Exact Large Charge in $\mathcal{N} = 4$ SYM and Semi-Classical String Theory

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Based on 2303.13207, 2209.06639
with Hynek Paul and Eric Perlmutter

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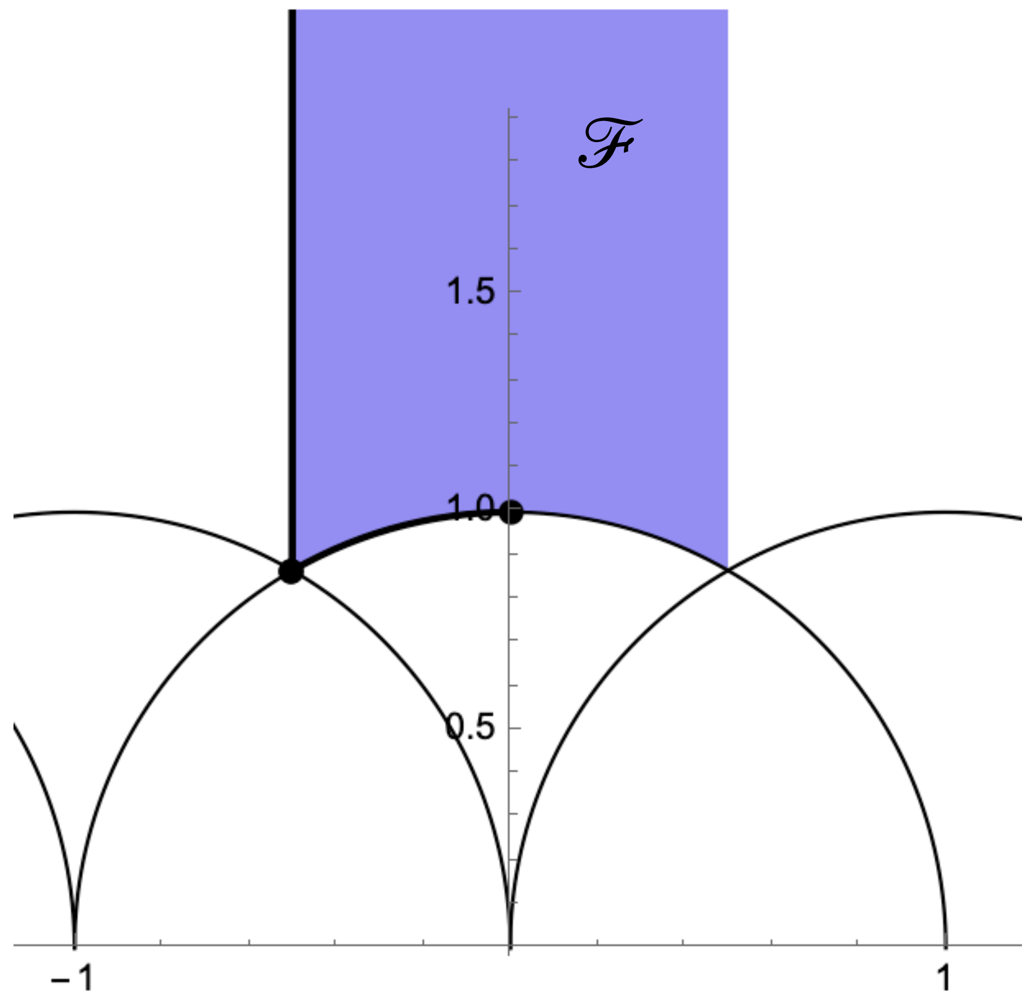
This talk is about **large charge limit** of four dimensional $\mathcal{N} = 4$ super Yang-Mills theory

What is the large charge limit of semiclassical string theory in AdS?

Correlation functions in large charge states

- Large charge perturbation theory, non-perturbative effects, $SL(2,\mathbb{Z})$ duality
- An emergent 't Hooft-like expansion parameter
- A large charge gravity regime

Setup



$\mathcal{N} = 4$ Super-Yang Mills theory on $\mathbb{R}^4$ with gauge group $SU(N)$

Exactly marginal gauge coupling $\tau = \dfrac{\theta}{2\pi} + \dfrac{4\pi i}{g^2} \equiv x + iy$

The theory enjoys S-duality — is $SL(2,\mathbb{Z})$ invariant — $\tau \in$ fundamental domain $\mathscr{F}$ of $SL(2,\mathbb{Z})$

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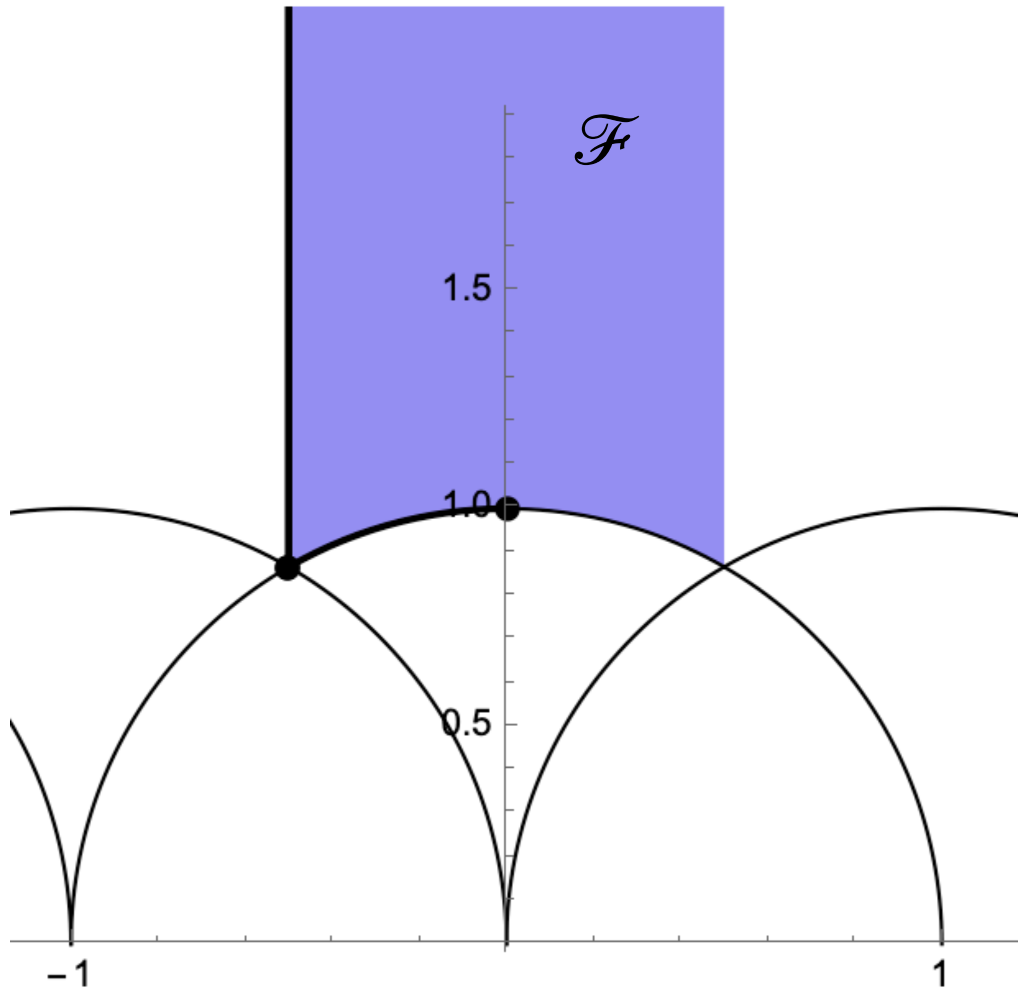

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1/2-BPS operators $\mathcal{O}_p$: Lorentz scalar , Δ = R-charge p

$$\mathcal{O}_p = \left[\mathrm{Tr}(\phi^2)\right]^{p/2}$$

Multi-trace composite of stress-tensor

We will be interested in **four-point correlation functions** of such operators

What exactly are we computing?

In particular we look at four-point correlators of the type:

$$\left\langle 0 \left| \mathcal{O}_p \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \right| 0 \right\rangle_c \sim \mathcal{H}_p^{(N)}(U, V; \tau, \bar{\tau})$$

U, V conformal cross ratios

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Generically hard to compute !

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Generically hard to compute !

Less ambitious : Average out the dependence on the cross-ratios and consider a simpler object

$$\mathcal{G}_p^{(N)}(\tau, \bar{\tau}) = \int dU dV \, \rho(U, V) \, \mathcal{H}_p^{(N)}(U, V; \tau, \bar{\tau})$$

"Integrated correlators"

$\mathcal{G}$ is non-trivial function: R-charge p, rank N and non-holomorphic in τ

Localization formula

$$\mathcal{G}_p^{(N)}(\tau) \sim \int_0^\infty dr \int_0^\pi d\theta \, \frac{r^3 \sin^2\theta}{U^2} \mathcal{H}_p^{(N)}(U, V; \tau)$$

$U = 1 + r^2 - 2r\cos\theta$
$V = r^2$

$\mathcal{G}_p^{(N)}(\tau)$ can be computed via **supersymmetric localization** of the $\mathcal{N} = 2^*$ theory on S^4

[Binder, Chester, Pufu, Wang], [Chester, Green, Pufu, Wang, Wen], [Dorigoni, Green, Wen], [Gerchkovitz, et. al.]

$$\mathcal{G}_p^{(N)}(\tau) \sim \partial_\tau^p \partial_{\bar{\tau}}^p \partial_m^2 \log \mathcal{Z}_{\mathbb{S}^4}(N; \tau, m) \Big|_{m=0}$$

Partition function $\mathcal{Z}_{\mathbb{S}^4}$ determined by supersymmetric localization

[Pestun], [Nekrasov], [Fucito, Morales, Poghossian], [Gerchkovitz, et. al.], …

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In practice, we exploit this connection to compute $\mathcal{G}_p^{(N)}(\tau)$ **exactly in all parameters !**

Exact solution

$\mathcal{G}_p^{(N)}(\tau)$ can be written as a spectral integral [Paul, Perlmutter, HR], [Collier, Perlmutter]

$$\mathcal{G}_p^{(N)}(\tau) = \left\langle \mathcal{G}_p^{(N)} \right\rangle + \frac{1}{4\pi i} \int_{\mathrm{Re}\, s=\frac{1}{2}} ds \frac{\pi}{\sin(\pi s)} s(1-s)(2s-1)^2 g_p^{(N)}(s) E_s^*(\tau)$$

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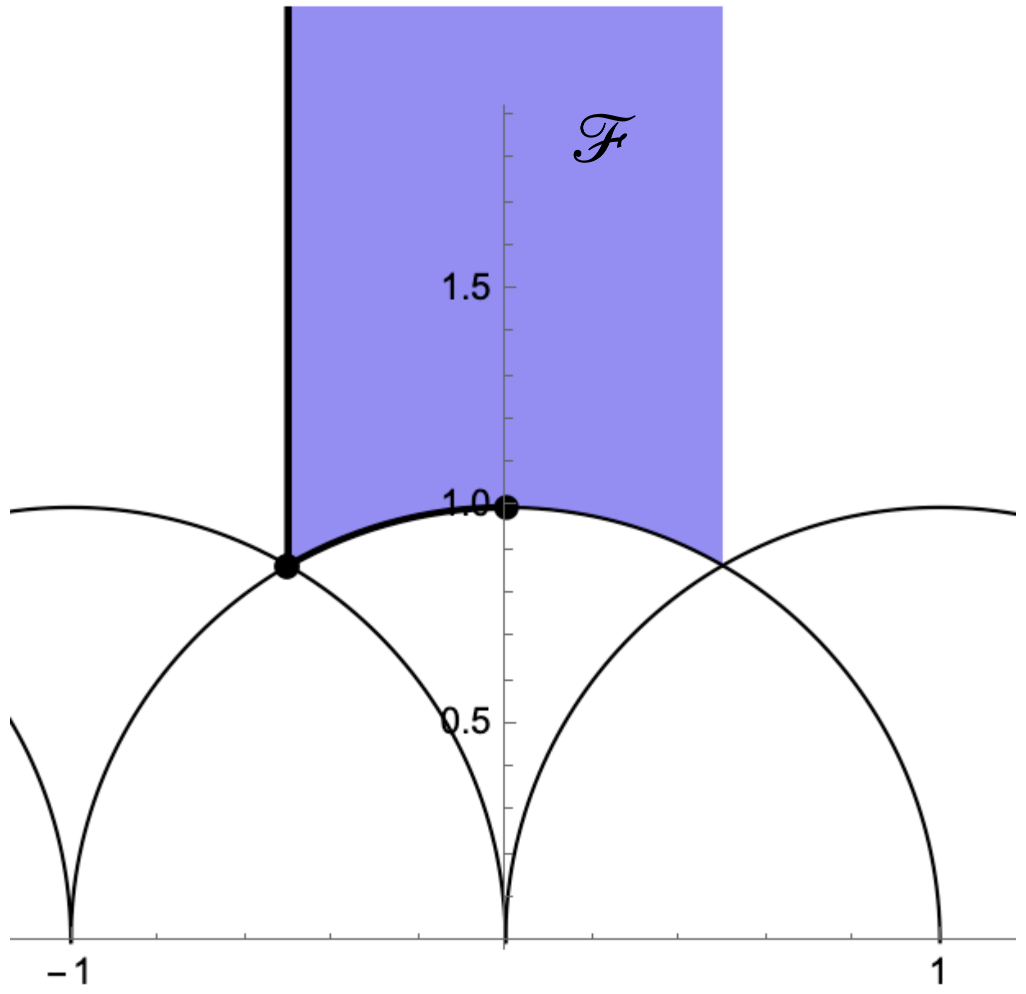
constant piece

Constant piece = average over the $\mathcal{N} = 4$ conformal manifold

$$\left\langle \mathcal{G}_p^{(N)} \right\rangle = \mathrm{vol}(\mathcal{F})^{-1} \int_{\mathcal{F}} \frac{dxdy}{y^2} \mathcal{G}_p^{(N)}(\tau)$$



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Real analytic completed Eisenstein series

The entire coupling dependence packaged into the Eisenstein series $E_s^*(\tau)$

Eigen function of the hyperbolic laplacian

$$\Delta_\tau E_s^*(\tau) = s(1-s) E_s^*(\tau)$$

$$\Delta_\tau = -y^2 \left(\partial_x^2 + \partial_y^2\right)$$

$$E_s^*(\tau) = E_{1-s}^*(\tau)$$

functional identity

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The entire coupling dependence packaged into the Eisenstein series $E_s^*(\tau)$

Fourier decomposition

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \equiv x + iy$$

$$E_s^*(\tau) = \underbrace{\Lambda(s) y^s + \Lambda(1-s) y^{1-s}} + \sum_{k=1}^{\infty} 4\cos(2\pi k x)\, \frac{\sigma_{2s-1}(k)}{k^{s-\frac{1}{2}}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi k y)$$

zero modes
perturbative series

non-zero modes
instanton corrections

$$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$$

Exact solution

$\mathcal{G}_p^{(N)}(\tau)$ can be written as a spectral integral [Paul, Perlmutter, HR], [Collier, Perlmutter]

$$\mathcal{G}_p^{(N)}(\tau) = \left\langle \mathcal{G}_p^{(N)} \right\rangle + \frac{1}{4\pi i} \int_{\mathrm{Re}\, s = \frac{1}{2}} ds \frac{\pi}{\sin(\pi s)} s(1-s)(2s-1)^2 g_p^{(N)}(s) E_s^*(\tau)$$

↑ Eisenstein overlap

All remaining info contained in Eisenstein overlap $g_p^{(N)}(s)$

1. Completely fixed from perturbation theory data (via the localization formula)
2. At finite N these are polynomials of s symmetric under $s \leftrightarrow 1-s$ (from $E_s^*(\tau) = E_{1-s}^*(\tau)$)

Exact solution

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Eisenstein overlap

Closed form solution

$$g_2^{(N)}(s) = \frac{N}{N+1} {}_3F_2(2-N, s, 1-s; 3, 2; 1)$$

$$g_p^{(N)}(s) = F_p(N,s)\; g_2^{(N)}(s)$$

$$F_p(N,s) = \frac{N^2-1}{2s(1-s)} \left[1 - {}_3F_2\left(-\frac{p}{2}, s, 1-s; 1, \frac{N^2-1}{2}; 1 \right) \right]$$

Products of ${}_3F_2$ Hypergeometric functions

Exact solution: A coupled harmonic system

$$\Delta_\tau \mathcal{Q}^{(N)}_{p-2}(\tau) = -\kappa_p \left(\mathcal{Q}^{(N)}_{p}(\tau) - \mathcal{Q}^{(N)}_{p-2}(\tau) \right) + \kappa_{p-2} \left(\mathcal{Q}^{(N)}_{p-2}(\tau) - \mathcal{Q}^{(N)}_{p-4}(\tau) \right)$$

$$\kappa_p := \frac{p}{4}\left(N^2 + p - 3\right)$$

Shifted correlator: $\mathcal{Q}^{(N)}_{p}(\tau) := \mathcal{G}^{(N)}_{p}(\tau) - \frac{1}{2}\left(N^2 - 1\right) \Delta_\tau^{-1} \mathcal{G}^{(N)}_{2}(\tau)$

$\mathcal{G}^{(N)}_{2}(\tau)$ satisfies a differential recursion in N [Dorigoni, Green, Wen]

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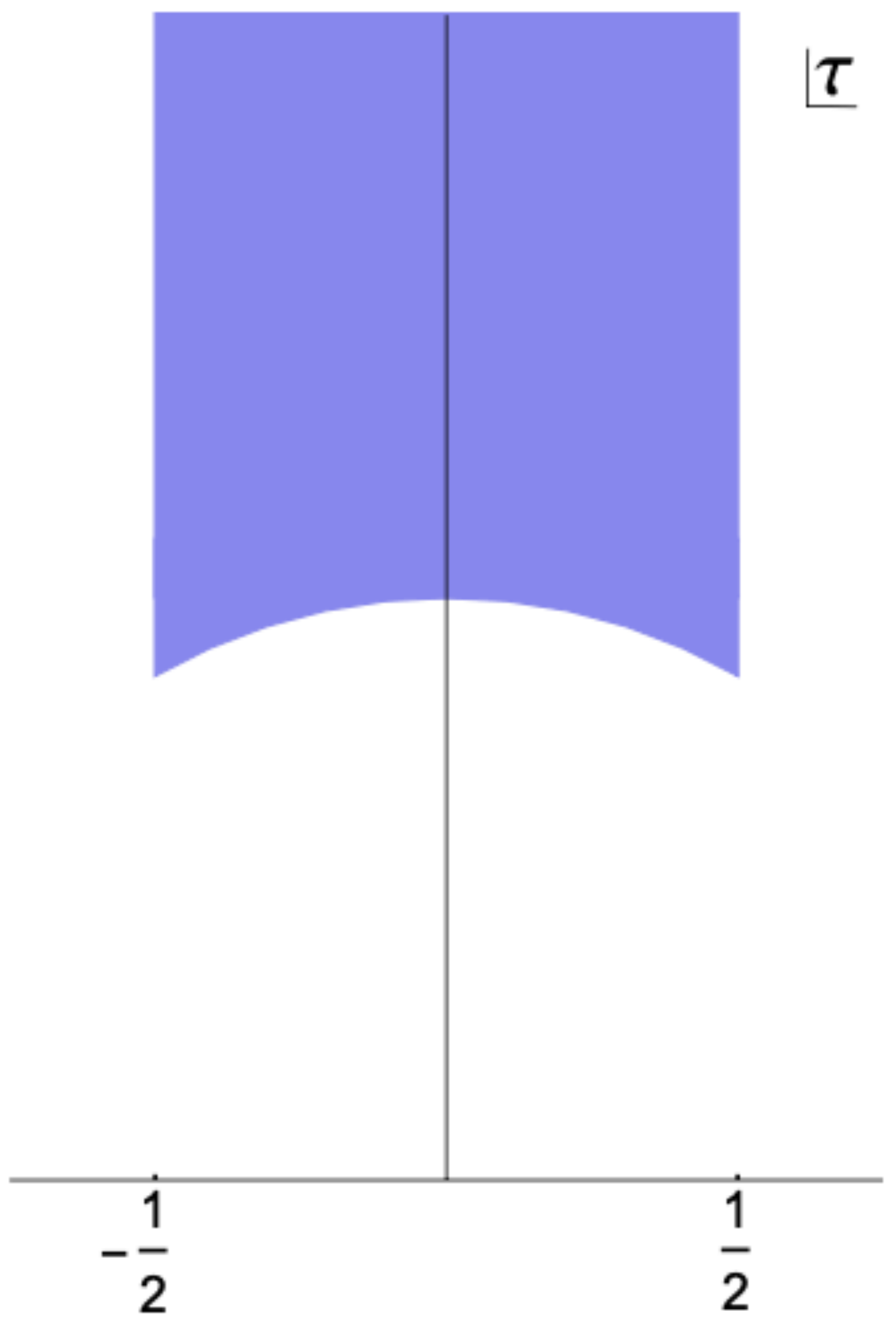

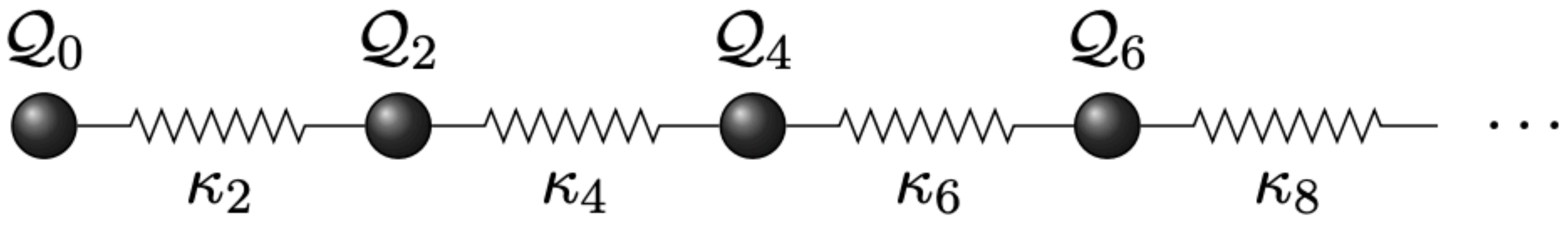

- Each lattice site = integrated correlator
- Harmonic interactions among lattice sites with a site- and N-dependent coupling κ_p
- This describes the evolution of a 1D semi-infinite lattice chain over the fundamental domain of $SL(2,\mathbb{Z})$

Towards large charge

Three parameters in $\mathcal{G}_p^{(N)}(\tau)$: coupling τ, R-charge p and rank N

$$\mathcal{G}_p^{(N)}(\tau) = \left\langle \mathcal{G}_p^{(N)} \right\rangle + \frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds \frac{\pi}{\sin(\pi s)} s(1-s)(2s-1)^2 g_p^{(N)}(s) E_s^*(\tau)$$

$$g_p^{(N)}(s) = F_p(N,s)\; g_2^{(N)}(s)$$

$$g_2^{(N)}(s) = \frac{N}{N+1}\, {}_3F_2(2-N, s, 1-s; 3, 2; 1)$$

entire p dependence here

$$F_p(N,s) = \frac{N^2-1}{2s(1-s)}\left[1 - {}_3F_2\left(-\frac{p}{2}, s, 1-s; 1, \frac{N^2-1}{2}; 1\right)\right]$$

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Three distinct regimes:

$p \gg N^2$ Both N finite or N large

$p = \alpha N^2$ Gravity regime: where p scales with N^2

$p \ll N^2$ Both p finite or p large

Towards large charge

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Large Charge at Finite N

$$p \gg 1, \qquad N \text{ fixed}$$

1. Large charge 't Hooft like limit: $p \to \infty, \quad \lambda_p := g^2 p$ fixed

   (non-trivial, previously seen to emerge in $\mathcal{N} = 2$ extremal correlators)

   [Bourget, Rodriguez-Gomez, Russo], [Beccaria], [Grassi, Komargodski, Tizzano]

2. Large charge at finite coupling $p \to \infty, \quad \tau$ fixed

   (always exists for any QFT with a global symmetry)

Large Charge at Finite N: 't Hooft-like limit

$$p \gg 1, \qquad N \ \text{fixed}$$
$$g^2 p = \lambda_p \ \text{fixed}$$

Gauge instantons are exponentially suppressed in p: e^{-p/λ_p}

Existence of this limit is manifest in the spectral representation of the correlator

$$\mathcal{G}_p^{(N)}(\tau) = \left\langle \mathcal{G}_p^{(N)} \right\rangle + \frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds \frac{\pi}{\sin(\pi s)} s(1-s)(2s-1)^2 g_p^{(N)}(s) E_s^*(\tau)$$

$$E_s^*(\tau) \sim g^{2s}$$
$$g_p^{(N)}(s) \sim p^s$$

Large Charge at Finite N: 't Hooft-like limit

$p \gg 1, \quad N$ fixed

$g^2 p = \lambda_p$ fixed

A genus-like expansion in the double-scaled large charge limit

$$\mathcal{G}_p^{(N)}(\tau) = \sum_{\mathfrak{g}=0}^{\infty} p^{-\mathfrak{g}} \mathcal{G}_{\mathfrak{g}}^{(N)}\left(\lambda_p\right)$$

Genus 0 result for $SU(2)$

$$\mathcal{G}_{\mathfrak{g}=0}^{(2)}\left(\lambda_p\right) = -\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=1+\epsilon} ds(2s-1)\Gamma^2(1-s)\zeta(2-2s)\left(\frac{\lambda_p}{2}\right)^{s-1}$$

deforming integration contour to the right

$$\mathcal{G}_{\mathfrak{g}=0}^{(2)}\left(\lambda_p\right) = \int_0^{\infty} dw \frac{w}{\sinh^2(w)}\left[1 - J_0\left(w\sqrt{2\lambda_p}/\pi\right)\right]$$

Independently obtained by [Caetano, Komatsu, Wang]

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Strong coupling expansion:

$$\mathcal{G}_{\mathfrak{g}=0}^{(2)}\left(\lambda_p\right) = \frac{1}{2}\log\left(\frac{\lambda_p}{8\pi^2}\right) + 1 + \gamma_E + \sqrt{2\pi}\left(2\lambda_p\right)^{1/4} \mathrm{Li}_{-\frac{1}{2}}\left(e^{-\sqrt{2\lambda_p}}\right) + O\left(\lambda_p^{-1/4}\right)$$

exponentially suppressed terms

Large Charge at Finite N: 't Hooft-like limit

$p \gg 1, \quad N$ fixed

$g^2 p = \lambda_p$ fixed

Similar results exist for the $SU(N)$ case with the interesting difference that:

For even N the strong λ_p expansion terminates at a finite order

For odd N the strong λ_p expansion does not terminate

In both cases the scale of non-perturbative corrections at large λ_p is the same $\sim e^{-\sqrt{2\lambda_p}}$

Large Charge at Finite N: Finite Coupling

$$p \gg 1, \qquad N \;\text{ fixed}$$
$$\tau \;\text{ fixed}$$

Gauge instantons are no longer suppressed

The spectral representation of the integrated correlator easily facilitates the large charge expansion

NP terms in the large charge expansion can be rigorously computed

Large Charge at Finite N: Finite Coupling

$p \gg 1, \quad N$ fixed

τ fixed

$$\mathcal{F}^{(2)}_{\rm NP}(p;\tau) \sim p^{1/4} \sum_{(m,n)\neq(0,0)} \exp\left(-2\sqrt{2pY_{mn}(\tau)}\right)\left(Y_{mn}(\tau)\right)^{\frac{1}{4}} + O\left(p^{-1/4}\right)$$

$$Y_{mn}(\tau) = \frac{1}{4}g^2|m+n\tau|^2$$

Encodes exponentially suppressed corrections in large charge $\sim e^{-\sqrt{p}}$

Exact NP scale

$$M = \sqrt{2}\sqrt{p}g\,|m+n\tau|$$

mass of a BPS-saturated dyonic states

First NP correction of its kind rigorously computed in the literature

(also discussed in [Grassi, Komargodski, Tizzano], [Hellerman] in the $\mathcal{N}=2$ SQCD context)

Gravity Regime: Large Charge at Large N

$$p \gg 1, \quad N \gg 1, \quad \alpha = \frac{p}{N^2} \text{ fixed }, \quad \alpha \in \mathbb{R}_{\geq 0}$$

A **triple-scaled limit:** $N \to \infty$ with $\lambda = g_{YM}^2 N$ and α finite

Semi-classical string theory

Gravity Regime: Large Charge at Large N

$p \gg 1, \quad N \gg 1, \quad \alpha = \frac{p}{N^2}$ fixed

$g_{YM}^2 N$ fixed

The correlators $\mathcal{G}^{(N)}_{\alpha N^2}(\lambda)$ in **triple-scaled limit** organizes into a genus expansion (!)

$$\mathcal{G}^{(N)}_{\alpha N^2}(\lambda) = \sum_{\mathfrak{g}=0}^{\infty} N^{2-2\mathfrak{g}}\ \mathcal{G}^{(\mathfrak{g})}_{\alpha}(\lambda)$$

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Genus 0 term has the spectral representation

$$\mathcal{G}^{(0)}_{\alpha}(\lambda) = \frac{\log(\alpha+1)}{4} + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds \frac{\pi}{\sin(\pi s)} s(1-s)(2s-1)^2 \Lambda(1-s) \left(\frac{\lambda}{4\pi}\right)^{s-1} h^{(0)}_{\alpha}(s)$$

$$h^{(0)}_{\alpha}(s) = \widetilde{h}_0(s) - \widetilde{h}_{\alpha}(s), \qquad \widetilde{h}_{\alpha}(s) = \frac{{}_2F_1(1-s, s; 1; -\alpha)}{2s(1-s)} g^{(0)}_2(s)$$

usual 't Hooft limit term $\to$ $\dfrac{2^{2s}\Gamma\left(s+\frac{1}{2}\right)}{\sqrt{\pi}(2s-1)\Gamma(s+1)\Gamma(s+2)}$

Gravity Regime: Large Charge at Large N

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We can learn about the NP physics of this regime by analysing the above equation

$$p \gg 1, \quad N \gg 1, \quad \alpha = \frac{p}{N^2} \text{ fixed}$$

$$g_{YM}^2 N \text{ fixed}$$

Gravity Regime: Large Charge at Large N

Strong coupling expansion:

$$\mathcal{G}_\alpha^{(0)}(\lambda \gg 1) = \frac{\log(\alpha+1)}{4} + 4\left[1 - {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}; 1; -\alpha\right)\right]\frac{\zeta(3)}{\lambda^{3/2}} + O(\lambda^{-5/2})$$

$O(\lambda^0)$: supergravity result

$O(\lambda^{-3/2})$: leading α' correction

The strong coupling expansion is asymptotic. There are exponentially small corrections

The scale of such corrections is controlled by weak coupling radius of convergence [Collier, Perlmutter]

Two sets of NP corrections at large λ

$$e^{-2\sqrt{\lambda}}$$

leading NP effect

$$e^{-2\sqrt{\lambda/R_\alpha}}$$

emergent subleading scale

$$R_\alpha = 1 + 2\alpha - 2\sqrt{\alpha(\alpha+1)}$$

Gravity Regime: Large Charge at Large N

$$p \gg 1, \quad N \gg 1, \quad \alpha = \frac{p}{N^2} \text{ fixed}$$

$$g_{YM}^2 N \text{ fixed}$$

New prediction for novel NP effects in the gravity regime of large charge

In $AdS_5 \times S^5$ holography

$e^{-2\sqrt{\lambda}}$ $\longleftrightarrow$ fundamental string worldsheet instantons $2\pi T_{F1} = \sqrt{\lambda}$

What about the emergent scale $e^{-2\sqrt{\lambda/R_\alpha}}$ $(R_\alpha = 1 + 2\alpha - 2\sqrt{\alpha(\alpha+1)})$

$e^{-2\sqrt{\lambda/R_\alpha}}$ $\longleftrightarrow$ fundamental string action in a background dual to $|\mathcal{O}_p\rangle$

R_α : large charge dressing factor

Conclusion

The recursion formulas for integrated correlators can be generalized to other 1/2 BPS operator insertions. Recently worked out and proven in [Brown, Wen, Xie [2303.13195]]

These exact results for integrated correlators in $\mathcal{N} = 4$ SYM serve as a useful benchmark for future EFT approach to computing correlators at large charge

An emergent double-scaled t' Hooft like limit: Interesting NP aspects, genus expansion and connections to RMT

Large charge large N regime $p \sim N^2$: novel NP effects, worthy of further investigation

Thank You!