RG flows on $2d$ Spherical Defects

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Based on "**RG flows on two-dimensional spherical defects**",

arXiv: 2212.08081[hep-th], with

Tom Sachar, and *Michael Smolkin*

Motivation

- All QFTs can be thought of as deformations of some underlying CFTs.
- Classifying the space of CFTs is therefore an important problem!
- Monotonic functions (or c- functions) provide a first step towards this classification.
- They assume a constant value at the UV and IR fixed points, such that the $c_{UV} > c_{IR}$ (weak form).
- Additionally, the monotonic function can have a gradient flow along the RG (strong form).
- Restricts flows of unstable UV f.p.s to only a subset of all available IR f.p.s.

What are we looking for?

The Usual Suspects:

From the standard Langrangian/partition function based approach:

1. The Weyl anomaly (or certain coefficients in it) of the stress tensor. (in even d)
2. The universal terms in the Free Energy of a sphere. (in both odd and even d)

Alternately, the information theoretic approach based on entanglement measures such as the entanglement entropy and the relative entropy have been highly successful in 2,3 and 4d.

Motivated by these ideas and techniques, one can look for similar functions to classify all possible defects in a given CFT.

General trivia about defects

(See Mezei's talk for a broad overview and references)

- Defects are usually submanifolds in a system such as lines, interfaces, hypersurfaces (can also be spin impurities on a lattice, Wilson lines, 't Hooft lines etc.), that can have their own localised set of degrees of freedom, as well as an RG flow.
- Simplest set-up for studying defects: d-dim. "bulk" CFT (conformal background) coupled to a p-dim. ($p < d$) defect manifold.
- We are exclusively interested in scenarios where the DRG (defect RG) flow terminates at an IR fixed point, described by a defect CFT (or DCFT).
- Note: The stress tensor for a DCFT vanishes up to anomalies. (believed to be true!)
- Away from the fixed point, the defect has a non-zero stress tensor.

A quick review

- Until recently, rigorous results regarding a systematic classification of the landscape of defect CFTs was missing from the literature.
- In a very interesting recent development, new monotonic function for the $1d$ defect, called the *defect entropy function* (DEF), was discovered.
  Cuomo, Komargodski, Raviv-Moshe '21, CK + Mezei '21

Candidate function:

$$\log g(\mu_0 R) = \log\left(\frac{Z_D}{Z_{CFT}}\right)$$

The finite defect entropy function (DEF):

$$s(\mu_0 R) = \left(1 - R\frac{\partial}{\partial R}\right)\log g(\mu_0 R)$$

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The DEF assumes constant values at the f.p.s of 1d DRG flows, and is monotonically decreasing along the entire flow from the UV to the IR (strong form). Thus,

$$\boxed{g_{UV} > g_{IR}}$$

Our goal is to generalise this approach to 2d defects embedded in a $d > 2$ dim. CFT.

Non-triviality of the problem

- Tempting to think that the generalisation to 2d defects is simple and straightforward.
- However, we know from trying to find monotonic functions in usual bulk CFTs, each dimension has its own peculiarities.
- In our case, 2d spheres have a higher set of allowed intrinsic and extrinsic curvature terms that spoil various properties from lower dimensions.
- We will see how that is the case, and how to bypass these problems in what follows!

Set-up

- We consider a d-dimensional Euclidean CFT which has an $SO(d+1,1)$ symmetry.
- Insertion of a p-dimensional *conformal* defect breaks the symmetry explicitly to $SO(p+1,1) \times SO(d-p)$.
- Turning on a relevant operator localised on the defect induces a DRG flow which terminates at an IR DCFT.
- We construct a *renormalised defect entropy* (RDE) function, analogous to the 1d case, which decreases along the DRG flow and assumes a constant value at the DCFT points.
- This is an alternate construction of the *b-theorem* for 2d defects due Jensen and O'Bannon. Jensen, O'Bannon '15
- Since we have the full RDE function along the flow, we will also comment on its strong monotonicity properties, which has not been done previously.

The Defect Entropy

We begin with the action,

$$I = I_{\text{DCFT}} + g^i \int_{\mathcal{D}} d^p\sigma \sqrt{\hat{\gamma}}\, \mathcal{O}_i \ ,$$

where I_{DCFT} is the defect CFT action, and $\mathcal{O}_i$ is a relevant operator $\Delta_i < p$ on the defect.

We define $\mathcal{F}$ as,

$$\mathcal{F} = -\log \frac{Z_{\mathcal{D}}}{Z_{\text{CFT}}} \ ,$$

where $Z_{\mathcal{D}}$ is the partition function for I, and Z_{CFT} is the partition function without defect.

The Defect Entropy

At a fixed point of the DRG flow on a 2d spherical defect,

$$\mathcal{F}_{\text{DCFT}} = c_0 + a_0(\mu R)^2 - \frac{b_0}{3} \log(\mu R) \, .$$

where R is the radius of the sphere, and μ is the running scale.

- c_0 not universal, rescaling μ.
- a_0 not universal, cosmological constant counter term.
- $\boxed{b_0 \text{ is universal}}$.

Need to extract b_0 from $\mathcal{F}_{\text{DCFT}}$.

The Defect Entropy

Motivated by the renormalised entanglement entropy in 4d Liu, Mezei '12

$$S = -R\partial_R \left(1 - \frac{1}{2}R\partial_R\right)\mathcal{F} = \frac{1}{2}\left(R^2\partial_R^2 - R\partial_R\right)\mathcal{F} \,.$$

Presence of the defect introduces an infinite number of new Ward identities into the system, derived using the dilaton effective action. See talk by M. Mezei
We only need the following one to prove our results.

$$\int_{S^2} \left\langle \hat{T}\,(\hat{n}) \right\rangle = \frac{1}{2}\int_{S^2}\int_{S^2} (\hat{n}_1 \cdot \hat{n}_2)\left\langle \hat{T}\,(\hat{n}_1)\,\hat{T}\,(\hat{n}_2) \right\rangle \,.$$

This relation is in fact true for any p-dimensional defect.

The Defect Entropy

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Our construction of the renormalised defect entropy (RDE) function,

$$\boxed{S = -\frac{1}{4R^2} \int_{S^2} \int_{S^2} s^2(\sigma_1, \sigma_2) \langle \hat{T}(\sigma_1) \hat{T}(\sigma_2) \rangle \ ,}$$

where $s^2(\sigma_1, \sigma_2) = 2R^2(1 - \hat{n}_1 \cdot \hat{n}_2)$ is the square of the chordal distance between the two points σ_1 and σ_2 on a 2-sphere.

UV and IR Finiteness

Is the RDE finite?

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UV and IR Finiteness

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In the IR,

- Finiteness is guaranteed due to size of the sphere R.

In the UV,

- since $\hat{T} = \beta^i \mathcal{O}_i$, $\langle \hat{T}\hat{T}\rangle = \beta^i\beta^j\langle \mathcal{O}_i\mathcal{O}_j\rangle$.
- $\beta_i = 0$ at the UV DCFT.
- Moreover, $\mathcal{O}_i$ is a relevant operator in the UV, so $\langle \hat{T}\hat{T}\rangle$ is less singular than $1/s^4$.

So, RDE is finite and positive up to contact terms.

Contact terms and the b-theorem

The RDE is designed such that the contact term gives rise to the UV anomaly. The UV anomaly and the manifestly positive and finite part of RDE can be written separately,

$$S = \frac{b_{\text{UV}}}{3} - \frac{1}{4R^2} \int_{S^2} \int_{S^2} s^2(\sigma_1, \sigma_2) \langle \hat{T}(\sigma_1) \hat{T}(\sigma_2) \rangle$$

As $R \to \infty$, RDE assumes the IR anomaly value, $S \to \frac{b_{IR}}{3}$ such that,

$$\frac{b_{\text{IR}} - b_{\text{UV}}}{3} = -\pi \int_{S^2} s^2(\sigma) \langle \hat{T}(\sigma) \hat{T}(0) \rangle \Big|_{R \to \infty} \leq 0 \quad \Leftrightarrow \quad \boxed{b_{\text{IR}} \leq b_{\text{UV}}} \, .$$

This is the $b -$ *theorem* of Jensen and O'Bannon proving the irreversibility of RG flows on 2d defects.

Monotonicity of the DRG flow

Is the DRG flow over 2d defects monotonic?
The RDE is a function of dimensionless couplings, g^i, and μR. The DRG flow can be probed by the running scale $\mu \sim 1/R$,

$$S\big(\mu R,\ g^i(\mu)\big)\Big|_{\mu\sim 1/R} = S\big(g^i(R^{-1})\big)\ .$$

Thus, under a change of radius R,

$$\begin{aligned} R\frac{d}{dR}S(g^i) &= -\beta^i\frac{\partial}{\partial g^i}S\big(g^i\big) = +\pi\beta^i\frac{\partial}{\partial g^i}\int_{S^2} s^2\,(\sigma)\,\beta^j\beta^k\,\langle\mathcal{O}_j(\sigma)\mathcal{O}_k(0)\rangle \\ &= \pi\beta^i\beta^j\Big(2\,\frac{\partial\beta^k}{\partial g^i}+\beta^k\frac{\partial}{\partial g^i}\Big)\int_{S^2} s^2\,(\sigma)\,\langle\mathcal{O}_j(\sigma)\mathcal{O}_k(0)\rangle = -2\pi^2\,\beta^i\beta^j h_{ij}\ . \end{aligned}$$

Monotonicity of the DRG flow

$$\boxed{R\frac{d}{dR}S(g^i) = -2\pi^2\,\beta^i\beta^j h_{ij}}$$

where,

$$h_{ij} = \left(2\,\frac{\partial\beta^k}{\partial g^i} + \beta^k\frac{\partial}{\partial g^i}\right)\int_{S^2} s^2\,(\sigma)\,\langle\mathcal{O}_j(\sigma)\mathcal{O}_k(0)\rangle$$

is a Zamolodchikov-like metric with the following properties,

- Since, $\beta_i = 0$ at the UV and IR f.p.s of the DRG flow, the first term changes sign.
- Second term proportional to 3 point function of $\mathcal{O}$, not manifestly positive.

So, the h_{ij} is not manifestly positive $\implies$ the flow of RDE is not necessarily monotonic.

Perturbative DRG flow

Consider a DCFT perturbed by a slightly relevant operator, $\Delta_i = 2 - \epsilon_i$ with $0 < \epsilon_i \ll 1$. At the UV f.p.,

$$\begin{aligned} \langle \mathcal{O}_i(\sigma_1)\mathcal{O}_j(\sigma_2)\rangle_{\text{UV}} &= \frac{\delta_{ij}}{s(\sigma_1,\sigma_2)^{2\Delta_i}} \ , \\ \langle \mathcal{O}_i(\sigma_1)\mathcal{O}_j(\sigma_2)\mathcal{O}_k(\sigma_3)\rangle_{\text{UV}} &= \frac{C_{ijk}}{s(\sigma_1,\sigma_2)^{\Delta_i+\Delta_j-\Delta_k}s(\sigma_1,\sigma_3)^{\Delta_i+\Delta_k-\Delta_j}s(\sigma_2,\sigma_3)^{\Delta_j+\Delta_k-\Delta_i}} \ . \end{aligned}$$

The form of the beta function under the perturbative DRG flow,

$$\beta^i = \mu\frac{dg^i}{d\mu} = -\epsilon_i g^i + \pi C^i_{jk} g^j g^k + \mathcal{O}(g^3) \ ,$$

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The form of the beta function under the perturbative DRG flow,

$$
\beta^i = \mu \frac{dg^i}{d\mu} = -\epsilon_i g^i + \pi C^i_{jk} g^j g^k + \mathcal{O}(g^3) \,,
$$

For $g^i \sim \mathcal{O}(\epsilon)$, $\Delta b = b_{IR} - b_{UV}$ can be computed using conformal perturbation theory,

$$
\begin{aligned}
\Delta b &= -3\pi \beta^i \beta^j \int d^2\sigma \sqrt{\hat{\gamma}}\, s^2(\sigma) \Big(Z_i{}^k Z_j{}^\ell \langle \mathcal{O}_k(\sigma)\mathcal{O}_\ell(0)\rangle_{\text{UV}} \\
&- \quad g^k \int d^2\sigma' \sqrt{\hat{\gamma}}\, \langle \mathcal{O}_i(\sigma)\mathcal{O}_j(0)\mathcal{O}_k(\sigma')\rangle_{\text{UV}} + \mathcal{O}(g^2) \Big) \,,
\end{aligned}
$$

Perturbative Monotonicity

Without loss of generality, we can assume deformation under a single relevant operator, to get,

$$\beta(g_{\text{IR}}) = 0 \quad \Rightarrow \quad g_{\text{IR}} = \frac{\epsilon}{\pi C} \quad \Rightarrow \quad \Delta b = -\frac{\epsilon^3}{C^2} < 0 \; .$$

This is a perturbative proof that RDE captures the irreversibility of the DRG flow.

Perturbative Monotonicity

Without loss of generality, we can assume deformation under a single relevant operator, to get,

$$\beta(g_{\text{IR}}) = 0 \quad \Rightarrow \quad g_{\text{IR}} = \frac{\epsilon}{\pi C} \quad \Rightarrow \quad \Delta b = -\frac{\epsilon^3}{C^2} < 0 \ .$$

This is a perturbative proof that RDE captures the irreversibility of the DRG flow.
Moreover, the matrix h_{ij} along the perturbative DRG flow is,

$$h_{ij} = \delta_{ij} + \mathcal{O}(g).$$

Hence, as long as the perturbative expansion is sensible, h_{ij} is positive definite in a small neighborhood of the UV DCFT. In fact, the RDE is perturbatively monotonic to all orders in the coupling constant, and plays the role of a C-function if the UV and IR fixed points are sufficiently close to each other.

Summary

Summary of the current status,

- RDE for 1d defects, completely monotonic along the flow
  Cuomo, Komargodski, Raviv-Moshe '21
- RDE for 2d spherical defects, perturbatively monotonic.
- RDE for $d > 2$ spherical defects, no manifest monotonicity (most likely!).
- However, a very interesting new result claims to prove monotonicity for p=2,3,4 dimensional defects using SSA and QNEC! Cassini, Landea, Torroba '22

RDE for Higher dimensional defects?

- In principle, possible to construct an RDE for higher dimensional spherical defects using the same algorithm.
- However, higher d-defects $\implies$ more divergences $\implies$ more derivatives w.r.t. R to define RDE.
- More derivatives of $R \implies$ dependence on higher point functions of $\hat{T}$.
- No manifest positivity of RDE guaranteed, let alone monotonicity.

Conclusions

- In light of the new information theoretic proofs for monotonicity of $p \geq 2$ defects, can the RDE be made non-perturbatively monotonic in 2d?
- Can integrated 3 pt. functions of the defect stress tensor be manifestly positive? Is there an analog of ANEC/QNEC for stress-tensors on defects?
- If true, can this idea be extended to define a monotonic RDE for $d > 2$ defects, as well?

Contact terms

To isolate the contribution of the contact term, we evaluate $\mathcal{F}$ at the UV fixed point of the DRG flow. To this end, we note that the UV DCFT satisfies,

$$\langle \hat{T} \rangle_{\text{UV}} = \frac{b_{\text{UV}}}{24\pi}\mathcal{R} \quad \Rightarrow \quad \langle \hat{T}(\sigma_1)\hat{T}(\sigma_2)\rangle_{\text{UV}} = -\frac{b_{\text{UV}}}{12\pi}(\mathcal{R}+\nabla^2)\frac{\delta(\sigma_1,\sigma_2)}{\sqrt{\hat{\gamma}(\sigma_1)}} \ ,$$

where the contact term on the right is obtained by varying the anomaly term on the left with respect to the induced metric on the defect. Substituting this expression into the expression for RDE, yields the expected result $S_{\text{UV}} = \frac{b_{\text{UV}}}{3}$.

Free Energy on the spherical defect

For a $2d$ defect of characteristic size R embedded in a flat Euclidean space, the $\mathcal{F}$-function at the UV fixed point of the RG flow takes the form (there are additional contributions if the ambient Euclidean space is curved),

$$\mathcal{F}^{UV}_{\text{DCFT}} = c_0 + \frac{a_0\,\mu_{UV}^2}{4\pi}\int d^2\sigma\,\sqrt{\hat{\gamma}} - \left(\frac{b_0}{24\pi}\int d^2\sigma\,\sqrt{\hat{\gamma}}\,\mathcal{R} + \frac{b_1}{24\pi}\int d^2\sigma\,\sqrt{\hat{\gamma}}\,\text{Tr}\big(\tilde{K}_\mu \tilde{K}^\mu\big)\right)\log(\mu_{UV} R)\ .$$

Here, $\mathcal{R}$ is the Ricci scalar of the defect, whereas $\tilde{K}^\mu_{ac} = K^\mu_{ac} - \frac{1}{2}\hat{\gamma}_{ac}\text{Tr}(K^\mu)$ is the traceless part of the defect extrinsic curvature K^μ_{ac}. The constants in the above expression are functions of the critical couplings. This ansatz is obtained by solving the Wess-Zumino consistency conditions at the fixed points of the DRG flow. Moreover, for a sphere in flat space $\tilde{K}^\mu_{ac} = 0$.

Ward Identities involving Defects

The total stress tensor of the theory is,

$$T^{\text{tot}}_{\mu\nu} = T_{\mu\nu} + \hat{T}_{\mu\nu}\,\delta_{\mathcal{D}}$$

The bulk stress tensor $T^{\mu}_{\mu} = 0$. Presence of the defect introduces an infinite number of new Ward identities into the system. We only need one out of those to prove our results.

$$\int_{S^2} \left\langle \hat{T}\,(\hat{n}) \right\rangle = \frac{1}{2} \int_{S^2} \int_{S^2} (\hat{n}_1 \cdot \hat{n}_2) \left\langle \hat{T}\,(\hat{n}_1)\, \hat{T}\,(\hat{n}_2) \right\rangle .$$

This relation is in fact true for any p-dimensional defect.
Also add the stress tensor, conservation, and displacement operator,

Ward Identities for Defects

- 2 important properties of the DQFTs: 1. Locality, 2. Unitarity (will feature later on)
- Locality: implies 1. all non-local interactions in the DQFT are through the bulk.
- 2. The Ward identity for the stress tensor, involving the displacement operator, is satisfied.
- The 1d DCFT trivially has no stress-tensor. Perturbing the DCFT by a massive/relevant parameter reintroduces the stress tensor and allows one to localise energy on the defect again.
- In 2d DCFT, we can show via explicit computation (???) that the stress tensor vanishes in the UV and the IR DCFTs. It re-appears along the flow since we know that T=beta O.
- What about topological defects???

Ward Identities for Defects

- Consider a 2d spherical defect embedded in a d-dimensional Euclidean bulk CFT. Think of it as an extended 2 dimensional non-local operator.
- Since we begin with a DCFT, the charges corresponding to the SO(p+1,1) symmetry group wrap the surface defect at a finite distance.
- Conformal invariance then implies that the expectation value of these charges with the DCFT vanish.
- Image of the charges wrapping the spherical defect.