Holographic Torus Entanglement and its RG flow

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KU LEUVEN

V Postgraduate Meeting On Theoretical Physics
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fwo Research Foundation Flanders
Opening new horizons

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Université de Montréal

Based on:

PB, Witczak-Krempa, arXiv:1611.01846

Outline

1- Introduction

Entanglement Entropy

Definition

Starting point: bipartite quantum system ⟷ $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Choose some state $\rho \in \mathcal{H}$

Integrate out d.o.f. in the complement ⟶ $\rho_A = \mathrm{Tr}_B\, \rho$

Compute Von Neumann entropy of ρ_A

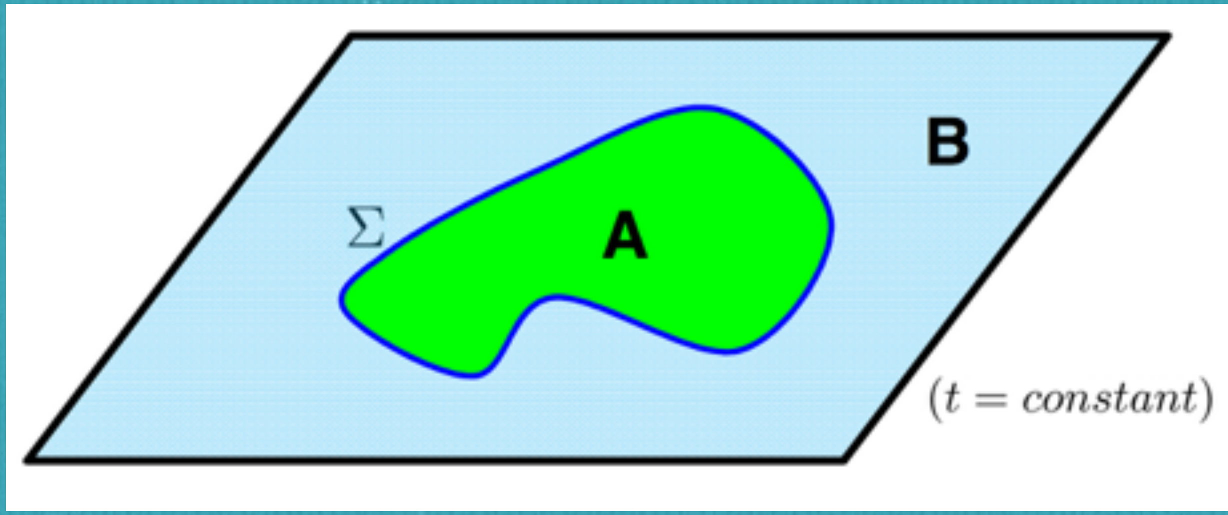
Entanglement Entropy

$$S_{EE} = -\,\mathrm{Tr}\left[\rho_A \log \rho_A\right]$$

Counts # of entangled bits between A and B

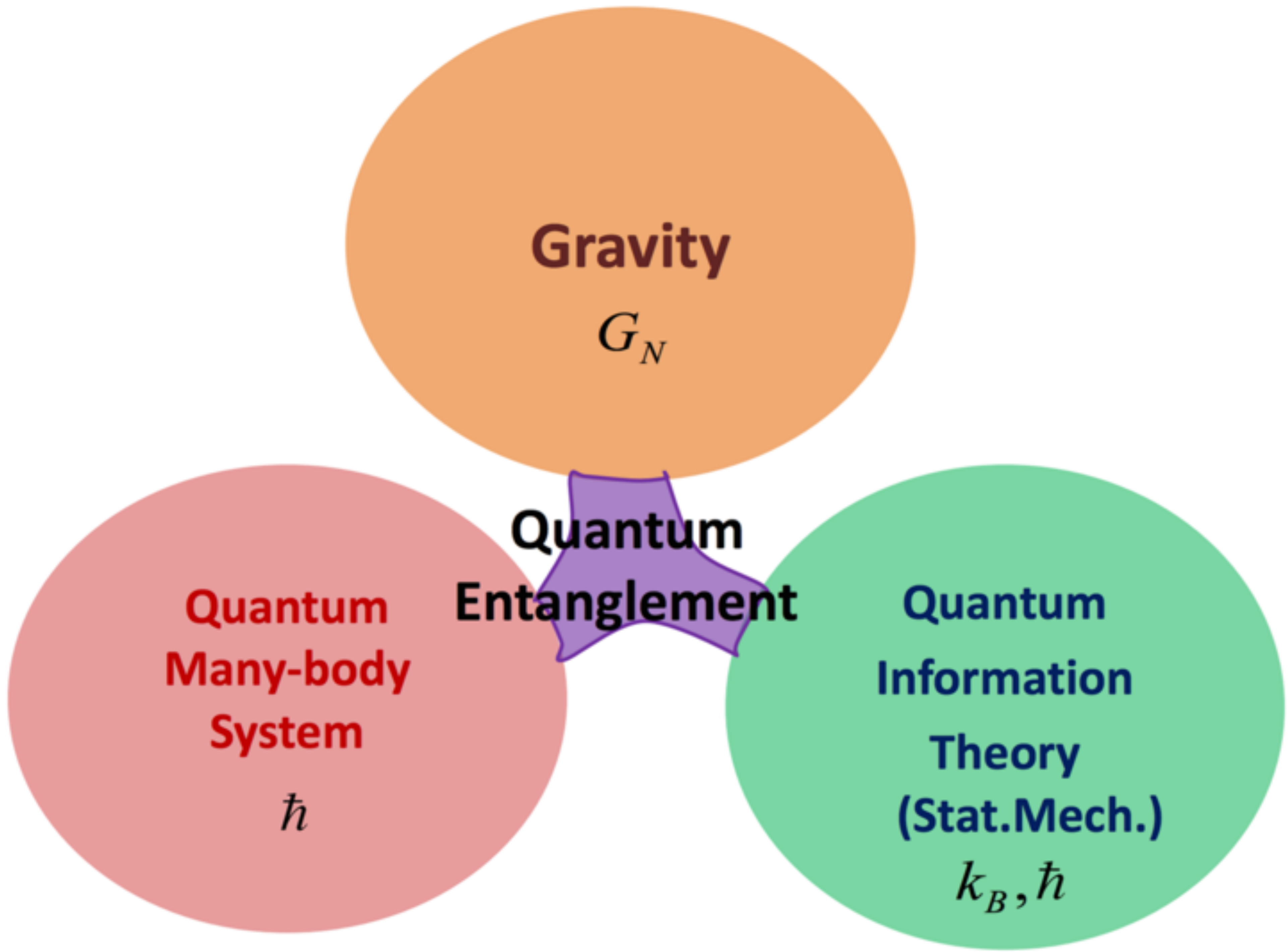
Geometric entropy ⟷ ***A*** **is a spatial region**

Focus in this talk



Entanglement Entropy

Why?



cred. T. Takayanagi

1.1- Entanglemet Entropy in (3d) QFTs

Entanglement Entropy

In QFTs

QFT$_d$ ⟶ **UV divergent** but **universal terms**

$$S_{EE}(A) = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_1 \frac{R^{d-3}}{\delta^{d-3}} + \dots$$

regulator dependent!

Leading term controlled by local correlations at both sides of Σ: Area law

(Volume law for excited states)

Some of the subleading terms encode well-defined information about the theory

Entanglement Entropy

In 3d CFTs

General structure:

$$S_{3d} = \mathcal{B}\frac{L}{\epsilon} + S_{\text{univ}}$$

"Area"-law term

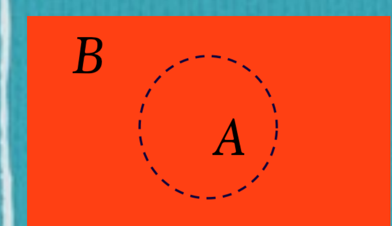
Relevant IR scale

Non-universal constant

Universal term

UV cut-off

Disk region



$$S_{\text{univ}} = -F$$

pure constant

equals Free energy on S^3

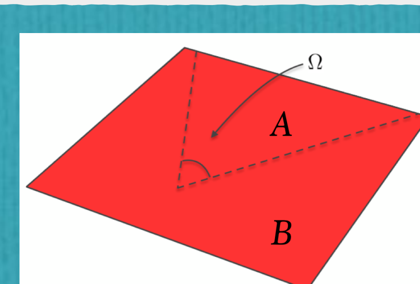
Casini, Huerta, Myers

F-theorem!

Casini, Huerta

3d version of "c-theorem"

Corner region

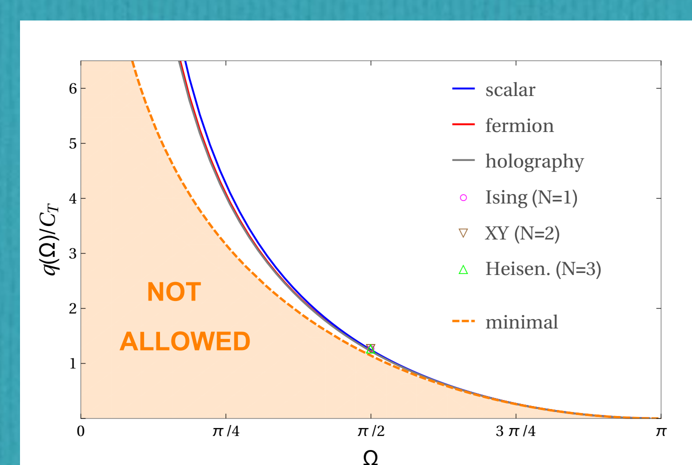


$$S_{\text{univ}} = -q(\Omega)\log\left(L/\epsilon\right)$$

controlled by local correlator

$$\langle T_{ab}(x)T_{cd}(0)\rangle = C_T\frac{\mathcal{I}_{ab,cd}}{x^6}$$

exact in almost smooth limit



$$q(\Omega\sim\pi) = \sigma\cdot(\pi-\Omega)^2$$

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24}$$

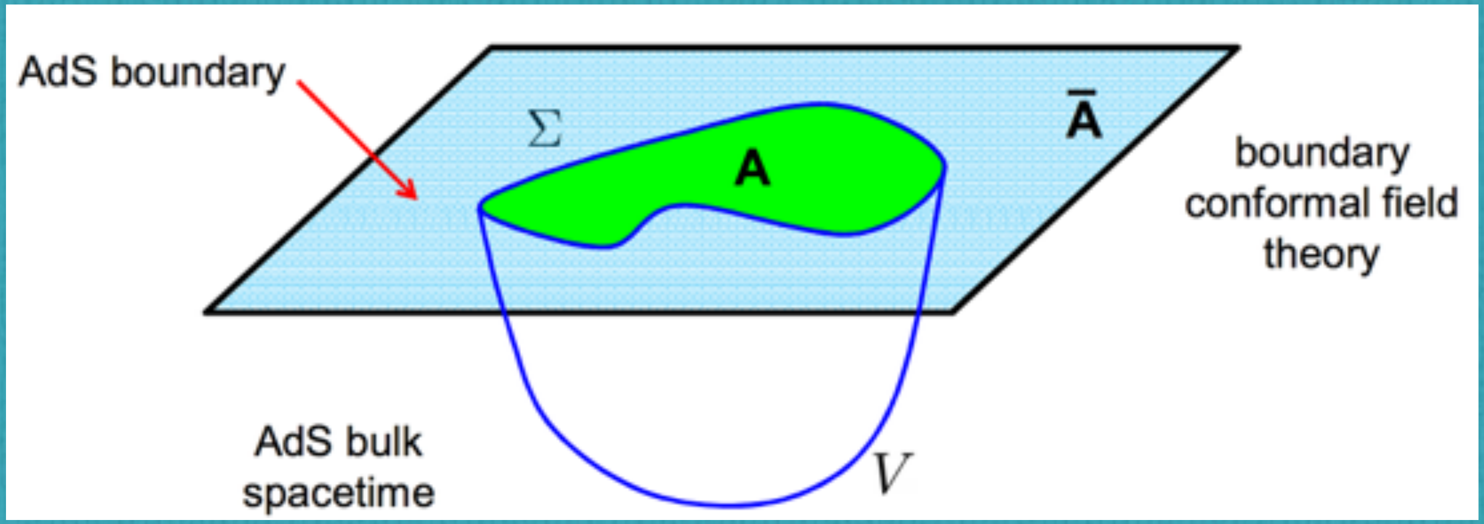
PB, Myers, Witczak-Krempa

1.2- Holographic Entanglement Entropy

Holographic Entanglement Entropy

Ryu-Takayanagi prescription

for CFTs dual to **Einstein gravity**



Ryu Takayanagi +1000 citations!

$$S_{EE}(A) = \min_{V \sim A} \left[\frac{\mathcal{A}(V)}{4G} \right]$$

Extremize area functional $\mathcal{A}(V)$ over all bulk surfaces V whose boundary coincides with Σ

Evaluate $\mathcal{A}(V)$ on the extremal V

Tons of consistency checks and applications *Many people*

Proof *Lewkowycz, Maldacena*

Generalized to: time dependence *Hubeny, Rangamani, Takayanagi*

higher-order gravities *Many people*

quantum corrections, etc. *Faulkner, Lewkowycz, Maldacena*

2- Torus Entanglement

Torus Entanglement

Smooth curved surfaces are not ideal for numerics

Pixelization leads to corners → pollutes the result with log. terms

Finite-size effects also pollute unless total space much larger than A

Alternative

↓

Flat but finite entangling surfaces

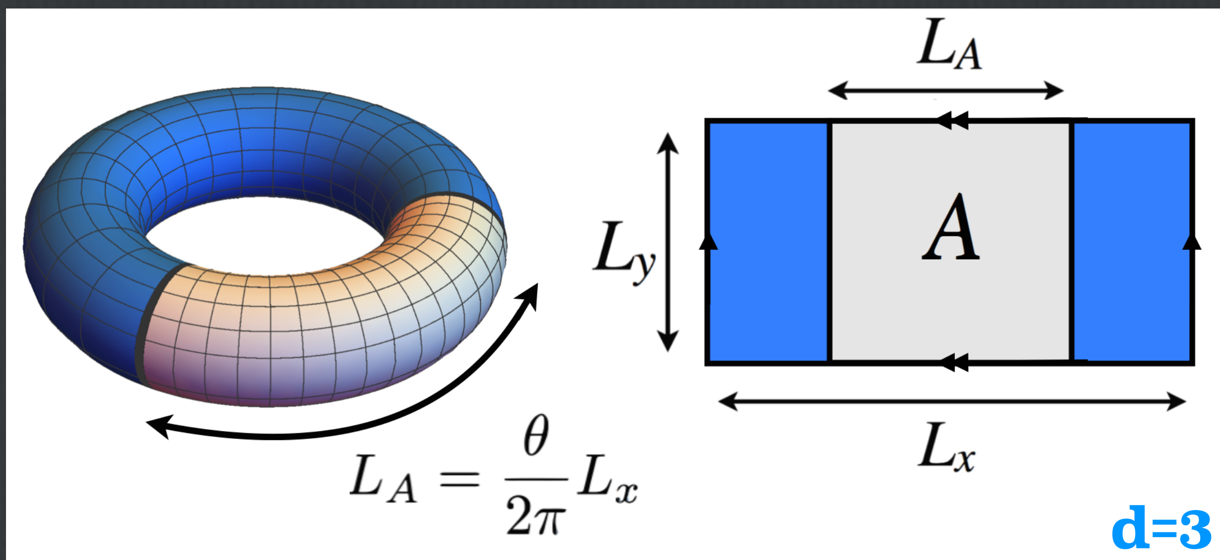
↓

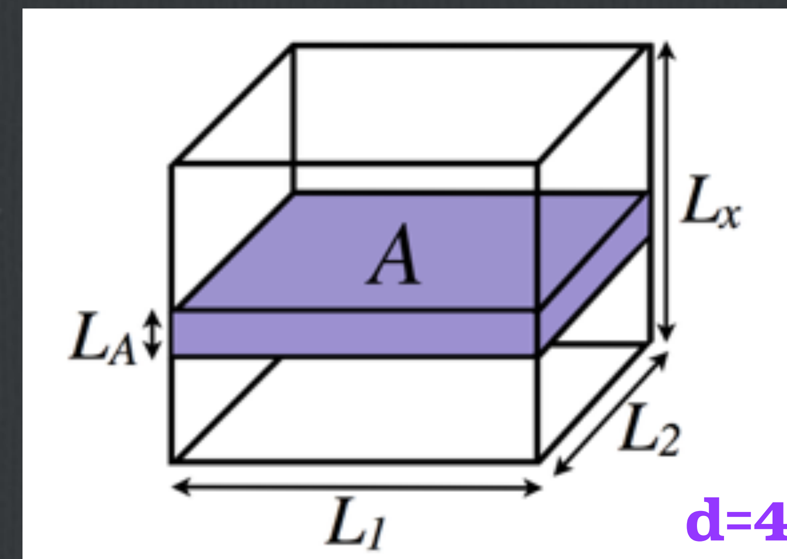
Spaces with non-trivial topology

less explored

2.1- General results & holography

Torus Entanglement





Spatial dimensions form a $T^{(d-1)}$

$$S(A) = B\,\frac{\text{Area}(\partial A)}{\epsilon^{d-2}} - \chi(\theta; b_i) + \cdots$$

$$b_i = L_x / L_i$$

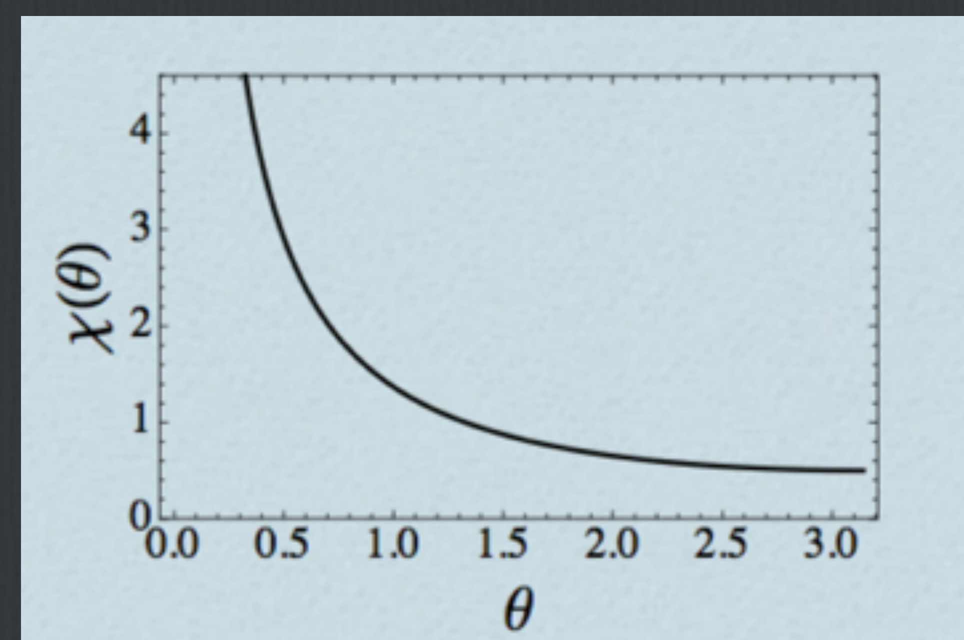
Regulator independent

SSA $\chi'(\theta) \leq 0\,, \qquad \chi''(\theta) \geq 0,$

$$\chi(\theta \to 0) = \frac{(2\pi)^{d-2}\,\kappa}{\theta^{d-2}\, b_1 \cdots b_{d-2}}$$

$$\chi(\theta \approx \pi) = \sum_{\ell=0} c_\ell \cdot (\pi - \theta)^{2\ell}$$

Witczak-Krempa, Hayward, Melko



Chen, Cho, Faulkner, Fradkin

Torus Entanglement

Holography $\xrightarrow{(T=0)}$ **AdS solitons**

Witten; Horowitz, Myers

$$ds^2 = \frac{1}{z^2}\left[\frac{dz^2}{f(z)} + f(z)\,dx^2 + d\vec{y}^{\,2}_{(d-2)} - dt^2\right]$$

$f(z) = 1 - (z/z_h)^d$ doubly-Wick-rotated black branes

$L_x = 4\pi z_h/d$ smallest dim. must be compact

If additional (d-2) spatial dimensions also periodic → **conformal boundary foliated by $T^{(d-1)}$**

AdS solitons dominate the semiclassical partition function at small temperatures

Torus Entanglement

Focus in this talk: d=3

Chen, Cho, Faulkner, Fradkin
PB, Witczak-Krempa

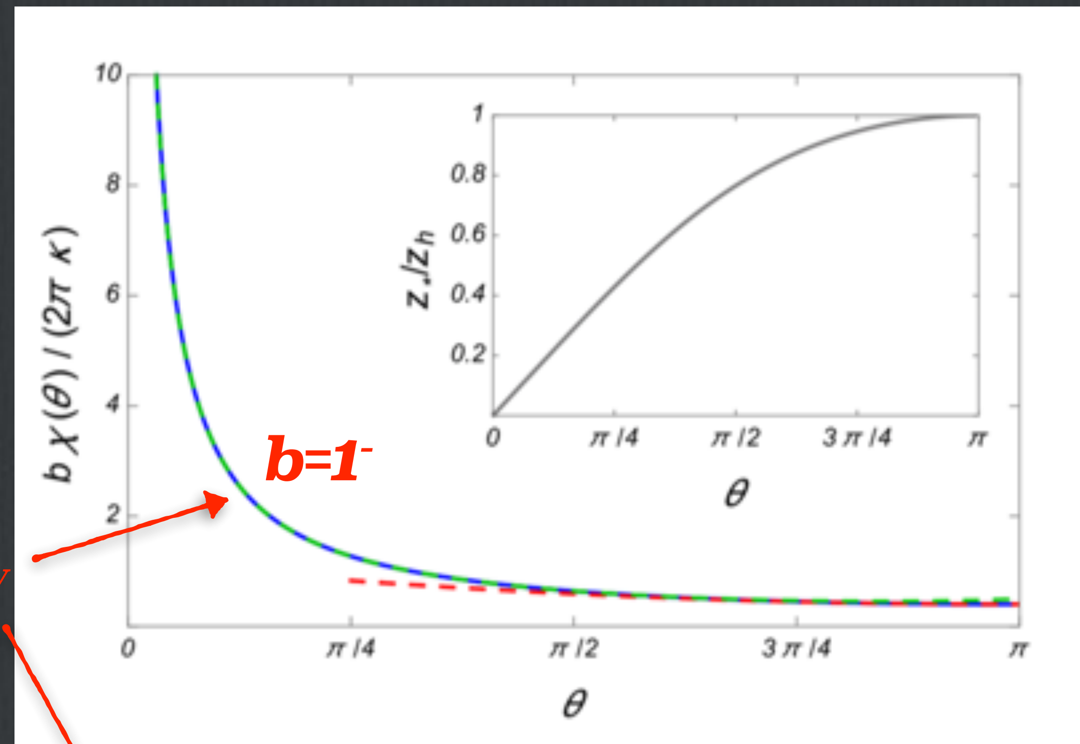
$L_y > L_x$

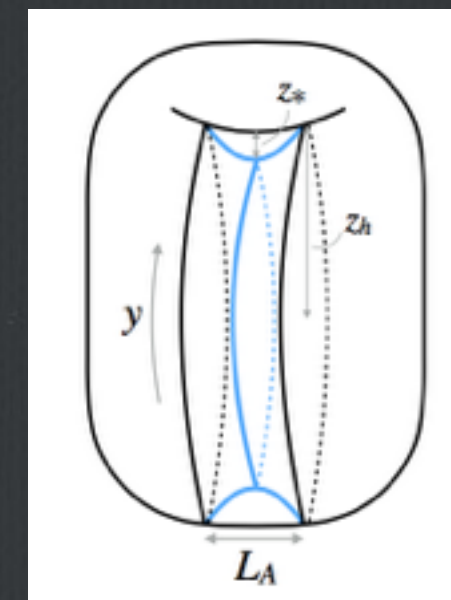
$$\chi(\theta) = \left[\frac{2\pi\kappa}{b}\right]\frac{1}{\theta} + \left[\frac{\Gamma(\frac{1}{4})^{12}}{1306368\pi^7\, b\, G}\right]\theta^5 + \cdots$$

Corner like

(Non-generic) jump at L_x=L_y

Chen, Cho, Faulkner, Fradkin



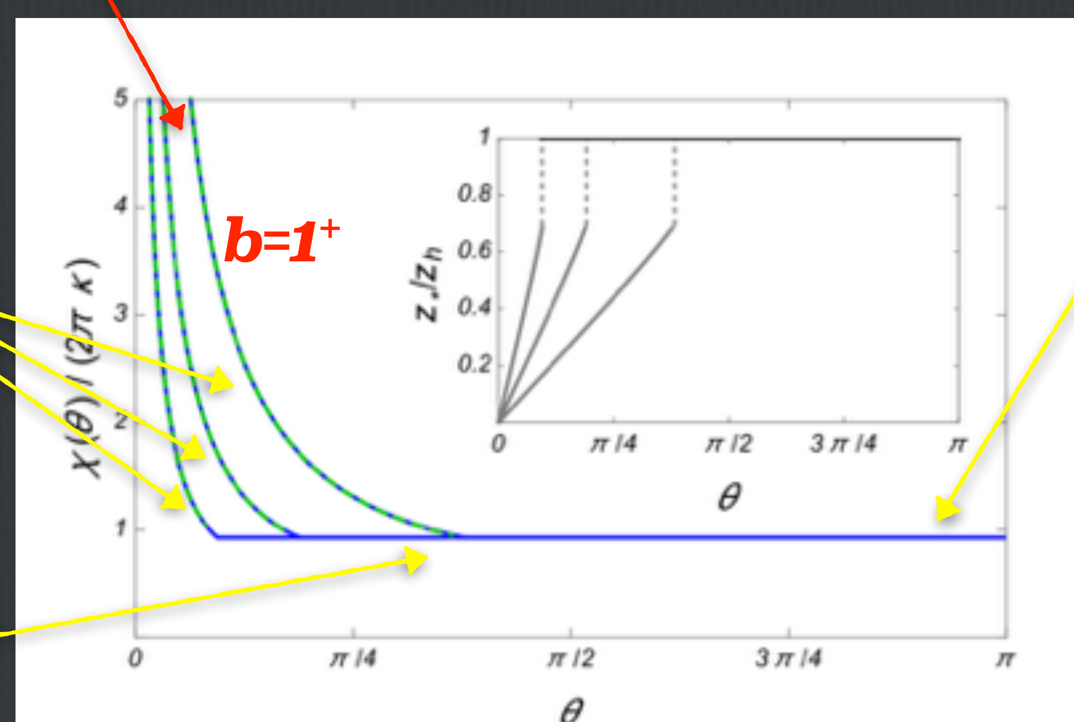


$L_x > L_y$

$$\chi(\theta) = \left[\frac{2\pi\kappa}{b}\right]\frac{1}{\theta} + \left[\frac{\Gamma(\frac{1}{4})^4 b^2}{432\pi\, G}\right]\theta^2 + \cdots$$

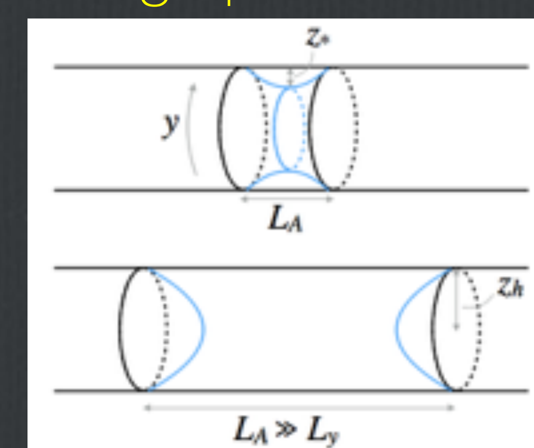
$$0 < \frac{\theta}{2\pi} \leq \frac{p}{b}$$

$$p \simeq 0.1889$$



$$\chi(\theta) = \frac{2\pi}{3G}, \qquad \frac{p}{b} < \frac{\theta}{2\pi} \leq \frac{1}{2}$$

Disconnected holographic surface



Non-smoothness (large-N)
Smooth for free scalar

Witczak-Krempa, Hayward, Melko

2.2- Renormalized EE

Torus Entanglement

PB, Witczak-Krempa

Focus on cylinder limit with A covering half of it

$$S = \frac{L_y}{4G\epsilon} - \gamma$$

$$\gamma = \frac{\pi}{3G}$$

Pure constant

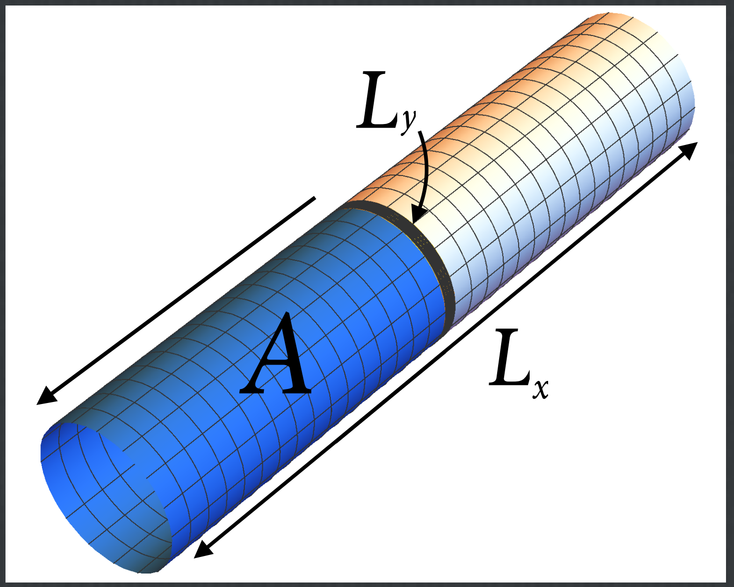
Very similar to disk entanglement

$$S_{\text{disk}} = \mathcal{B}R/\epsilon - F$$

F-theorem

Casini, Huerta

Casini, Huerta, Myers, Yale



Monotonicity theorem for γ?

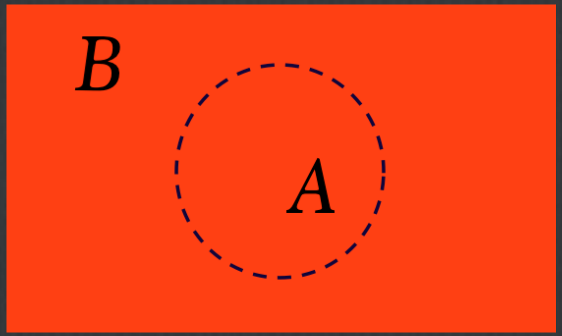
Renormalized EE

PB, Witczak-Krempa

Away from the fixed point, additional divergences

Solution → **use "Renormalized" EE** Liu, Mezei

Disk



$$S_{\text{disk}} = \mathcal{B}R/\epsilon - F$$

↓

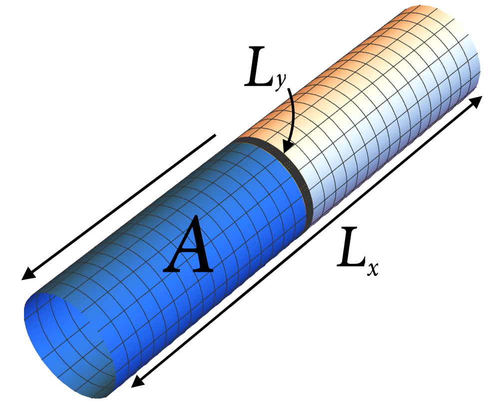
$$\mathcal{F}(R) = -S(R) + R\frac{\partial S(R)}{\partial R}$$

Liu, Mezei

Removes divergences
Isolates ***F*** at fixed point

This is in fact the quantity that satisfies the **F-theorem**

Half cylinder



$$S_{\text{half cyl.}} = \mathcal{B}L_y/\epsilon - \gamma$$

↓

$$\gamma_r^{(\alpha)}(L_y) = -S + L_y\frac{\partial S}{\partial L_y} + \alpha L_y^2\frac{\partial^2 S}{\partial L_y^2}$$

Removes divergences
Isolates $\boldsymbol{\gamma}$ at fixed points

(d=4 case is more interesting)

2.3- Holographic RG flow

Holographic RG flow

Deform CFT with relevant scalar perturbation

$$S = S_{\rm CFT} + \lambda \int d^d x\, O(x) \quad \longrightarrow \quad I = \int \frac{d^4x\sqrt{-g}}{16\pi G}\left[6 + R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \cdots\right] \qquad m^2 = \Delta(\Delta - 3)$$

Expansion parameter → $\lambda z_{h0}^{3-\Delta} \propto \lambda L_y^{3-\Delta}$

unitary bound $1/2 < \Delta < 3$ marginality

Perturbed metric

$$ds^2 = \frac{1}{z^2}\left[\frac{dz^2}{f\cdot g_1(z)} + dx^2 + f\cdot g_2(z)\,dy^2 - dt^2\right]$$

Regularity

↓

$$z_h = \frac{3L_y}{4\pi}\sqrt{g_1(z_h)g_2(z_h)}$$

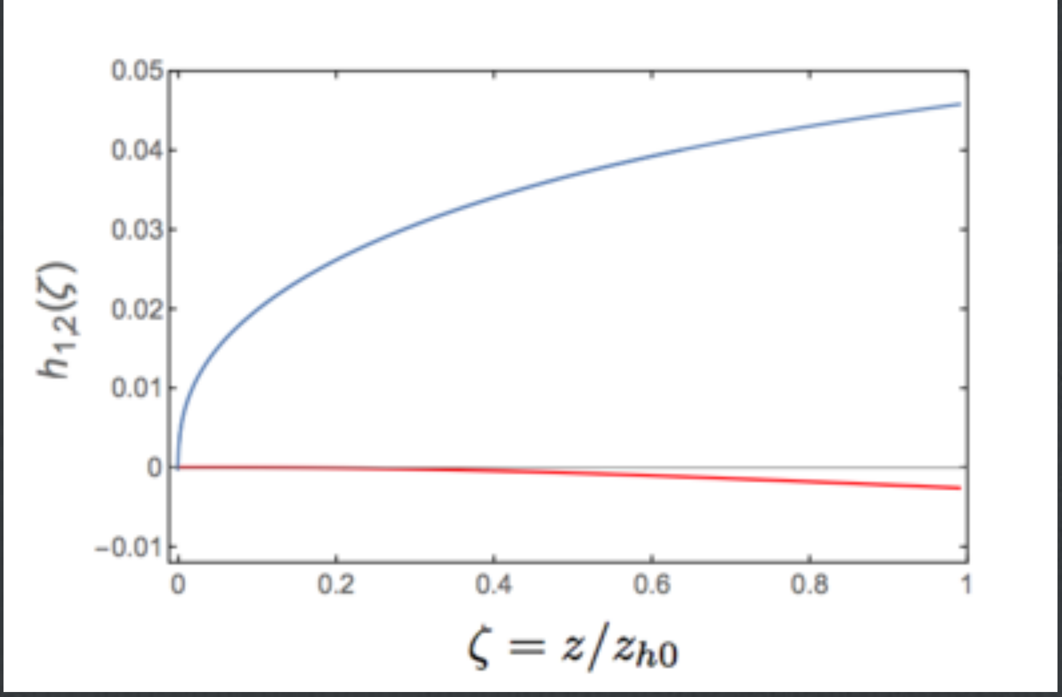
Scalar field back-reaction at leading order fixes

do some calculations!

↓

$$g_i(z) = 1 + h_i(\zeta)\; z_{h0}^{2(3-\Delta)}\lambda^2$$

→ **where** →



↓

EE gets corrected in the perturbed geometry

Holographic RG flow

PB, Witczak-Krempa

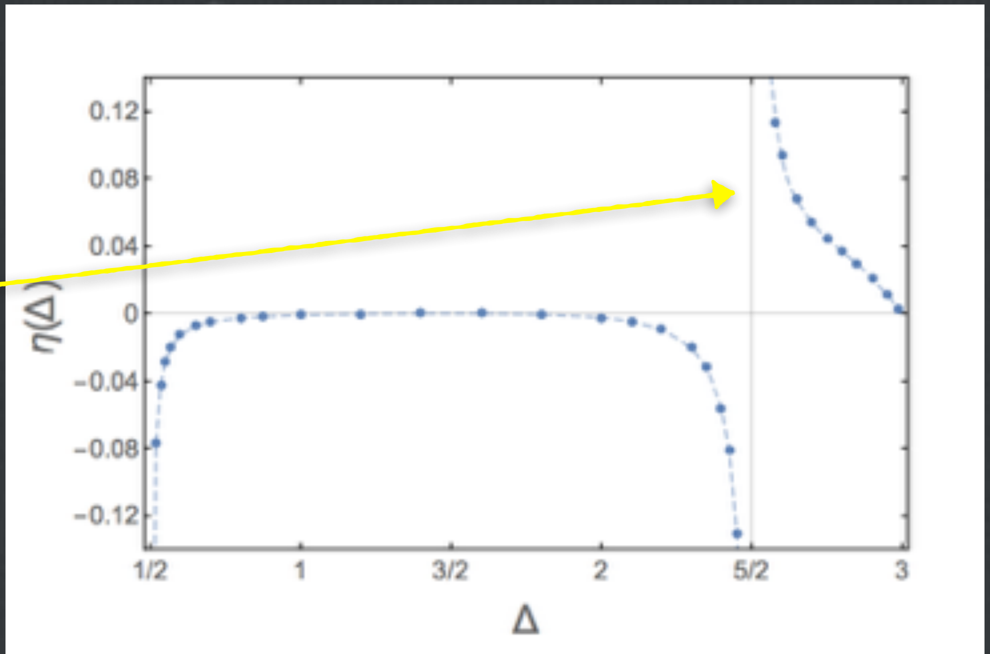
$$S = \frac{L_y}{2G\epsilon} - \frac{(3-\Delta)}{32(\Delta-5/2)}\frac{L_y\lambda^2}{\epsilon^{2\Delta-5}G} + \frac{(\frac{3}{4\pi})^{6-4\Delta}D_{0\Delta}}{4(2\Delta-1)}\frac{L_y^{7-4\Delta}\lambda^2}{\epsilon^{1-2\Delta}G} - 2\gamma + \cdots$$

Extra divergences
Naive subtraction of area law fails

Corrected universal constant

$$\gamma = \frac{\pi}{3G}\left[1-\eta(\Delta)L_y^{2(3-\Delta)}\lambda^2\right]$$

Unphysical divergence at Δ=5/2



$$\eta(\Delta) = \left(\frac{3}{4\pi}\right)^{2(3-\Delta)}\left[\frac{h_1(1)+h_2(1)}{2} + \frac{3-\Delta}{16(\Delta-5/2)} - \frac{D_{0\Delta}}{2(2\Delta-1)} - \int_0^1 \frac{d\zeta}{2\zeta^2}\left(h_1(\zeta) - \frac{3-\Delta}{4}\zeta^{2(3-\Delta)} - D_{0\Delta}\zeta^{2\Delta}\right)\right]$$

Holographic RG flow

PB, Witczak-Krempa

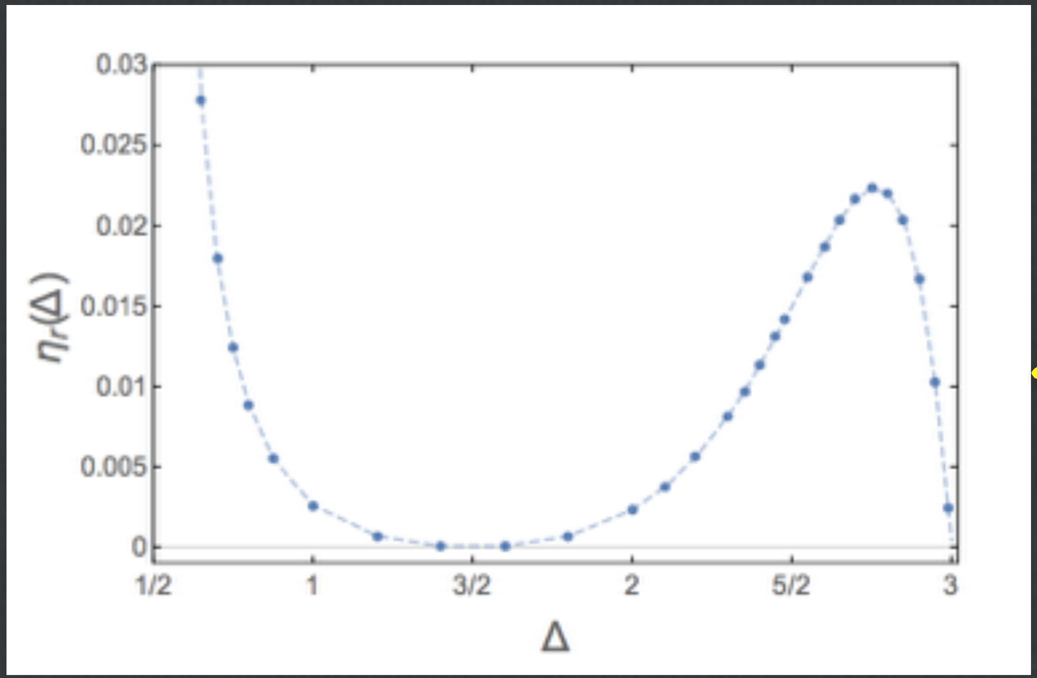
Use REE!

$$\gamma_r(L_y) = -S(L_y) + L_y \frac{\partial S(L_y)}{\partial L_y}$$

Removes all divergences and issue at Δ=5/2

$$\gamma_r(L_y) = \frac{\pi}{3G}\left[1 - \eta_r(\Delta) L_y^{2(3-\Delta)}\lambda^2\right]$$

$$\eta_r(\Delta) = (2\Delta - 5)\eta(\Delta)$$



Decreasing for all Δ!

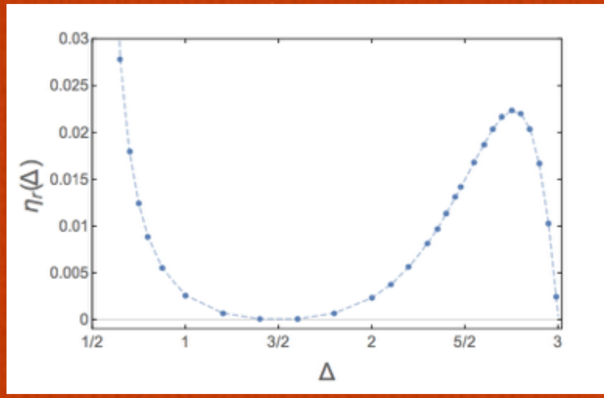
3- Summary

Defining good measures of degrees of freedom in CFTs is challenging

F-theorem for disk REE achieves it. But smooth regions not nice for simulations

Torus EE quite unexplored. Alternative to smooth regions.

In d=3 always decreasing at leading order for a particular holographic RG flow.



$$\gamma_r(L_y) = \frac{\pi}{3G}\left[1-\eta_r(\Delta)L_y^{2(3-\Delta)}\lambda^2\right]$$

What happens for more complicated flows and outside holography?

REE not uniquely defined in d=4.

$$\gamma_r^{(\alpha)}(L_1;r) = -S(L_1) + (1-\alpha)\frac{L_1}{2}\frac{\partial S(L_1)}{\partial L_1} + \alpha\frac{L_1^2}{2}\frac{\partial^2 S(L_1)}{\partial L_1^2}$$

Thank you

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Opening new horizons

Delta ITP visitor programme

d=4

$$S = \frac{L_1 L_2}{8G\epsilon^2} - \gamma$$

$$\gamma = \frac{\pi^2}{8G}\frac{L_2}{L_1}$$

Depends on aspect ratio

Very different from smooth surface entanglement in flat space

$$S_{\text{univ}} = I(a, c)\log(l/\epsilon)$$

Free en. on S^4 when A sphere

↓

a-theorem

Myers, Sinha

Komargodski, Schwimmer (no use of EE)

Smooth (sphere)

$$S_{\text{sphere}} = \mathcal{B}(R/\epsilon)^2 - 4a\log(R/\epsilon)$$

$$\mathcal{S}(R) = \frac{1}{2}\left[R^2\frac{\partial^2 S(R)}{\partial R^2} - R\frac{\partial S(R)}{\partial R}\right]$$

Liu, Mezei

Removes divergences
Isolates **a** at fixed points

Half cylinder

Scalable region fix

$$r = L_2/L_1$$

Removes divergences
Isolates **γ** at fixed points

$$\gamma_r^{(\alpha)}(L_1; r) = -S(L_1) + (1-\alpha)\frac{L_1}{2}\frac{\partial S(L_1)}{\partial L_1} + \alpha\frac{L_1^2}{2}\frac{\partial^2 S(L_1)}{\partial L_1^2}$$

Family of REE