Bootstrapping the AdS Virasoro-Shapiro amplitude

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Strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$

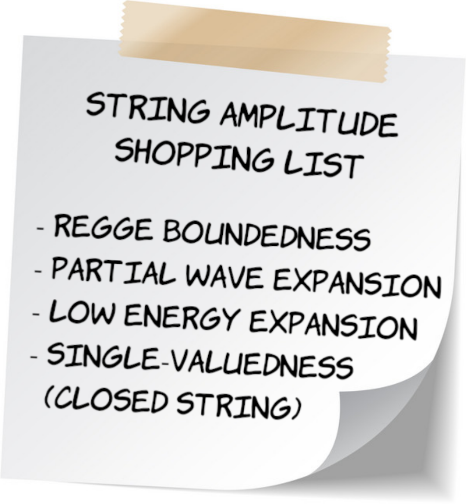
type IIb string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ = $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group

What is the (usable) worldsheet theory?

What is the 4pt tree level string amplitude?

Can we bootstrap it from target space arguments?

Flat space review



I will review these first for
the Virasoro-Shapiro amplitude
(4 gravitons in the type IIb superstring):

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\,\Gamma(T+1)\,\Gamma(U+1)}$$

$$S = -\frac{\alpha'}{4}(p_1+p_2)^2\,, \qquad T = -\frac{\alpha'}{4}(p_1+p_3)^2$$

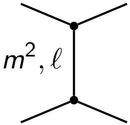
$$S+T+U=0$$

Regge boundedness (flat space)

String amplitudes have soft UV (Regge) bahaviour

$$\lim_{|S|\to\infty} A^{(0)}(S,T) \sim S^{\alpha' T + \alpha_0}$$

and higher spin resonances



$$= \frac{P_\ell(S)}{T^2 - m^2} \qquad\qquad P_\ell(S) = S^\ell + O(S^{\ell-1})$$

Regge bahaviour places strong constraints on the coefficients $a_{\delta,\ell}$ in

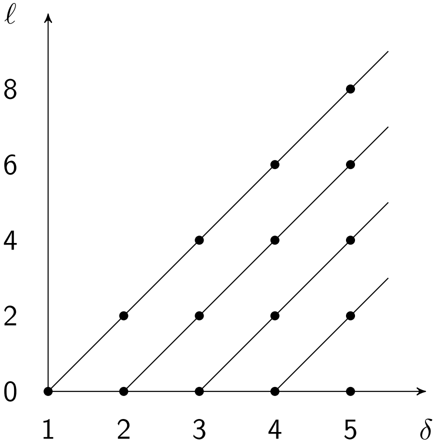
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T^2 - \delta}$$

The spectrum (flat space)

The exchanged massive string spectrum is extracted via the partial wave expansion

$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T^2 - \delta}$$

It forms linear Regge trajectories.



Single-valuedness (flat space)

The sphere worldsheet integrand is (and has to be) single-valued:

$$A^{(0)}(S,T) = -\frac{(S+T)^{-2}}{2\pi i}\int |z|^{-2S-2}|1-z|^{-2T-2}dzd\bar{z}$$

This implies that the Wilson coefficients $\alpha^{(0)}_{a,b}$ in the low energy expansion

$$A^{(0)}(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty}(\tfrac{1}{2}(S^2+T^2+U^2))^a(STU)^b\alpha^{(0)}_{a,b}$$

are single-valued multiple zeta values [Stieberger;2013],[Brown,Dupont;2018]

Example: $\alpha^{(0)}_{a,0} = \zeta(3+2a)\,, \qquad \alpha^{(0)}_{a,1} = \sum_{\substack{i_1,i_2=0\\ i_1+i_2=a}}^{a}\zeta(3+2i_1)\zeta(3+2i_2)$

The AdS amplitude

4 graviton amplitude in $AdS_5 \times S^5 \leftrightarrow \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$ in $\mathcal{N} = 4$ SYM theory at

$$g_s \ll \alpha'/R^2_{\text{AdS}} \ll 1 \qquad \Leftrightarrow \qquad N \gg \sqrt{\lambda} \gg 1$$

$\mathcal{O}_2$ = superconformal primary of stress-tensor multiplet

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$$

$\downarrow$ superconformal Ward identity

$$H(u, v)$$

$\downarrow$ Mellin transform

$$M(s, t)$$

$\downarrow$ Borel transform (flat space limit [Penedones;2010])

$$A^{(0)}(S, T) + \frac{1}{\sqrt{\lambda}} A^{(1)}(S, T) + \dots$$

$M(s,t)$ has only OPE poles:

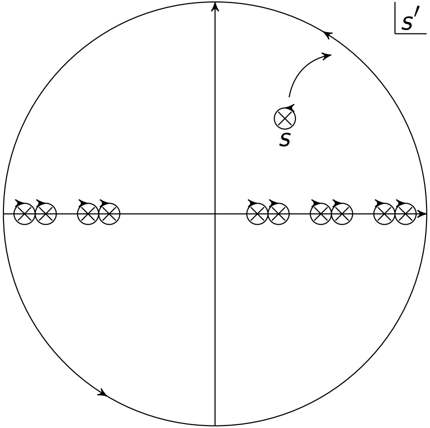
$$\text{poles} \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s' - (\Delta - \ell + 2m)}$$

[Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Regge bounded due to bound on chaos:

$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}$$

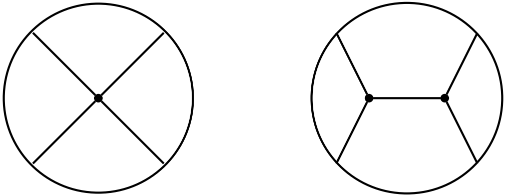
[Maldacena,Shenker,Stanford;2015]



$$M(s,t) = \oint_s \frac{ds'}{2\pi i} \frac{M(s', -s'-u)}{(s'-s)} = \sum_{\text{operators}} f(s,t,\text{OPE data})$$

The low energy expansion

The low energy (or large λ) expansion is an expansion into tree level Witten diagrams.



The corrections to SUGRA are polynomials in s, t, u.

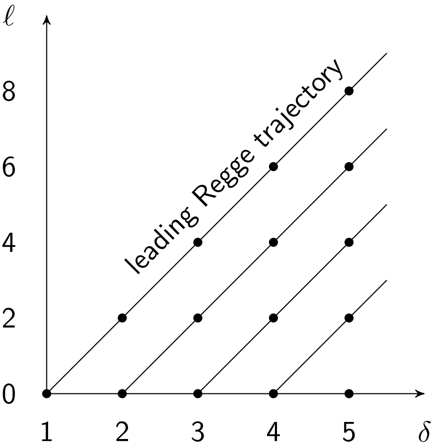
$$M(s,t) = \text{SUGRA} + \sum_{a,b=0}^{\infty} \Gamma(2a+3b+6) \left(\frac{s^2+t^2+u^2}{8\lambda} \right)^a \left(\frac{stu}{8\lambda^{\frac{3}{2}}} \right)^b \left(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \cdots \right)$$

Combining this with the dispersion relation gives:

$$\alpha_{a,b}^{(k)} = \sum_{\text{operators}} F(\text{OPE data})$$

Data in the dispersive sum rules

Exchanged operators: short single-trace operators of $\mathcal{N}=4$ SYM theory



	$A^{(0)}$ data		$A^{(1)}$ data
	$\alpha^{(0)}_{a,b}$	+	$\lambda^{-\frac{1}{2}}\alpha^{(1)}_{a,b}$
$\Delta_{\delta,\ell} =$	$\lambda^{\frac{1}{4}}\Delta^{(0)}_{\delta,\ell}$	+	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$
$C^2_{\delta,\ell} =$	$C^{2(0)}_{\delta,\ell}$	+	$\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}$

known from flat space (→ $A^{(0)}$ data)

2 known from localisation [Chester,Pufu;2020] (→ $\lambda^{-\frac{1}{2}}\alpha^{(1)}_{a,b}$)

leading trajectory known from integrability [Gromov,Serban, Shenderovich,Volin; 2011] (→ $\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$)

related by sum rules

$$\alpha^{(k)}_{a,b} = \sum_{\delta,\ell} F(\text{OPE data}) = \sum_{\delta=1}^{\infty} \text{nested sums} = \text{MZVs}$$

Single-valued multiple zeta values

multiple polylogs $\log(1-z), \mathrm{Li}_n(z), \ldots$ $\xrightarrow{z=1}$ MZVs $\zeta(n_1, n_2, \ldots)$

$\downarrow$ single-valued combinations

sv multiple polylogs [Brown;2004] $\xrightarrow{z=1}$ sv MZVs [Brown;2013]

$\zeta(2n+1)$ are single-valued, $\zeta(2n)$ are not.

Example at weight 6:

MZV basis:	$\zeta(3)^2, \zeta(2)^3$	$\zeta(3,2,1) = 3\zeta(3)^2 - \frac{29}{30}\zeta(2)^3$
sv MZV basis:	$\zeta(3)^2$	$\zeta^{\mathrm{sv}}(3,2,1) = 12\zeta(3)^2$

Solving the sum rules

The sum rule for $A^{(1)}(S, T)$ has unknown data on both sides

$$\alpha_{a,b}^{(1)} = \sum_{\delta,\ell} F(\Delta_{\delta,\ell}^{(1)}, C_{\delta,\ell}^{2(1)})$$

We find a unique solution by imposing

$$\alpha_{a,b}^{(1)} = \sum_{\delta=1}^{\infty} \text{nested sums} = \text{sv MZVs}$$

Solution reproduces all known data from localisation and integrability!

Degeneracies in the spectrum

The amplitude encodes OPE data of multiple degenerate superprimaries.

We determined the degeneracies in the spectrum starting from type IIb strings in flat 10d:

$$SO(9) \to SO(4) \times SO(5) \overset{KK}{\to} SO(4) \times SO(6)$$

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

Number of superconformal long multiplets with superprimary $\mathcal{O}_{\delta,\ell}$ ($SO(6)$ singlet with $\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}} + O(\lambda^0)$)

Example: $\mathcal{O}_{1,0} = \text{Konishi} \sim \text{Tr}(\phi^I \phi_I)$

OPE data

We computed analytically for many Regge trajectories:

$$\langle C^{2(0)}_{\delta,\ell} \Delta^{(1)}_{\delta,\ell} \rangle \qquad \text{and} \qquad \langle C^{2(1)}_{\delta,\ell} \rangle$$

Leading Regge trajectory:

$$\Delta^{(1)}_{\frac{\ell+2}{2},\ell} = \frac{3\ell^2 + 10\ell + 16}{4\sqrt{2(\ell+2)}}\,, \qquad C^{2(1)}_{\frac{\ell+2}{2},\ell} = \ldots$$

$\Delta^{(1)}_{\frac{\ell+2}{2},\ell}$ agrees with integrability result!

[Gromov,Serban,Shenderovich,Volin;2011]

Expression for $A^{(1)}(S,T)$

Resumming the low energy expansion reveals the poles and residues of $A^{(1)}(S,T)$:

$$A^{(0)}(S,T) = \frac{1}{STU} + \sum_{\delta=1}^{\infty} \frac{1}{\delta^3} \frac{y+2}{1-x-y} \binom{z+\delta-1}{\delta-1}^2$$

[Zagier,Zerbini;2019]

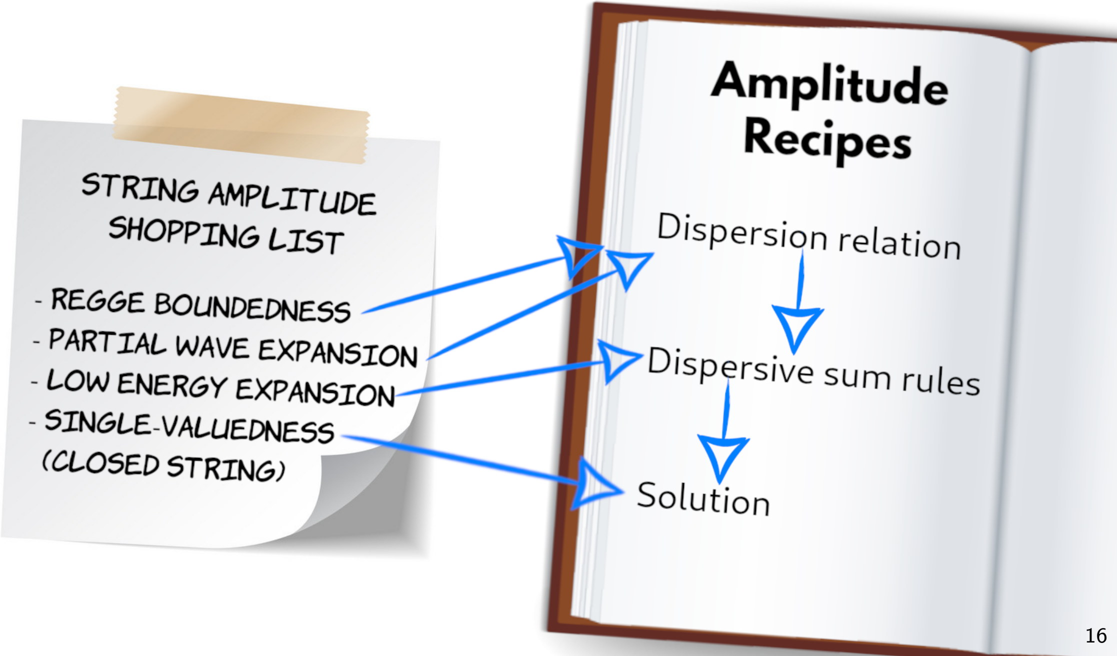
$$A^{(1)}(S,T) = -\frac{S^2+T^2+U^2}{3(STU)^2} + \sum_{\delta=1}^{\infty} \sum_{n=0}^{\delta-1} \frac{1}{\delta^4} \mathcal{D}_n(\delta) \frac{y+2}{1-x-y} \binom{z+\delta-\frac{n}{2}-1}{\delta-n-1}^2$$

$$\frac{y+2}{1-x-y} = 2 - \frac{S}{S-\delta} - \frac{T}{T-\delta} - \frac{U}{U-\delta}, \quad z = \frac{\delta}{2}\left(\sqrt{1-4STU/\delta^3} - 1\right)$$

$$\mathcal{D}_n(\delta) = \text{ degree 3 differential operator in } x, y, z$$

$A^{(1)}(S,T)$ has poles up to 4th order.

Summary


STRING AMPLITUDE SHOPPING LIST
- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- LOW ENERGY EXPANSION
- SINGLE-VALUEDNESS (CLOSED STRING)
Amplitude Recipes
Dispersion relation
Dispersive sum rules
Solution

Future directions

- Fully determining $A^{(2)}(S, T)$ with the same method seems to require unmixing $\Delta^{(1)}_{\delta,\ell}$. How?
  - Studying more general correlators of 1/2-BPS operators is not enough because they have the same flat space limit.
  - Correlators of massive string states?
  - Could integrability come to the rescue?
- Make contact with worldsheet theory by writing $A^{(1)}(S, T)$ as worldsheet integral.
- Can $A^{(1)}(S, T)$ be computed from the σ-model [Metsaev,Tseytlin;1998] by expanding around flat space?

Thank you!

Questions?