The gravitational path integral for N=4 BPS black holes from black hole microstate counting

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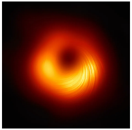
with Suresh Nampuri, Martí Rosselló, arXiv:2007.10302, arXiv:2112.10023

with Abhiram Kidambi, Suresh Nampuri, Valentin Reys, Martí Rosselló, arXiv: 2211.06873

Precision counting of BPS black hole microstates

An important research endeavour in string theory: obtaining a



precision counting of black hole microstates

Strominger + Vafa, arXiv:hep-th/9601029

Recently achieved for four-dimensional $\frac{1}{4}$ BPS black holes

in $N = 4$ heterotic string theory on T^6

Ferrari +Reys, arXiv:1702.02755, C+Nampuri+Rosselló, arXiv:2112.10023

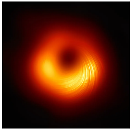
Single-centre, asymptotically flat, dyonic (q, p) , supersymmetric. Near horizon geometry is $AdS_2 \times S^2$. Charge bilinears (m, n, ℓ).

Microstates $d(m, n, \ell)$: $m, n, \ell \in \mathbb{Z}$, $\Delta \equiv 4mn - \ell^2 > 0$

$\log\, d(m, n, \ell) = \pi\sqrt{\Delta} + \dots = \frac{A_H}{4} + \dots$ A_H: area of event horizon

$$\Rightarrow S_{BH} = \log\, d(m, n, \ell) = \frac{A_H}{4} + \cancel{c_1} \log A_H + \frac{c_2}{A_H} + \dots + \alpha_n\, e^{-\beta_n A_H} + \dots$$

Three approaches to BPS black hole entropy



1. Number theory:
   $d(m, n, \ell)$: meromorphic Siegel modular form.
   Exact expression as a Rademacher type expansion. C, Nampuri, Rosselló, arXiv: 2112.10023

2. Quantum entropy function: Ashoke Sen, arXiv:0805.0095
   $d(m, n, \ell)$ from a quantum gravity path integral: sum over space-time geometries that asymptote to Euclidean AdS_2.

   Lin, Maldacena, Rozenberg, Shan, arXiv:2207.00408 suggests a space-time interpretation involving 2D wormholes in AdS_2. In the 'covariant' picture.

3. AdS_2/CFT_1 correspondence

   Maldacena, arXiv:9711200

Number theory: meromorphic Siegel modular form

Heterotic string theory on T^6: $\frac{1}{4}$ BPS states with unit torsion.

Microstate degeneracies $d(m,n,\ell)$ given in terms of the Fourier coefficients of $1/\Phi_{10}$. Φ_{10} Igusa cusp form of weight 10.

Dijkgraaf, Verlinde, Verlinde, arXiv: 9607026

$$d(m,n,\ell) = \int_C d\sigma\, dv\, d\rho\, \frac{e^{-2\pi i(m\rho + n\sigma + \ell v)}}{\Phi_{10}(\rho,\sigma,v)}$$

Three contour integrations. Since $1/\Phi_{10}$ is meromorphic Siegel modular form, $d(m,n,\ell)$ depends on the choice of the integration contour C. $\Delta = 4mn - \ell^2$.

$1/\Phi_{10}$ captures degeneracies of single-centre ($\Delta > 0$) as well as of two-centre black holes ($\Delta < 0$). Ashoke Sen, arXiv:0705.3874

Need to select a contour C that only captures single-centre degeneracies $d(m,n,\ell)$, with $\Delta > 0$. Cheng+Verlinde, arXiv:0706.2363, 0806.2337

Poles of $1/\Phi_{10}$

Poles of $1/\Phi_{10}$: $\quad n_2\,(\rho\sigma - v^2) + j\,v + n_1\,\sigma - m_1\rho + m_2 = 0, \quad n_2 \geq 0$

- Can be parametrized in terms of two distinct $SL(2,\mathbb{Z})$:

  Murthy+Pioline, arXiv:0904.4253

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \quad , \quad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

- Two types: quadratic ($n_2 > 0$) and linear ($n_2 = 0$) poles.
- Linear poles: Decay of BH bound state when crossing $jv_2 + n_1\sigma_2 - m_1\rho_2 = 0$.

  Cheng+Verlinde, arXiv:0706.2363, 0806.2337

Focus on the quadratic poles ($n_2 > 0$).

Contour integration and Rademacher path

Steps: 1) Evaluate the first integral (over ρ) as a sum over residues associated with the quadratic poles $n_2 > 0$.

2) Integrating over v: error functions, continued fraction structure.

3) The integration contour Γ_σ for σ: a union of Ford circles, anchored at rational points $0 \leq -\delta/\gamma < 1$.

Rademacher, Proc. Lond. Math. Soc. (2), 43(4): 241-254, 1937

Greek $SL(2,\mathbb{Z})$, Ford circles: Rademacher expansion for $d(n), n > 0$, of a meromorphic modular form of negative weight.

Encoded in $d(n)$ with $n < 0$. Example: $1/\eta^{24}(\tau) = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n \tau}$

$$d(n) = d(-1) \frac{2\pi}{n^{13/2}} \sum_{c>0} \frac{K(n,-1,c)}{c} \; I_{13}\left(\frac{4\pi\sqrt{n}}{c}\right) \quad , n > 0$$

Here: a Rademacher type expansion for a Siegel modular form. $d(m,n,\ell)$ with $\Delta > 0$ encoded in $d(m,n,\ell)$ with $\Delta < 0$.

Exact expression for $d(m,n,\ell)$ with $\Delta = 4mn - \ell^2 > 0$

Theorem: $d(m,n,\ell) \in \mathbb{N}, \quad n > m, \quad \tilde{\Delta} = 4m\tilde{n} - \tilde{\ell}^2 < 0$

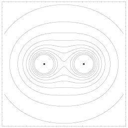
$$d(m,n,\ell) = (-1)^{\ell+1} \sum_{\gamma=1}^{+\infty} \sum_{\tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}} \Bigg(2\pi \sum_{\substack{\tilde{n}\geq -1, \\ \tilde{\Delta}<0}} c_m^F(\tilde{n},\tilde{\ell}) \frac{\mathrm{Kl}(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}, \gamma, \psi)_{\ell\tilde{\ell}}}{\gamma} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde{\Delta}|}\right)$$

$$-\delta_{\tilde{\ell},0}\sqrt{2m}\, d(m) \frac{\mathrm{Kl}(\frac{\Delta}{4m}, -1; \gamma, \psi)_{\ell 0}}{\sqrt{\gamma}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{\gamma}\sqrt{\frac{\Delta}{m}}\right)$$

$$+\frac{1}{2\pi} d(m) \sum_{\substack{g\in\mathbb{Z}/2m\gamma\mathbb{Z} \\ g=\tilde{\ell} \bmod 2m}} \frac{\mathrm{Kl}(\frac{\Delta}{4m}, -1-\frac{g^2}{4m}; \gamma, \psi)_{\ell\tilde{\ell}}}{\gamma^2}$$

$$\left(\frac{4m}{\Delta}\right)^{25/4} \int_{-1/\sqrt{m}}^{1/\sqrt{m}} dx'\, f_{\gamma,g,m}(x')(1-mx'^2)^{25/4} I_{25/2}\left(\frac{2\pi}{\gamma\sqrt{m}}\sqrt{\Delta(1-mx'^2)}\right)\Bigg),$$

with



$$c_m^F(\tilde{n},\tilde{\ell}) = \sum_{\substack{a>0,c<0 \\ b\in\mathbb{Z}/a\mathbb{Z},\ ad-bc=1 \\ 0\leq \frac{b}{a}+\frac{\tilde{\ell}}{2m} < -\frac{1}{ac}}} \Big((ad+bc)\tilde{\ell} + 2ac\tilde{n} + 2bdm\Big)\, d(c^2\tilde{n} + d^2 m + cd\tilde{\ell})\, d(a^2\tilde{n} + b^2 m + ab\tilde{\ell})$$

$\frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i \tau n}$, two $SL(2,\mathbb{Z})$, continued fraction decomposition of $\tilde{\ell}/2m$

Exact Rademacher type expansion for $1/\Phi_{10}$

- Area law:
  $\gamma = 1,\ \tilde{n} = -1,\ \tilde{\ell} = m : \qquad d(m,n,\ell) \approx e^{\pi\sqrt{\Delta}} = e^{\frac{1}{4}A_H}$
- Expansion encoded in degeneracies of the perturbative $\frac{1}{2}$ BPS states! $\quad c_m^F(\tilde{n},\tilde{\ell}) = \sum L\, d(M)\, d(N)$ bound state degeneracy
- Exponentially suppressed corrections: $e^{\pi\sqrt{\Delta|\tilde{\Delta}|}/\gamma m}$
- Contributions I_{12} and $I_{25/2}$: reflect underlying Mock modular behaviour that is encoded in the Fourier-Jacobi decomposition of $1/\Phi_{10}$,

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m=-1}^{\infty} \psi_m(\sigma,v)\, e^{2\pi i m\rho}$$

Dabholkar + Murthy+ Zagier, arXiv:1208.4074; Ferrari + Reys, arXiv:1702.02755

‘Covariant picture’

Integrate $1/\Phi_{10}$ over ρ , add total derivative term that vanishes on integration contour, change of variables $(\sigma, v) \to (\tau_1, \tau_2)$:

$$d(m,n,\ell)_{\Delta>0} = \sum_{SL^2(2,\mathbb{Z}),\,\Sigma} \frac{e^{i\pi\,\varphi}}{\gamma}\,\frac{1}{(ac)^{13}} \int_{\Gamma_2} \frac{d\tau_2}{\tau_2^2} \left(\int_{\Gamma_1} d\tau_1\, f(\tau_1,\tau_2) \right) + \ldots$$

$$f(\tau_1,\tau_2) = \left[26 + \frac{2\pi}{n_2}\,\frac{\left(m(\tau_1^2+\tau_2^2)+n-\ell\tau_1\right)}{\tau_2} \right] \frac{e^{\frac{\pi}{n_2}\,\frac{m(\tau_1^2+\tau_2^2)+n-\ell\tau_1}{\tau_2}}}{\eta^{24}(\rho'_*)\,\eta^{24}(\sigma'_*)\,\tau_2^{12}}$$

$$\begin{aligned} \rho'_* &= -\frac{a}{c}\,\frac{\tau_1 + i\tau_2}{\gamma} - \frac{b}{c}\,\frac{\alpha}{\gamma} - \frac{a}{c}\,\Sigma \\ \Gamma_1 : \quad \tau_1 &= \frac{\ell}{2m} + i\tau_2\,(-1+2y) \quad , \quad 0 < y < 1 \\ \Gamma_2 : \quad \tau_2 &= \frac{\sqrt{\Delta}}{2m} + i\,t \quad , \quad -\infty < t < \infty \end{aligned}$$

Invariance under $\tau_1 \to \tau_1 + 1$ of $SL(2,\mathbb{Z})$ manifest.

Quantum entropy function (QEF) / 'covariant picture'

Reproduce $d(m,n,\ell)$ by a suitable quantum gravity path integral, quantum entropy function (QEF). Ashoke Sen, arXiv:0805.0095, 0809.3304

- Finite-dimensional integral in an Euclidean background B that asymptotes to a specific Euclidean AdS_2 solution fixed by the attractor mechanism. $W = \sum_B W_B$. Dabholkar,Gomes,Murthy,arXiv:1012.0265
- QEF$\rightarrow$ Rademacher picture. $c^F_m(\tilde{n},\tilde{\ell})$ in measure.

'Covariant picture': Semi-classical interpretation

- $\mathbb{Z}_{n_2}$ orbifolds of $EAdS_2$ Murthy+Pioline, arXiv:0904.4253

$$
\begin{aligned}
ds^2 &= v_* \left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1} \right), \qquad 0 \leq \theta < \frac{2\pi}{n_2} \\
A^I_\theta &= -ie^I_* \,(r-1)\, d\theta + \mathrm{Re} A^I_\theta
\end{aligned}
$$

$(\mathrm{Re}A_\theta, \mathrm{Re}\tilde{A}_\theta)$ expressed in terms of $(q_I, p^I; m_1, n_1, j)$, S-duality

- Terms $\frac{1}{\eta^{24}(\rho'_*)\,\eta^{24}(\sigma'_*)\,\tau_2^{12}}$. No bound state structure here.

Macroscopic interpretation of ‘covariant picture’

Consider $n_2 = 1$: $\frac{1}{\eta^{24}(\tau)\,\eta^{24}(-\bar{\tau})}\,\frac{1}{(\tau-\bar{\tau})^{12}}$

Global Euclidean AdS_2: supported by constant dilaton field $\Phi_0 = v_*$

$$ds^2 = \frac{v_*}{\sin^2\sigma}\left(dT^2 + d\sigma^2\right) \quad , \quad -\pi < \sigma < 0 \quad , \quad T \cong T + 2\pi\tau_{2\,*} \; ,$$

Proposal:

Add 24 chiral + 24 antichiral periodic scalar fields (critical closed bosonic string), time-independent classical configuration: $T^{\rm cl}_{\mu\nu} = 0$.

$< T^{\rm quan}_{\mu\nu} > \neq 0$: 1-loop partition function of periodic scalars,

$$Z^{1-\text{loop}} = \frac{1}{\eta^{24}(\tau)\,\eta^{24}(-\bar{\tau})}\,\frac{1}{(\tau-\bar{\tau})^{12}}$$

$< T^{\rm quan}_{\mu\nu} > \neq 0$ backreacts on the dilaton,

$$\Phi_0 + \Phi = \Phi_0 - 24\,\mathcal{E}\left[2\pi\tau_{2\,*}\right]\left(1 - \frac{\sigma + \frac{\pi}{2}}{\tan\sigma}\right) \quad , \quad -\pi < \sigma < 0$$

The resulting solution (trumpet + dilaton) is interpreted as an 2D Euclidean wormhole solution.

Garcia-Garcia, Godet, arXiv:2010.11633

Holographic description:

Euclidean 1D Liouville type action

Mertens, Turiaci, Verlinde, arXiv:1606.03438

Lin, Maldacena, Rozenberg, Shan, arXiv:2207.00408

Lorentzian DFF type action de Alfaro, Fubini, Furlan, 1976

Gibbons, Townsend, hep-th/9812034

$$\begin{aligned} S_{\rm Liouv} &= \int dt \left[\frac{1}{2}(l')^2 + 2e^{-l}\right] \\ S_{\rm DFF} &= \int dt \left[\frac{(\nu')^2}{\alpha\,\nu} + \alpha\left(\frac{1}{\nu} + \nu\right)\right] \end{aligned}$$

by means of reparametrization $dt \to \alpha(t)\,dt$.

Thanks!