Holographic Torus
Entanglement and its RG flow

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Based on:
PB, Witczak-Krempa, arXiv:1611.01846
1- Introduction
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3- Summary
1- Introduction
**Entanglement Entropy**

**Definition**

**Starting point:** bipartite quantum system  \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \)

Choose some state  \( \rho \in \mathcal{H} \)

Integrate out d.o.f. in the complement

Compute Von Neumann entropy of  \( \rho_A \)

\[ S_{EE} = - \text{Tr} [\rho_A \log \rho_A] \]

**Entanglement Entropy**

Counts # of entangled bits between A and B

**Focus in this talk**

Geometric entropy  \( A \) is a spatial region

\[ (t = \text{constant}) \]
Hard to compute!
Numerics
Free fields

CM: Order parameter
QFT: F-theorem
Quantum gravity

Wen, Levine; Kitaev, Preskill…
Useful in
Melko et al…
Islam et al.
Measurable?
Casini, Huerta…
A lot of people!

Disk entanglement
cred. T. Takayanagi

Entanglement Entropy

Why?

Gravity
$G_N$

Quantum Entanglement

Quantum Many-body System
$\hbar$

Quantum Information Theory
(Stat.Mech.)
$k_B, \hbar$

cred. T. Takayanagi
1.1- Entanglement Entropy in (3d) QFTs
Entanglement Entropy

In QFTs

$\text{QFT}_d \quad \longrightarrow \quad \text{UV divergent but universal terms}$

$$S_{EE}(A) = c_0 \frac{R^{d-2}}{\delta_{d-2}} + c_1 \frac{R^{d-3}}{\delta_{d-3}} + \ldots$$

Leading term controlled by local correlations at both sides of $\Sigma$: **Area law**

(Regulator dependent!

Some of the subleading terms encode well-defined information about the theory

(Volume law for excited states)
Entanglement Entropy

In 3d CFTs

General structure:

\[ S_{3d} = B \frac{L}{\epsilon} + S_{\text{univ}} \]

“Area”-law term

Non-universal constant

Relevant IR scale

Universal term

UV cut-off

Disk region

\[ S_{\text{univ}} = -F \]

pure constant

equals Free energy on \( S^3 \)

F-theorem!

Casini, Huerta

Corner region

\[ S_{\text{univ}} = -q(\Omega) \log \left( \frac{L}{\epsilon} \right) \]

controlled by local correlator

exact in almost smooth limit

\[ q(\Omega \sim \pi) = \sigma \cdot (\pi - \Omega)^2 \]

Universal term

PB, Myers, Witczak-Krempa
1.2- Holographic Entanglement Entropy
Holographic Entanglement Entropy

**Ryu-Takayanagi prescription**

for CFTs dual to *Einstein* gravity

Extremize area functional \( \mathcal{A}(V) \) over all bulk surfaces \( V \) whose boundary coincides with \( \Sigma \)

Evaluate \( \mathcal{A}(V) \) on the extremal \( V \)

Tons of consistency checks and applications

Generalized to: time dependence

higher-order gravities

quantum corrections, etc.

Many people

Proof

Lewkowycz, Maldacena.
2- Torus Entanglement
Smooth curved surfaces are not ideal for numerics

Pixelization leads to corners → pollutes the result with log. terms

Finite-size effects also pollute unless total space much larger than $A$

Alternative

↓

Flat but finite entangling surfaces

↓

Spaces with non-trivial topology

less explored
2.1- General results & holography
Torus Entanglement

Spatial dimensions form a $T^{(d-1)}$

$$S(A) = B \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} - \chi(\theta; b_i) + \cdots$$

$$b_i = L_x / L_i$$

**Regulator independent**

**SSA**

$$\chi'(\theta) \leq 0, \quad \chi''(\theta) \geq 0,$$

$$\chi(\theta \to 0) = \frac{(2\pi)^{d-2} \kappa}{\theta^{d-2} b_1 \cdots b_{d-2}}$$

$$\chi(\theta \approx \pi) = \sum_{\ell=0} c_{\ell} \cdot (\pi - \theta)^{2\ell}$$

Witczak-Krempa, Hayward, Melko

Chen, Cho, Faulkner, Fradkin
Torus Entanglement

(T=0)

Holography $\rightarrow$ AdS solitons

Witten; Horowitz, Myers

$$ds^2 = \frac{1}{z^2} \left[ \frac{dz^2}{f(z)} + f(z) \, dx^2 + dy_{(d-2)}^2 - dt^2 \right]$$

$$f(z) = 1 - (z/z_h)^d$$

doubly-Wick-rotated black branes

$$L_x = 4\pi z_h/d$$

smallest dim. must be compact

If additional (d-2) spatial dimensions also periodic $\rightarrow$ conformal boundary foliated by $T^{(d-1)}$

AdS solitons dominate the semiclassical partition function at small temperatures
Focus in this talk: \(d=3\)

**Torus Entanglement**

For \(L_y > L_x\):

\[
\chi(\theta) = \left[ \frac{2\pi \kappa}{b} \right] \frac{1}{\theta} + \left[ \frac{\Gamma(\frac{1}{4})^{12}}{1306368\pi^7 b G} \right] \theta^5 + \cdots
\]

Corner like

(Non-generic) jump at \(L_x = L_y\)

For \(L_x > L_y\):

\[
\chi(\theta) = \left[ \frac{2\pi \kappa}{b} \right] \frac{1}{\theta} + \left[ \frac{\Gamma(\frac{1}{4})^4 b^2}{432\pi G} \right] \theta^2 + \cdots
\]

\[
0 \leq \frac{\theta}{2\pi} \leq \frac{p}{b} \quad p \simeq 0.1889
\]

Non-smoothness (large-N)

Smooth for free scalar

Disconnected holographic surface

Chen, Cho, Faulkner, Fradkin

PB, Witczak-Krempa

Witczak-Krempa, Hayward, Melko
2.2- Renormalized EE
Torus Entanglement

Focus on cylinder limit with A covering half of it

\[ S = \frac{L_y}{4G\epsilon} - \gamma \]
\[ \gamma = \frac{\pi}{3G} \]

\[ S_{\text{disk}} = \mathcal{B}R/\epsilon - F \]

F-theorem
Casini, Huerta
Casini, Huerta, Myers, Yale

Pure constant

Very similar to disk entanglement

Monotonicity theorem for \( \gamma \)?
Renormalized EE

Away from the fixed point, additional divergences

Solution $\rightarrow$ use “Renormalized” EE

Disk

$$S_{\text{disk}} = B R / \epsilon - F$$

$$F(R) = -S(R) + R \frac{\partial S(R)}{\partial R}$$

Removes divergences
Isolates $F$ at fixed point

Half cylinder

$$S_{\text{half cyl.}} = BL_y / \epsilon - \gamma$$

$$\gamma_{\alpha}^{(L_y)} = -S + L_y \frac{\partial S}{\partial L_y} + \alpha L_y^2 \frac{\partial^2 S}{\partial L_y^2}$$

Removes divergences
Isolates $\gamma$ at fixed points

This is in fact the quantity that satisfies the **F-theorem**

(d=4 case is more interesting)
2.3- Holographic RG flow
Holographic RG flow

Deform CFT with relevant scalar perturbation

\[ S = S_{\text{CFT}} + \lambda \int d^d x O(x) \]
\[ I = \int \frac{d^4 x \sqrt{-g}}{16\pi G} \left[ 6 + R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \ldots \right] \]
\[ m^2 = \Delta (\Delta - 3) \]

Expansion parameter

\[ \lambda z_{h0}^{3-\Delta} \propto \lambda L_y^{3-\Delta} \]

Unitary bound

\[ 1/2 < \Delta < 3 \]

Marginality

Perturbed metric

\[ ds^2 = \frac{1}{z^2} \left[ \frac{dz^2}{f \cdot g_1(z)} + dx^2 + f \cdot g_2(z) dy^2 - dt^2 \right] \]

Regularity

\[ z_h = \frac{3L_y}{4\pi} \sqrt{g_1(z_h)g_2(z_h)} \]

Scalar field back-reaction at leading order fixes

\[ g_i(z) = 1 + h_i(\zeta) \frac{z_{h0}^{2(3-\Delta)}}{\lambda^2} \]

Where

EE gets corrected in the perturbed geometry
Holographic RG flow

Unphysical divergence at $\Delta = 5/2$

Extra divergences
Naive subtraction of area law fails

Corrected universal constant

\[
S = \frac{L_y}{2G\epsilon} - \frac{(3 - \Delta)}{32(\Delta - 5/2)} \frac{L_y \lambda^2}{\epsilon^{2\Delta - 5} G} + \frac{(3/4\pi)^6}{4(2\Delta - 1)} D_0 \Delta \frac{L_y^{7-4\Delta} \lambda^2}{\epsilon^{1-2\Delta} G} - 2\gamma + \ldots
\]

\[
\gamma = \frac{\pi}{3G} \left[ 1 - \eta(\Delta) L_y^{2(3-\Delta)} \lambda^2 \right]
\]

\[
\eta(\Delta) = \left( \frac{3}{4\pi} \right)^{2(3-\Delta)} \left[ \frac{h_1(1) + h_2(1)}{2} + \frac{3 - \Delta}{16(\Delta - 5/2)} - \frac{D_0 \Delta}{2(2\Delta - 1)} - \int_0^1 \frac{d\zeta}{2\zeta^2} \left( h_1(\zeta) - \frac{3 - \Delta}{4} \zeta^{2(3-\Delta)} - D_0 \Delta \zeta^{2\Delta} \right) \right]
\]

PB, Witczak-Krempa
Holographic RG flow

Use REE!

\[ \gamma_r(L_y) = -S(L_y) + L_y \frac{\partial S(L_y)}{\partial L_y} \]

\[ \gamma_r(L_y) = \frac{\pi}{3G} \left[ 1 - \eta_r(\Delta) L_y^{2(3-\Delta)} \lambda^2 \right] \]

\[ \eta_r(\Delta) = (2\Delta - 5) \eta(\Delta) \]

Removes all divergences and issue at \( \Delta = 5/2 \)

Decreasing for all \( \Delta \)!
3- Summary

Defining good measures of degrees of freedom in CFTs is challenging.

F-theorem for disk REE achieves it. But smooth regions not nice for simulations.

Torus EE quite unexplored. Alternative to smooth regions.

In $d=3$ always decreasing at leading order for a particular holographic RG flow.

\[ \gamma_{r}(L_{y}) = \frac{\pi}{3G} \left[ 1 - \eta_{r}(\Delta)L_{y}^{2(3-\Delta)} \right] \]

What happens for more complicated flows and outside holography?

REE not uniquely defined in $d=4$.

\[ \gamma^{(\alpha)}(L_{1}; r) = -S(L_{1}) + (1 - \alpha) \frac{L_{1}}{2} \frac{\partial S(L_{1})}{\partial L_{1}} + \alpha \frac{L_{1}^{2}}{2} \frac{\partial^{2} S(L_{1})}{\partial L_{1}^{2}} \]
Thank you
Very different from smooth surface entanglement in flat space.

\[ S = \frac{L_1 L_2}{8G\epsilon^2} - \gamma \]

\[ \gamma = \frac{\pi^2}{8G} \frac{L_2}{L_1} \]

Depends on aspect ratio

Free en. on \( S^4 \) when A sphere

**a-theorem**

Myers, Sinha
Komargodski, Schwimmer (no use of EE)

\[ S_{\text{univ}} = I(a, c) \log(l/\epsilon) \]

Smooth (sphere)

\[ S_{\text{sphere}} = B(R/\epsilon)^2 - 4a \log(R/\epsilon) \]

\[ S(R) = \frac{1}{2} \left[ R^2 \frac{\partial^2 S(R)}{\partial R^2} - R \frac{\partial S(R)}{\partial R} \right] \]

Removes divergences
Isolates \( a \) at fixed points

**Half cylinder**

Removes divergences
Isolates \( \gamma \) at fixed points

\[ r = \frac{L_2}{L_1} \]

Scalable region
fix

Family of REE

\[ \gamma_r^{(\alpha)}(L_1; r) = -S(L_1) + (1 - \alpha) \frac{L_1}{2} \frac{\partial S(L_1)}{\partial L_1} + \alpha \frac{L_1^2}{2} \frac{\partial^2 S(L_1)}{\partial L_1^2} \]