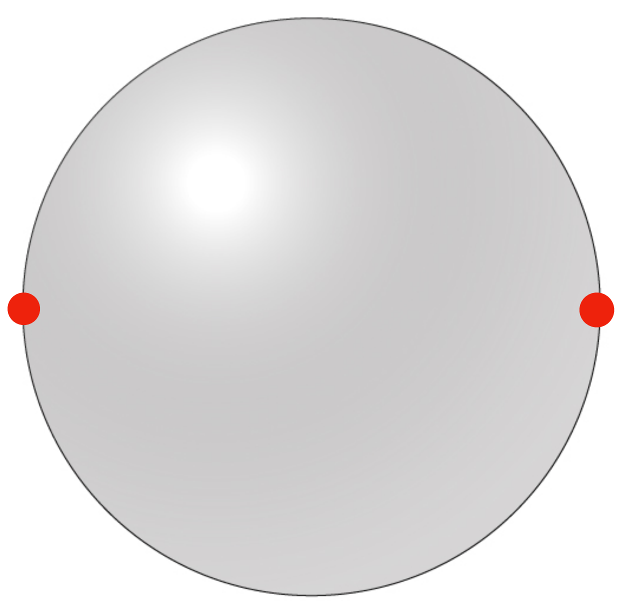

$\mathbf{R}^d$

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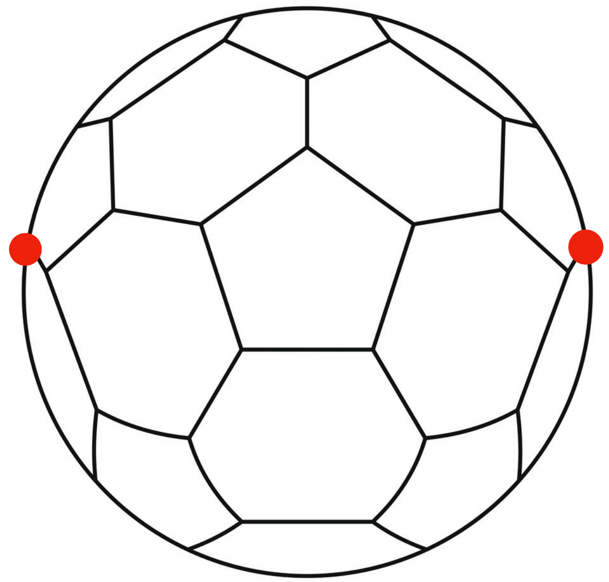

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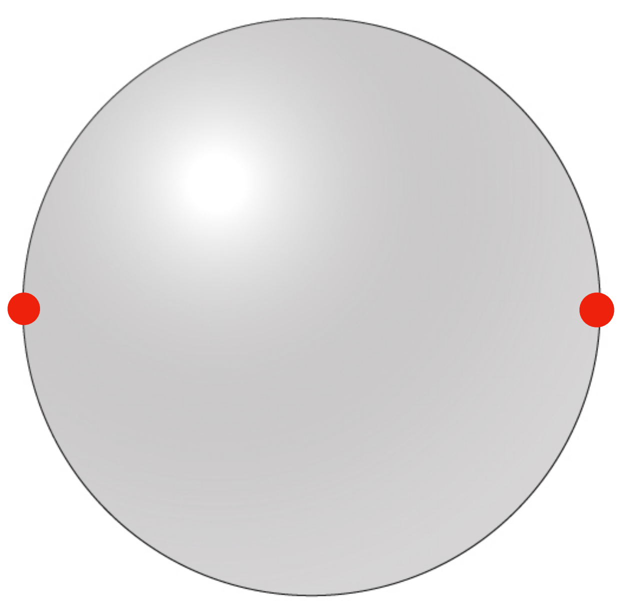

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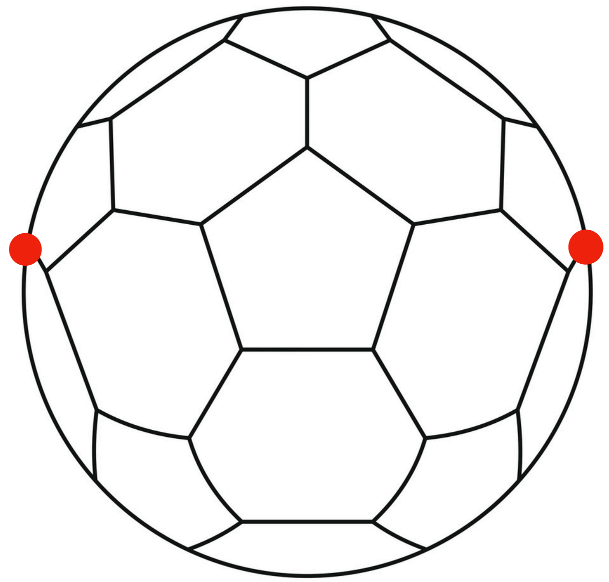
Mapping QFT to string amplitudes



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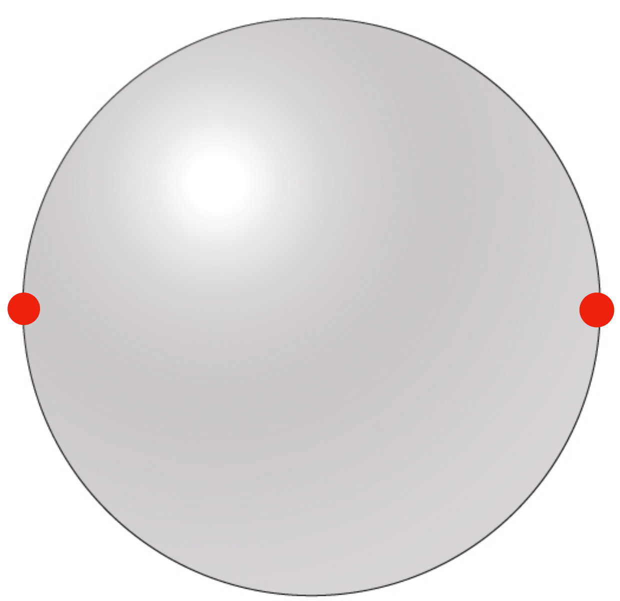

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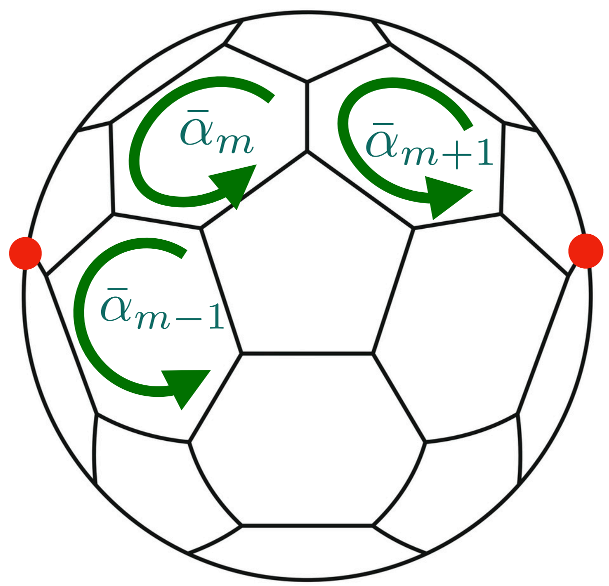

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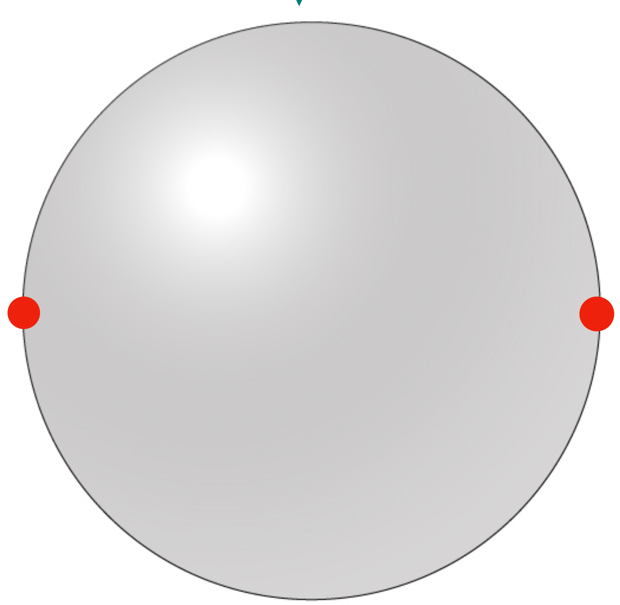
Continuum limit $\ell \to \infty$

$\bar{\alpha}_m \leftrightarrow e^{2A(\tau(\sigma))}$



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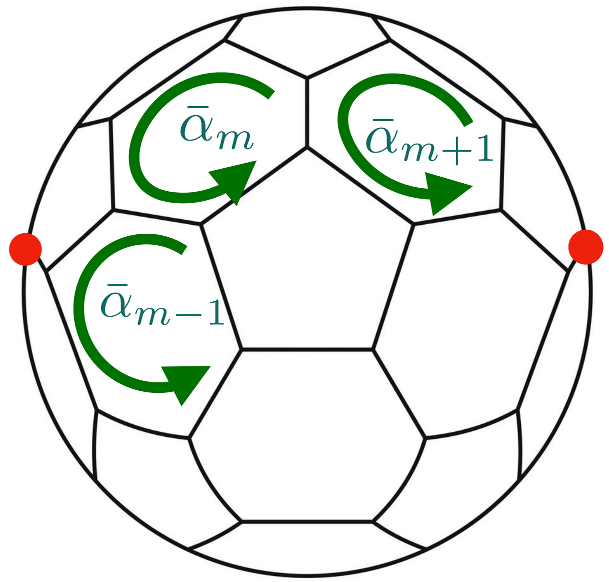
Mapping QFT to string amplitudes



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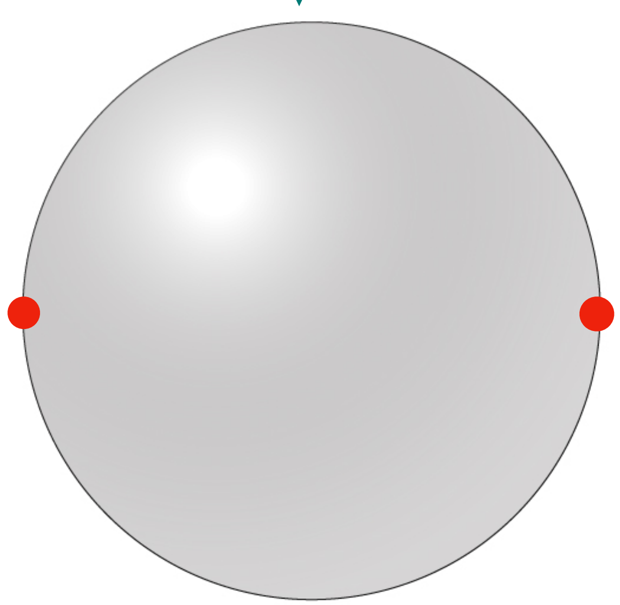

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Holographic coordinate emerges as continuum limit of Schwinger parameters

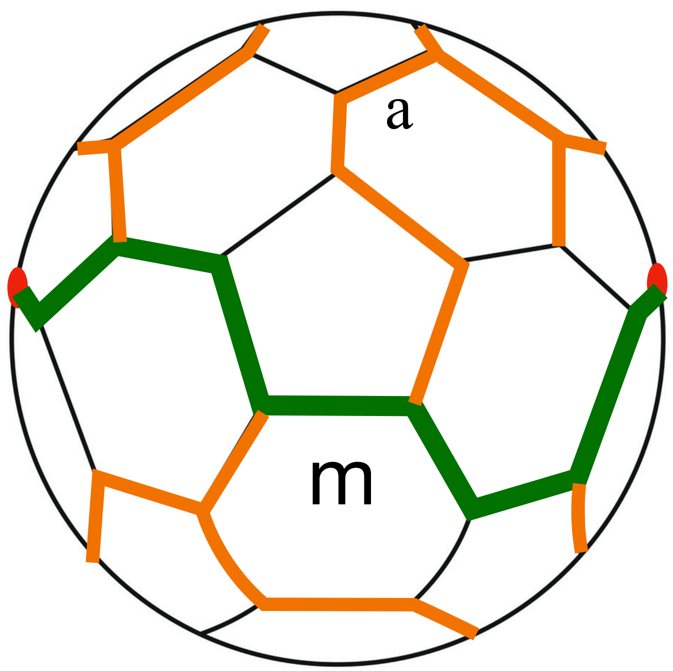
Gallegos, Zinnato, UG `22

generalizing `t Hooft `74

Part III: A primitive form of open-closed duality

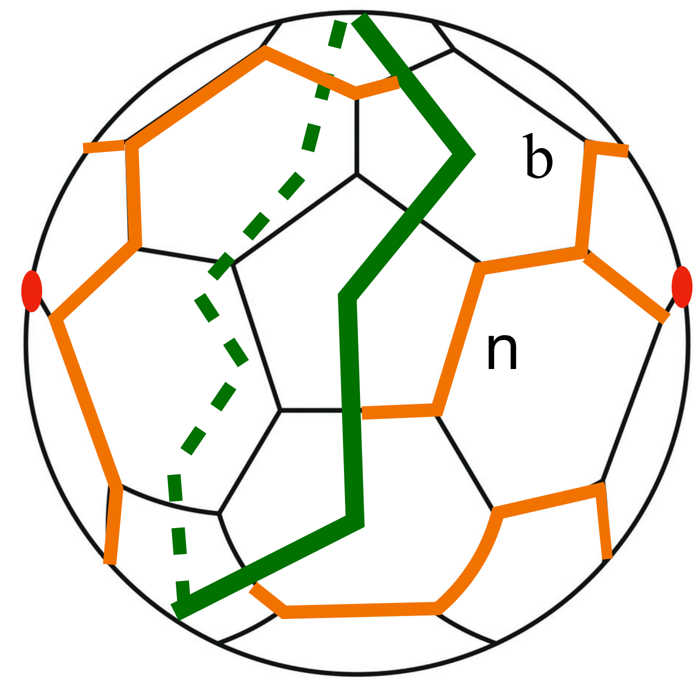
Symanzik polynomials as "stringy cuts"

$$\Gamma(k) = \int_0^\infty \prod_{r=1}^{I} d\alpha_r\, \mathcal{U}^{-d/2}(\alpha_r) \exp[-\mathcal{U}(\alpha_r)^{-1}\mathcal{A}(\alpha_r)k^2]$$



$$\mathcal{U} = \sum_{\mathcal{T}_1} \prod_{r=1}^{\ell} \alpha_r$$

"Open string" cuts



$$\mathcal{A} = \sum_{\mathcal{T}_2} \prod_{r=1}^{\ell+1} \alpha_r$$

"Closed string" cuts

A primitive form of open-closed duality

Compute $\Gamma_F(x,y)$ in two equivalent ways:

$$\Gamma_F(x,y) = \int d^d k\, e^{-ik\cdot(x-y)} \int \prod_r d\alpha_r\, \mathcal{U}_F^{-d/2} e^{-\frac{\mathcal{A}_F}{\mathcal{U}_F} k^2}$$

I. Compute Gaussian k integral, rescale $\alpha_r \to (x-y)^2 \alpha_r$

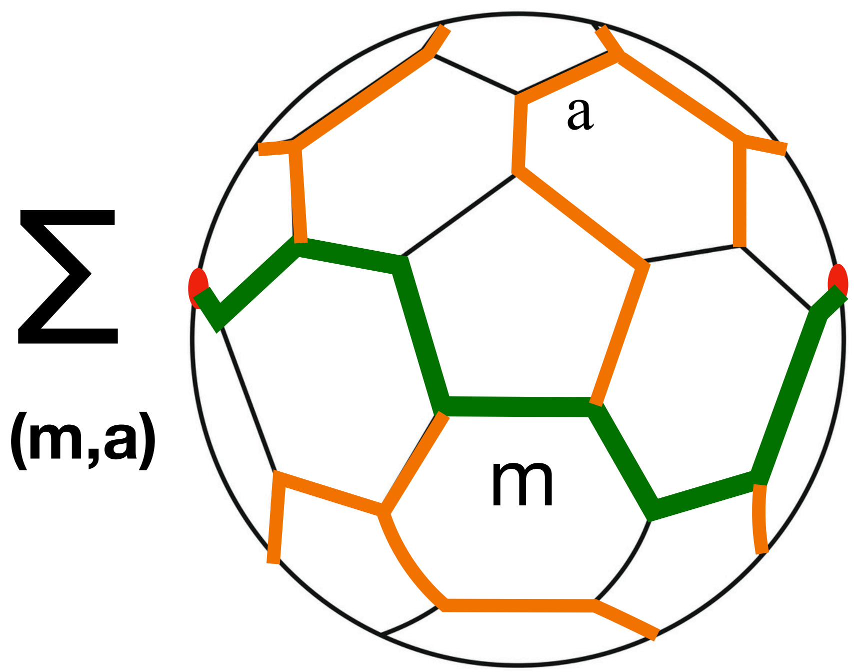
$$\Gamma_F(x,y) \propto |x-y|^{-2(\Delta+2)} \int \prod_r d\alpha_r\, \mathcal{A}_F^{-d/2} e^{-\frac{\mathcal{U}_F}{4\mathcal{A}_F}}$$

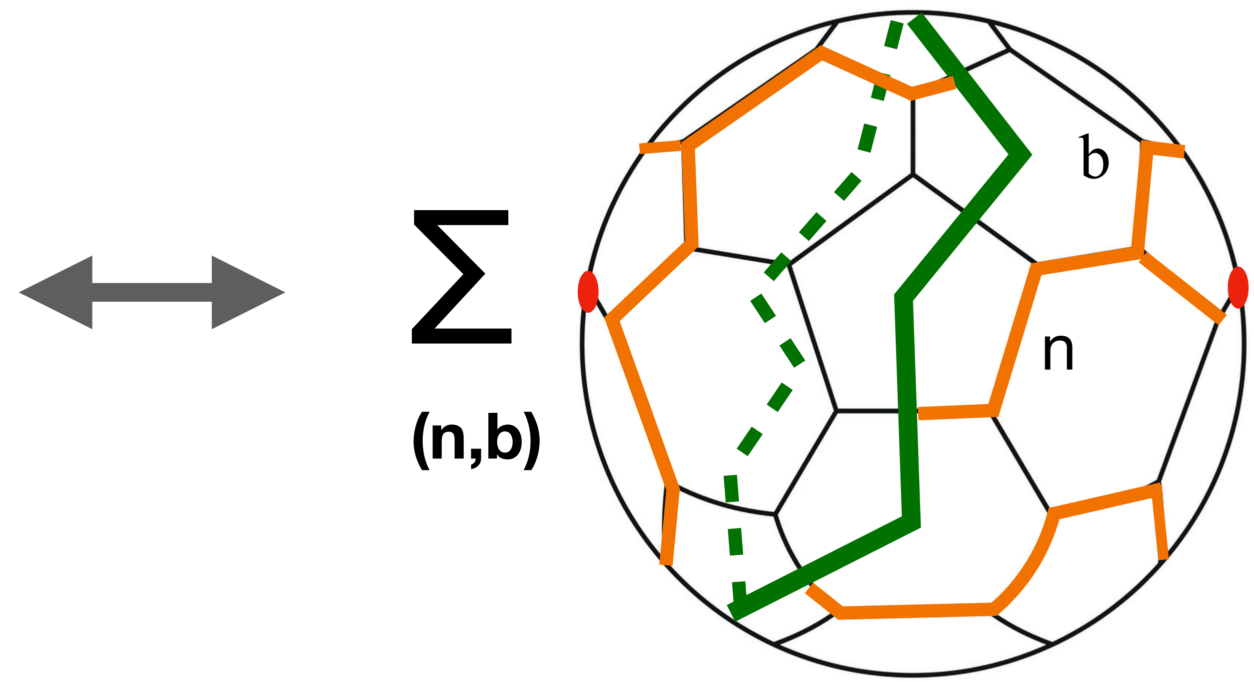
II. Rescale $\alpha_r \to \alpha_r/k^2$, carry out k-integral:

$$\Gamma_F(x,y) \propto |x-y|^{-2(\Delta+2)} \int \prod_r d\alpha_r\, \mathcal{U}_F^{-d/2} e^{-\frac{\mathcal{A}_F}{\mathcal{U}_F}}$$

We find a duality $\mathcal{A}_F \leftrightarrow \mathcal{U}_F$

A primitive form of open-closed duality





$$\mathcal{U} = \sum_{\mathcal{T}_1} \prod_{r=1}^{\ell} \alpha_r$$

"Open string" cuts

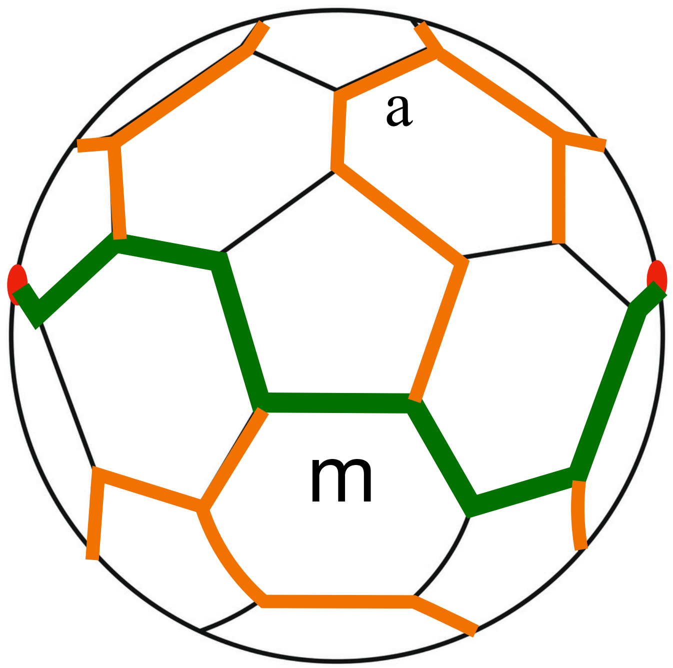
$$\mathcal{A} = \sum_{\mathcal{T}_2} \prod_{r=1}^{\ell+1} \alpha_r$$

"Closed string" cuts

Microscopic origin of holographic duality?

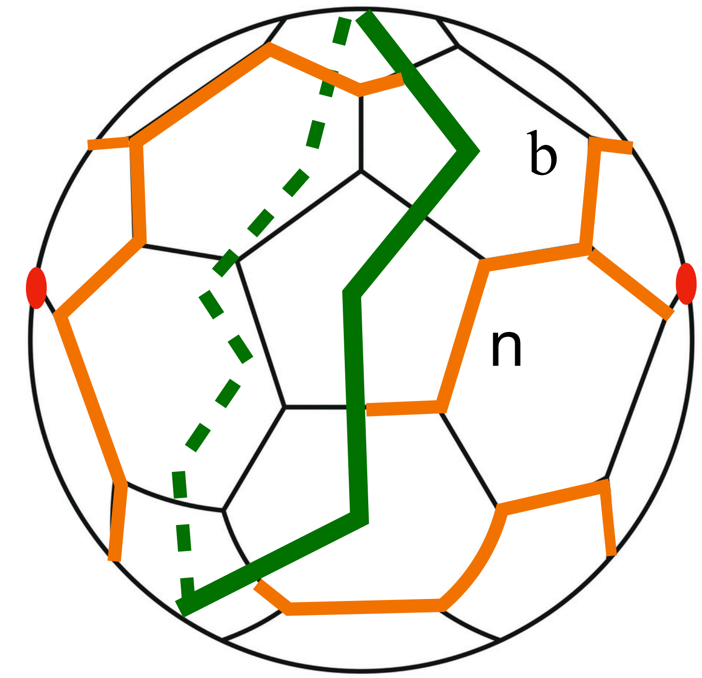
A primitive form of open-closed duality

$\sum$ **(m,a)**



$\longleftrightarrow$

$\sum$ **(n,b)**



$$\mathcal{U} = \sum_{\mathcal{T}_1} \prod_{r=1}^{\ell} \alpha_r$$

"Open string" cuts

$$\mathcal{A} = \sum_{\mathcal{T}_2} \prod_{r=1}^{\ell+1} \alpha_r$$

"Closed string" cuts

Microscopic origin of holographic duality?

Requires existence of continuum limit of Feynman diagrams

Further results

- Continuum limit necessitates critical `t Hooft coupling λ_c
- Value of λ_c determined by saddle point of Schwinger integrals
- Check 1: quantum $CFT_d \Longrightarrow AdS_{d+1}$
- Check 2: AdS_5 radius determined as $\lambda_c \sim \left(\frac{\ell_{AdS}}{\ell_s}\right)^4$

Gallegos, Zinnato, UG `22

Summary

- Witten diagrams from Feynman graphs

- Holographic coordinate from continuum limit of Schwinger parameters

- Primitive open-closed duality based on symmetry of interacting worldlines

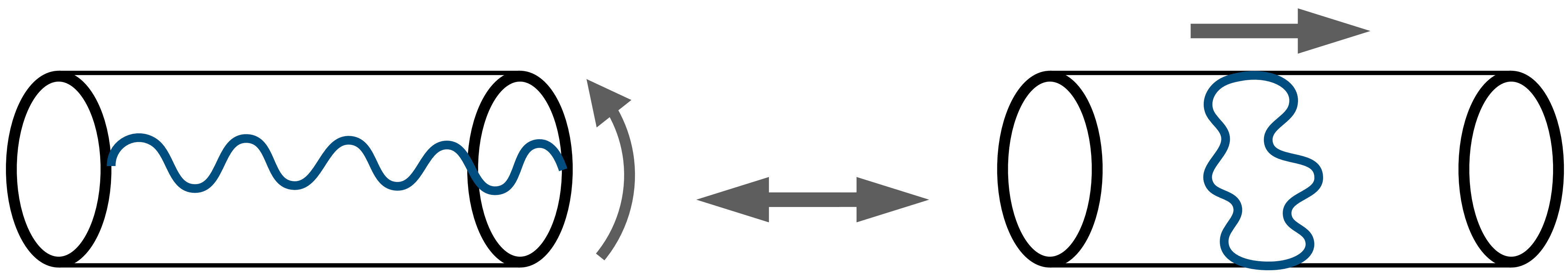
Summary

- Witten diagrams from Feynman graphs
- Holographic coordinate from continuum limit of Schwinger parameters
- Primitive open-closed duality based on symmetry of interacting worldlines

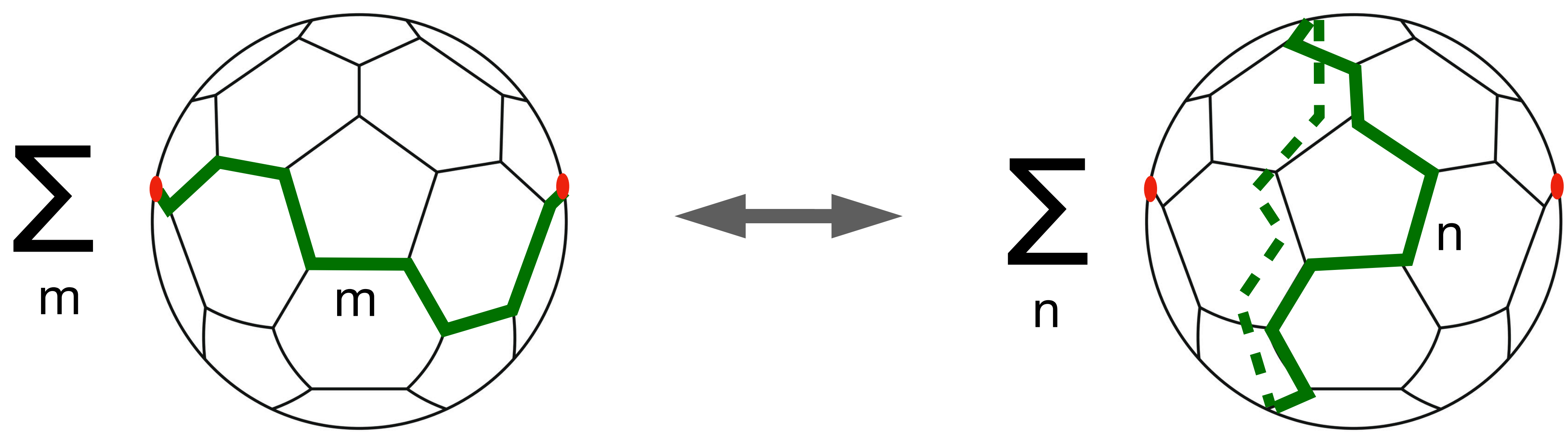
Outlook

- Derive target space geometry from the saddle of Schwinger integrals?
- A better controlled version of bottom-up holography? QCD?
- Worldline formalism works well for small N: non-perturbative string theory?

A primitive version of open-closed duality



Emerging from a duality of Feynman diagrams
in the worldline formalism



Based on: arXiv 2211.16514 w/ D. Gallegos and N. Zinnato

Regularization

Regularization and renormalization in the worldline formalism

Bogoulibov, Zimmerman, Epstein, Glaser, Hepp, Itzykson, Zuber, … `60s, `70s

UV divergences that arise from subgraphs σ regularized as:

$$\Gamma_F(k_i) = \int_0^\infty \prod_r da_r \lim_{\gamma\to 1} \prod_\sigma \left(1 - \mathcal{T}_{\gamma_\sigma}^{-2I_\sigma}\right) \left[\mathcal{U}(a_r)^{-d/2}\, e^{-P(a_r,k_i)}\right]$$

$$\mathcal{T}^k f(\rho) = \gamma^{-p_1} \sum_{s=0}^{k+p_1} \frac{\gamma^s}{s!} \frac{d^s}{d\gamma^s} \left[\gamma^{p_1} f(\gamma)\right]_{\gamma=0} \quad \rightarrow \quad \cdots - \sum_{i=1}^{i_F} (k^2)^{n_i} F_i(a)$$

counterterms

Example: two-point function

$$\Omega_F(k_1) = \delta(k_1 + k_2)\, \mathcal{V}_F^R \int_0^\infty \frac{d\tau}{\tau} \tau^{-2-\Delta+\frac{d}{2}} e^{-\tau k_1^2}\,, \qquad \mathcal{V}_F^R = \frac{1}{\Gamma(\frac{d}{2} - 2 - \Delta)} \int_0^\infty \prod_{r=1}^{I} da_r \left[\cdots - \sum_{i=1}^{i_F} (k^2)^{n_i} F_i(a)\right]\Big|_{k\to 1}$$

$\Longrightarrow$ Earlier expressions hold with the replacement $\quad \mathcal{V}_F \rightarrow \mathcal{V}_F^R$

Continuum limit

Claim:
- Continuum limit exists at and only at a critical coupling $\lambda_h = \lambda_c$
- Value of λ_c is determined by extremizing over Schwinger parameters

Reconsider the two-point function:

$$\Omega(z,k) = \sum_{\ell} (k^2)^{\Delta+2-\frac{d}{2}} z^{-\ell} v_{\ell}\,, \qquad z = \lambda_h^{-2/(h-2)}$$

Inverse unilateral z-transform:

$$v_{\ell} = \sum_{F\in F_{\ell}} \frac{1}{\sigma_F} \prod_{r=1}^{I} \int_0^{\infty} da_r \frac{e^{-\frac{\mathcal{A}_F(a)}{\mathcal{U}_F(a)}}}{\mathcal{U}_F(a)^{d/2}}$$

$$(k^2)^{\Delta+2-\frac{d}{2}} = \frac{1}{v_{\ell}\, 2\pi i} \oint_{\mathcal{C}} dz\, \Omega(z,k) z^{\ell-1} \qquad \Delta = \left(\frac{d}{2} - \frac{h}{h-2}\right)\ell + \frac{d}{2} - 1$$

In the marginal case:

$$v_{\ell} = v_0(z_c) z_c^{\ell}$$

Continuum limit

Claim: Continuum limit exists at and only at a critical coupling $\lambda_h = \lambda_c$

$$\Omega(z,k) = (k^2)^{\Delta+2-\frac{d}{2}} \left(\sum_{\ell<\ell_{cut}} + \sum_{\ell\geq\ell_{cut}} \right) \lambda_h^{\frac{2\ell}{h-2}} v_\ell \,,$$

$\Omega_{\text{non-geo}}$ Ω_{geo}

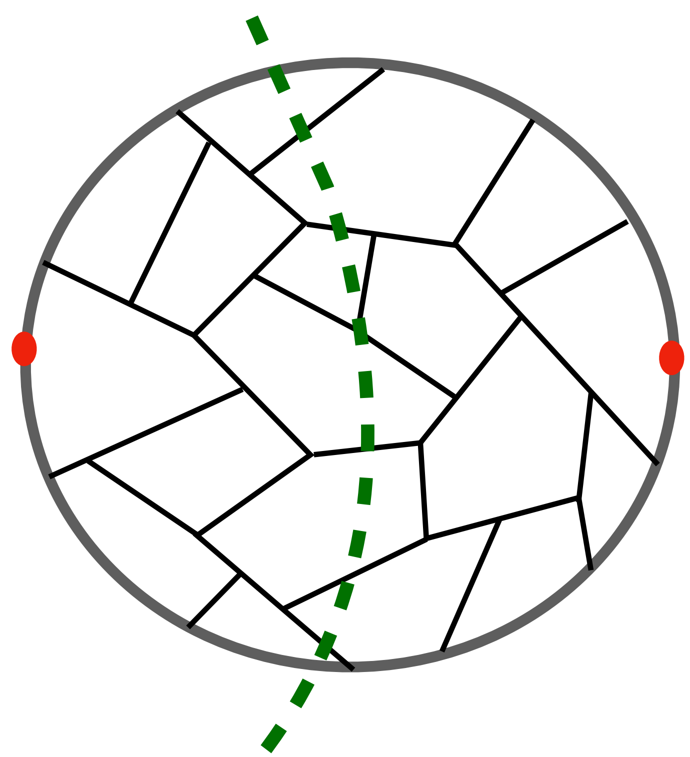
As $\ell_{cut} \to \infty$ $\quad \Omega_{\text{geo}} \propto \sum_{\ell\geq\ell_{cut}} \left(\frac{\lambda_h}{\lambda_c}\right)^{\frac{2\ell}{h-2}}$ finite only at criticality

In the continuum limit $\Omega_{\text{geo}}(\lambda_c)$ while full $\Omega(\lambda_c, \lambda_h)$

Full marginal 2pf: $\quad \Omega = k^2 \dfrac{v_0(z_c)}{1 - \left(\frac{\lambda_h}{\lambda_c}\right)^{\frac{2}{h-2}}}$ always a single pole

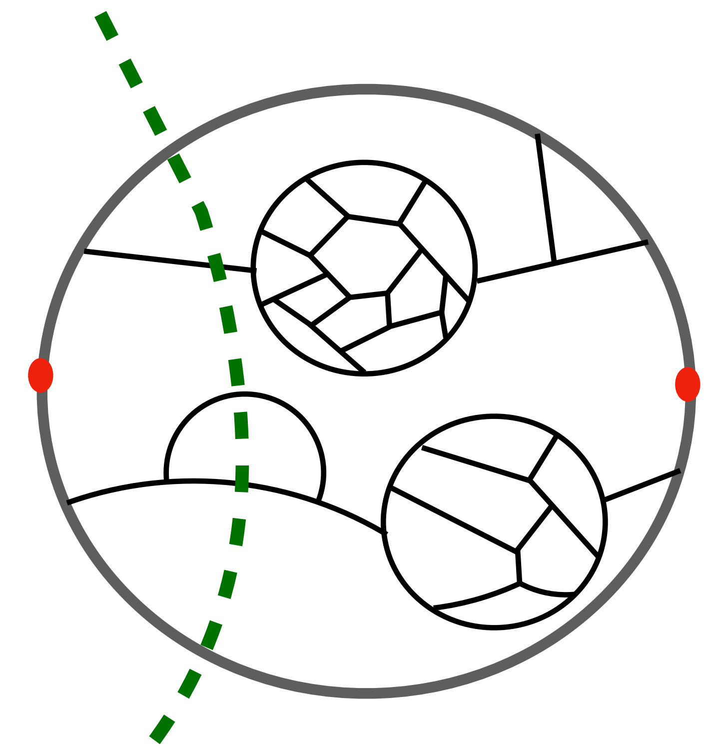
Continuum limit

Uniform: O(I) particle reducible



Dominate the continuum limit

Non-uniform: n≪I particle reducible



Sum over $\mathcal{F}_\ell$ $\implies$ emergent S(I) permutation symmetry of $\{a_m\}$

$\implies$ emergent worldsheet diffeomorhphism in the $\ell \to \infty$ limit

$\implies$ Dominant saddle of $\{a_m\}$ integrals $\implies$ $a_m = a_c$, $\forall$ m

Continuum limit

Claim: Value of λ_c is determined by extremizing over Schwinger parameters ✓

$$v_\ell = \sum_{F \in F_\ell} \frac{1}{\sigma_F} \prod_{r=1}^{I} \int_0^\infty da_r \, \frac{e^{-\frac{\mathcal{A}_F(a)}{\mathcal{U}_F(a)}}}{\mathcal{U}_F(a)^{d/2}}$$

For $a_r = a_c$ $\qquad \mathcal{U}_F \to N_1 a_c^\ell \,, \qquad \mathcal{A}_F \to N_2 a_c^{\ell+1}$

A rough estimate: $\qquad \lim_{\ell \to \infty} v_\ell \sim N_F(\ell)\, a^{-d\ell/2} e^{-a\kappa\sqrt{\ell}}$

Recalling $\qquad v_\ell = v_0(z_c) z_c^\ell$

$$\lambda_c \sim a_c^{h/2}$$

Related to curvature scale in the dual background

Emerging geometry - AdS

Consider a CFT:

$$\Gamma_F(k_1,k_2) \propto \delta^d(k_1+k_2) \prod_{r=1}^{I} \int_0^\infty da_r\, \mathcal{U}_F(a)^{-d/2} e^{-\frac{\mathcal{A}_F(a)}{\mathcal{U}_F(a)} k_1^2}$$

Scale symmetry:

$$k^\mu \to k^\mu \Lambda\,, \qquad x^\mu \to x^\mu/\Lambda \qquad a_r \to a_r \Lambda^2$$

Dual geometry: $$ds^2 = e^{2A(R)} \left(dR^2 + \delta_{\mu\nu} dx^\mu dx^\nu\right)$$

Recall $$a_r \leftrightarrow e^{2A(R)}$$

$$A(R) \sim -\log(R/\ell_{AdS})$$

Requires full quantum scale invariance, counterterms would violate

Emerging geometry - AdS

$$k^\mu \to k^\mu \Lambda\,, \qquad x^\mu \to x^\mu/\Lambda \qquad a_r \to a_r \Lambda^2$$

Relation between $\lambda_c \sim a_c^{h/2}$ and curvature radius:

$$\lambda_c = a_c^2 \left(\frac{\Lambda_{UV}}{E}\right)^4 \sim e^{4A}\left(\frac{\Lambda_{UV}}{E}\right)^4 \sim \left(\frac{\ell_{AdS}\Lambda_{UV}}{R\,E}\right)^4 \sim \left(\frac{\ell_{AdS}}{\ell_s}\right)^4$$

using R ~ 1/E

Recall in ordinary AdS/CFT from IIB this comes from D3 brane form factor:

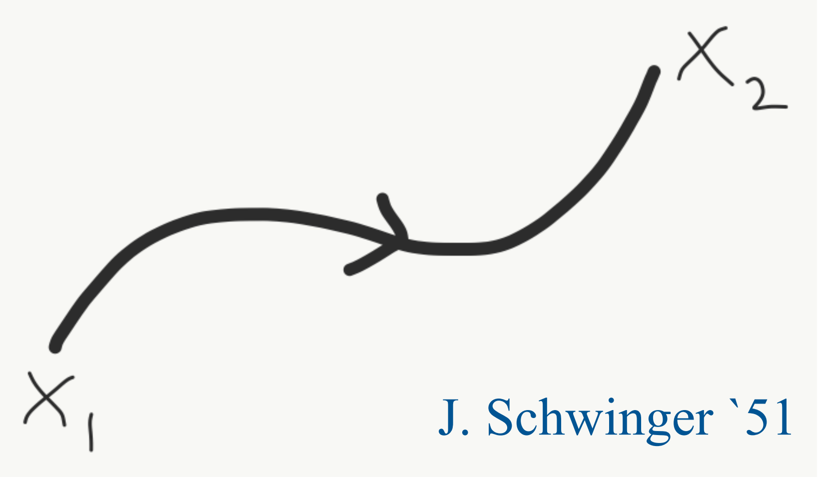
$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2)$$
$$F_5 = (1 + *)dt dx_1 dx_2 dx_3 df^{-1}\,,$$
$$f = 1 + \frac{L^4}{r^4}\,, \qquad L^4 \equiv 4\pi g_s \alpha'^2 N\,.$$

Worldline formalism of QFT

R. Gopakumar `03

Worldline formalism of QFT:

$$\langle \phi(x_1)\phi(x_2)\rangle = i\int d^4k \frac{e^{ik(x_1-x_2)}}{k^2+m^2-i\epsilon}$$
$$= \int_0^\infty d\tau \langle x_1|e^{-i\tau(\partial^2+m^2)}|x_2\rangle$$

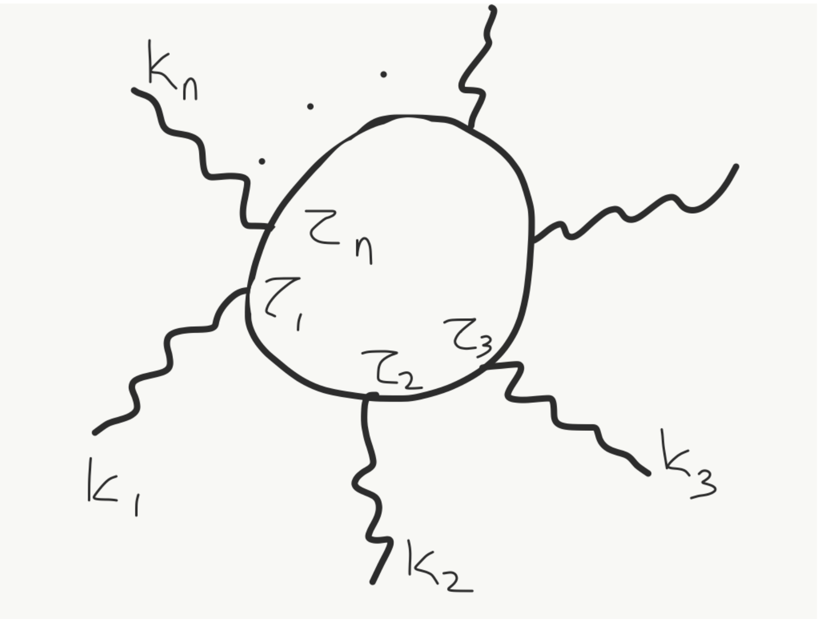


More generally:

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \int_0^\infty \frac{d\tau}{\tau}\prod_{i=1}^{n} d\tau_i \, \langle e^{ik_1\cdot\hat{X}(\tau_1)}\dots e^{ik_n\cdot\hat{X}(\tau_n)}\rangle_{q.m.}$$

$\underbrace{\qquad\qquad}$

Integral over moduli



Marginal case

Φ^h interaction non-renormalizable for $h > 2d/(d-2)$

$\Delta = \left(\frac{d}{2} - \frac{h}{h-2}\right)\ell + \frac{d}{2} - 1 \implies$ M = Δ(d-Δ) arbitrarily -ve for $h < 2d/(d-2)$

$\implies$ A simple class: marginal theories $h = 2d/(d-2)$

(d,h) = (2,∞), (3,6), (4,4), (6,3), (∞,2)

$$\Gamma(x,y) = \frac{C}{|x-y|^{d+2}}$$

$$C = 4^{\frac{d}{2}+1}\pi^{d/2}\Gamma(\tfrac{d}{2}+1)\sum_g N^{2-2g}\sum_{\ell=1}^{\infty}\lambda^{\ell\frac{d-2}{2}}\,\ell\sum_{F\in\mathcal{F}_{\ell,g}}\frac{\mathcal{V}_F(\Delta)}{\sigma_F}$$

Composite operators

Immediate to generalize to composite operators $\mathrm{tr}\,\Phi^J$

$$\Delta_J = \left(\frac{d}{2} - \frac{h}{h-2}\right)\ell + \frac{2J-h}{h-2} + \frac{d}{2} - 2\,.$$

In the marginal case:

$$\langle \mathrm{tr}\,\Phi^J(x)\,\mathrm{tr}\,\Phi^J(y)\rangle = \frac{C_J}{|x-y|^{\frac{4J}{h-2}}}$$

$$C_J = 4^J \pi^{d/2}\Gamma(J) \sum_g N^{2-2g} \sum_{\ell=1}^{\infty} \lambda^{\frac{2(\ell+1-J)}{h-2}}\,\ell \sum_{F\in\mathcal{F}_{\ell,g}} \frac{\mathcal{V}_F(\Delta)}{\sigma_F}$$

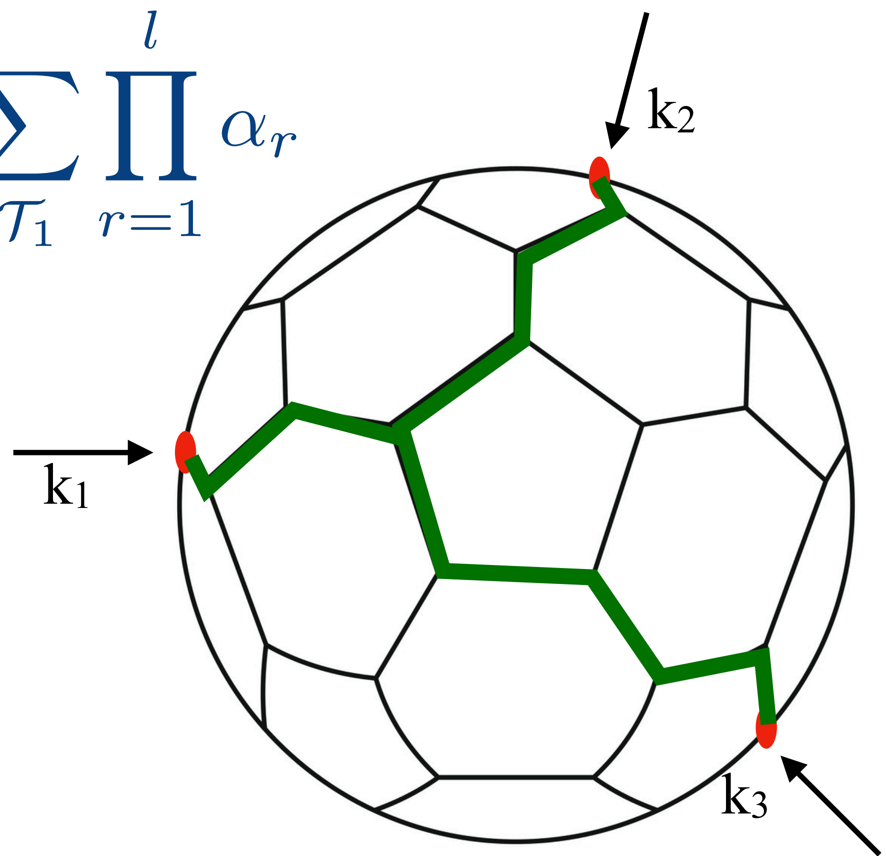
Three point function

Much more involved to generalize to $\langle \Phi(k_1)\Phi(k_2)\Phi(k_3)\rangle$

Consider the massless theory m = 0

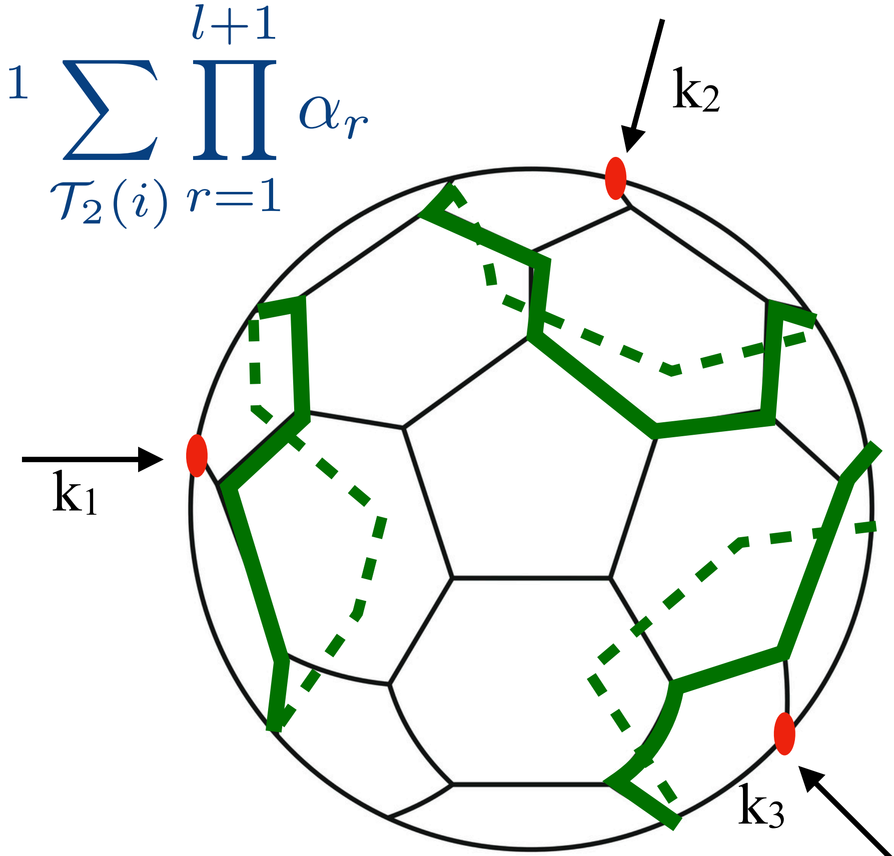
$$\Gamma(k_1,k_2,k_3) = \delta(k_1+k_2+k_3)\int_0^\infty \prod_{r=1}^{I} d\alpha_r\, \mathcal{U}^{-d/2}(\alpha_r)\exp[-\sum_i \frac{\mathcal{A}_i(\alpha_r)}{\mathcal{U}(\alpha_r)}k_i^2]$$

$$\mathcal{U} = \sum_{\mathcal{T}_1}\prod_{r=1}^{l}\alpha_r$$



“Open string” cuts

$$\mathcal{A}_i = \mathcal{U}^{-1}\sum_{\mathcal{T}_2(i)}\prod_{r=1}^{l+1}\alpha_r$$



“Closed string” cuts