Worldsheet-Induced Corrections to the Holographic Veneziano Amplitude

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10/11/2016
Based on: A. Armoni (1509.03077)
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1.Introduction

- Explaining the linear Regge trajectory of mesons helped birth String Theory through the Veneziano amplitude, the 4-point open string amplitude in flat space. Improving upon this relation is a long-standing goal: asymptotic freedom forbids the Regge trajectory to stay linear at all energies.
- Holography provides a bridge between the two setups but involves highly-curved backgrounds in which the Veneziano amplitude is not easily seen. Some work has been done already to recover it.
- In pure field theory, can prove that assuming area-law behaved (i.e. confining) Wilson loops at all energies reproduces Veneziano behaviour (Makeenko, Olesen).
- We notice that in some string backgrounds, can force *any* worldsheet hanging from a Wilson loop at the boundary to exhibit area-law behaviour, recovering the Veneziano amplitude. This destroys almost all contributions from holographic coordinates, but can be systematically be improved upon.

Q: How does curvature of a "realistic" string theory affect the Regge trajectory of mesons in QCD? Does it match observed phenomena?

2. The Worldline Formalism

First we need a set-up reproducing the Veneziano amplitude in holography. To map the QCD amplitude into holography we use the Worldline Formalism. Rewrite the path integral as a sum over Wilson loops:

$$\mathcal{Z} = \int DA\, \exp(-S_{\mathrm{YM}}) \exp\left(-\frac{N_f}{2}\mathrm{Tr}\int_0^\infty \frac{dT}{T}\mathcal{W}_T[A]\right) \tag{1}$$

$$\mathcal{W}_T[A] = \int DxD\psi e^{-\frac{1}{2}\oint_0^T d\tau\ \dot{x}^\mu \dot{x}_\mu + \psi^\mu \dot{\psi}_\mu} e^{i\oint_0^T d\tau\ \dot{x}^\mu A_\mu - \frac{1}{2}\psi^\mu F_{\mu\nu}\psi^\mu} \tag{2}$$

In the large N_c, fixed N_f linearise the exponential. Inserting 4 meson operators then restricts the Wilson loops to pass through those 4 points.

$$\left\langle \prod_{i=1}^{4} q\bar{q}(x_i) \right\rangle = \int DA\, \exp(-S_{\mathrm{YM}}) \left(-\frac{N_f}{2}\mathrm{Tr}\int_0^\infty \frac{dT}{T}\mathcal{W}_T[A]\Big|_{x_1,x_2,x_3,x_4}\right) \tag{3}$$

At this stage, *assuming* the Wilson loops are all area-behaved yields a Veneziano amplitude, can prove without string theory, but strings give a framework for improvement.

This quantity we map to a computation in String Theory via the gauge-gravity duality. For a suitable dual to Yang-Mills, of target space metric G_{MN}:

$$\mathcal{A}\left(k_{1\ldots 4}\right) = \oint \prod_{i=1}^{4} d\sigma_i \int [DX] W \exp\left(ik_i^{\mu} X_{\mu}(\sigma_i)\right) \exp\left(-\int d^2\sigma \, G_{MN} \partial_{\alpha} X^M \partial^{\alpha} X^N\right) \tag{4}$$

Where $W(\sigma_i) \exp\left(ik_i^{\mu} X_{\mu}(\sigma_i)\right)$ is a generic ansatz for the meson operator in that space, where W is usually unknown.
The flat open string 4-point amplitude reproduces the Veneziano so set the space up such that the strings are mostly flat. This relies on several assumptions:

- No quark masses,
- No higher genus corrections $g_{\alpha\beta} = \eta_{\alpha\beta}$,
- Dual background exhibits confinement (known conditions on G_{MN}),
- Ignore additional compact coordinates unrelated to holographic direction,
- Implement by hand the effects of W on the amplitude.

Impose that the characteristic depth of the space is infinitely small, such that all Wilson loops exhibit an area law

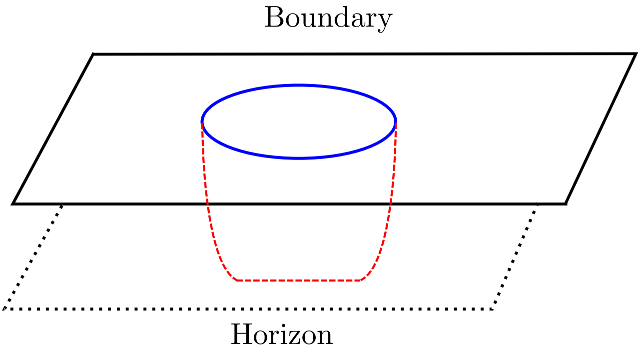


Figure: A confining string worldsheet accreting on the end of space.

Practical example: Witten's model of $D4$ branes wrapped around a circle. Where $f(U) = 1 - \frac{U_{\mathrm{KK}}^3}{U^3}$,

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(dX^2 + \mathrm{d}\tau^2 f(U)\right) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4\right) \tag{5}$$

- Taking $U_{\mathrm{KK}} \to \infty\,,\ \frac{U_{\mathrm{KK}}}{R} = \mathrm{cst.} = \lambda\,,\ dU = 0$ brings the end of space (Euclidean horizon) up towards the boundary:

  $$ds^2 = \lambda^{3/2} dX^2 + \dots \tag{6}$$

  This makes loop size far exceed depth of space, "most" loops confine. Free worldsheet action, obtain Veneziano amplitude.
- Can relax our assumptions a little. Instead of assuming depth of space infinitely small, we take it to be a small finite parameter with which we build interactions, by expanding the metric order by order, turning on interactions between X and U. We still need to neglect contributions from the edges of the sheet, very subleading in the classical action.

2. Preparing the worldsheet field theory

To prepare a workable worldsheet field theory, we need to perform the following steps.

- Creating the interaction terms would be easier if the action was not singular. Some thought is required to find the correct way to regularise it.

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}\left(dx^2 + \mathrm{d}\tau^2\left(1 - \frac{U_{\mathrm{KK}}^3}{U^3}\right)\right) + \left(\frac{R}{U}\right)^{3/2}\left(\left(\frac{1}{1-\frac{U_{\mathrm{KK}}^3}{U^3}}\right)dU^2 + U^2 d\Omega_4\right)$$

- A good change of coordinates needs to make this metric regular around the origin, but also to have a regular and non-vanishing Jacobian, because the parametrisation-invariant NLSM measure is $[DX] = \sqrt{\det(G)}DX$.
- At some point, expand metric locally around end of space, creating an interacting QFT for modes on the worldsheet.

- For regularity of the metric, change coordinates. writing $U = U_{KK}(1 + \frac{u^2}{U_{KK}^2})$ this regularises the coordinate system around the horizon. The form is standard for such Euclidean black hole-like metrics.
- For regularity of the determinant of the metric, we use a Kruskal-like procedure. Branes wrap a compact direction, resulting in a cone-like ("cigar") submanifold with vanishing subdeterminant at $u = 0$:

$$\begin{aligned} ds^2 &= \cdots a(u^2)du^2 + u^2 b(u^2)d\tau^2 + \cdots \\ &= C(Y^2 + Z^2)\left(dY^2 + dZ^2\right) + \cdots \end{aligned} \tag{7}$$

  The Kruskal procedure "unwraps" the warped cone to warped Cartesian coordinates, but crucially, requires to compute Tortoise coordinate, $\int G_{UU} dU$. Very impractical to do globally given form of integrand.
- We only need information from the metric locally around $u = 0$: in an expansion around U_{KK} assuming small fluctuations, change variables from (u, τ) to Kruskal-like coordinates (Y, Z). This naturally preserves shift symmetry in τ i.e. a $U(1)$ global symmetry.

- With a metric expanded to first order the exact change is

$$\frac{Y^2+Z^2}{2U_{KK}^2} = \frac{u^2}{2U_{KK}^2}\exp(\frac{u^2}{2U_{KK}^2}) \tag{8}$$

  Relation can be inverted and new metric expanded to first order to obtain an effective Lagrangian for the fluctuations.

- Defining $\lambda = \frac{U_{KK}}{R}$ and the doublet $\Upsilon = (Y, Z)$, the (bosonic) Lagrangian in these coordinates is then

$$L = \lambda^{3/2}\left(1 + \frac{3\Upsilon^2}{2U_{KK}^2}\right)\partial_\alpha X^\mu \partial^\alpha X_\mu + \frac{4}{3\lambda^{3/2}}\partial_\alpha \Upsilon \cdot \partial^\alpha \Upsilon + \dots \tag{9}$$

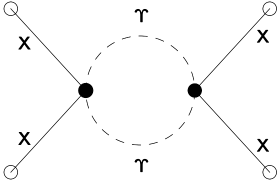
- But the integration measure also has to change, as it is proportional to det(G): by this change of coordinates

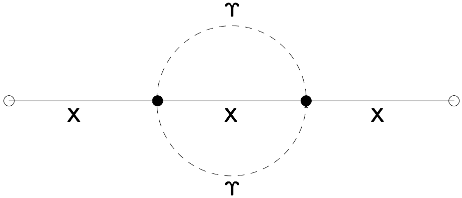
$$\det(G) = U_{KK}^8 \frac{16}{9\lambda^3}\left(1 + \frac{6\Upsilon^2}{U_{KK}^2} + \dots\right) \tag{10}$$

  This can be exponentiated to give a small mass to the new radial field.

Some comments about the path integration:

- Compute the string 4-point function $\left\langle \exp\left(\sum_{i=1}^{4} k^i \cdot X(\sigma_i)\right)\right\rangle$, by the usual trick of writing this as a current $\mathcal{J}(\sigma) = \sum_{i=1}^{4} k^i \cdot X(\sigma)\delta(\sigma - \sigma_i)$.
- Then, sufficient to compute partition function with a non-zero current. Introduces a 1-leg vertex in the Feynman rules (ending a propagator with a Fourier kernel), care taken for loop order vs. expansion order:





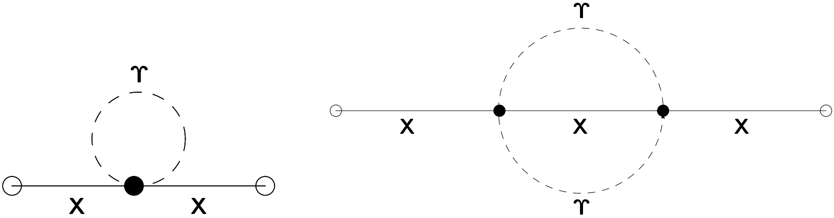
- Despite worldsheet having supersymmetry, ignore superpartners. They communicate less directly with the X fields than Υ does.

3. Computing and analysing the correction

- 2D QFTs have technical particularities, notably related to regularisation. Propagator is UV divergent in two dimensions. We use analytic regularisation: introducing an arbitrary mass scale μ,

$$\left[\frac{1}{p^2+m^2}\right]_{AR} = \lim_{x\to 0}\frac{d}{dx}\left(x\mu^x \frac{1}{(p^2+m^2)^{1+x}}\right) \tag{11}$$

- This also method also works in the massless case, dealing with the IR divergence identically.
- With correct variant of $\overline{MS}$, difficulties are dealt with automatically.



Two different situations:

- First diagram has no internal momentum transfer, integral factorises into independent terms, in our regularisation scheme the "bubble" is finite and body subsumes to a propagator.
- $\rightarrow$ Finite correction to the effective string tension, anyway set to right value *a posteriori*
- Second diagram has momentum transfer, therefore is much more interesting. However, keeping the Υ propagators massive makes it very difficult to compute even sub-integrals within the bigger computation.
- $\rightarrow$ Since mass parametrically small (same order as coupling) and massless diagram well-defined, justified to compute the latter for broadest behaviour.

- The 4-point function we compute is then summed over all possible positions of the 4 points, which subsumes to one Beta-type integral to leading order as shown previously. With the addition of the new diagram, we obtain (schematically)

$$\mathcal{A}(s,t) = \int_0^1 dz z^{s-1}(1-z)^{t-1}\left(1-\rho\left(s\log^3(z)+t\log^3(1-z)\right)\right)$$
$$= \left(1-\rho\left(s\left(\frac{\partial}{\partial s}\right)^3 + t\left(\frac{\partial}{\partial t}\right)^3\right)\right)B(s,t) \qquad (12)$$

- Now analyse the behaviour of this amplitude in order to find a corrected form of the Regge function. Several ways of obtaining the Regge trajectory out of the standard Beta function, depending on which regimes one investigates.

- By taking the $s \gg t$ limit of our result we get an approximate behaviour for the Regge trajectory to be (in normalised units)

$$\alpha(s) = s^{1-\rho \log^2(s)} \tag{13}$$

- Plotting this behaviour we find a surprisingly natural behaviour:

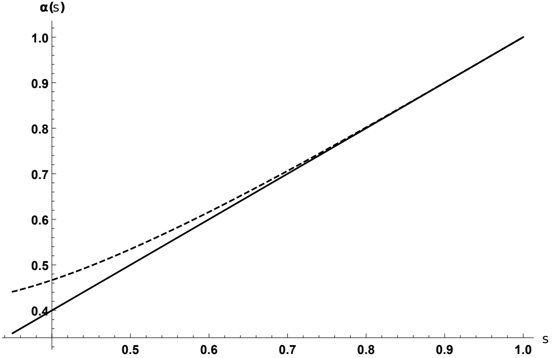


Figure: A plot of the Regge function for $\rho = 0.2$

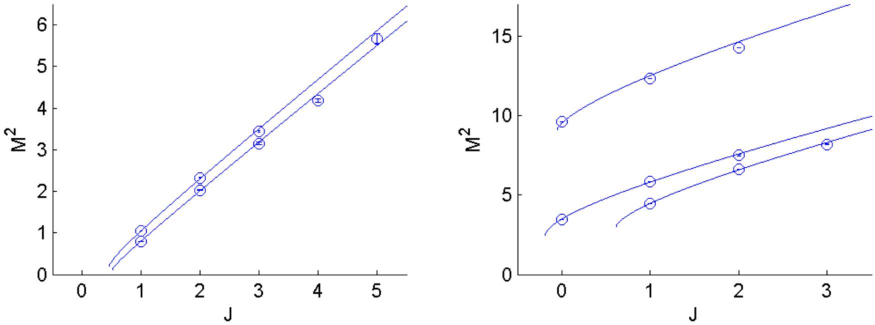


Figure: Regge fits for mesons modelled by spinning strings (1602.00704, J. Sonnenschein)

Note that $J = \alpha(M^2)$, plots flipped from previous layout.

4. Extending the computation to generic cases

The procedure we describe is generically applicable to a large class of dual string theories. Can check for other individual cases e.g. Klebanov-Strassler, re-obtain a term $\Upsilon^2 \partial X \partial X$. Indeed recall theorem classifying spaces with confinement (Kinar, Schreiber, Sonnenschein):

Theorem

Let $f = G_{00} G_{XX}$, $g = G_{00} G_{UU}$ *smooth positive over* $0 < U < \infty$, *and at* 0

$$f(U) = f(0) + U^k a_k + O(U^{k+1}) \;,\; g(U) = U^j b_j + O(s^{j+1}) \tag{14}$$

with $f(0) \geq 0$, $k > 0$, $a_k > 0$, $b_j > 0$ *and* $j \geq -1$, $\int^{\infty} \frac{g}{f^2} < \infty$
Then, $k \geq 2(j+1)$ *and an even worldsheet solution exists, furthermore confinement exists iff.* $f(0) > 0$.

Barring pathological cases with non-integer powers this subsumes to two cases:

- g has a pole and by the above only a simple pole works.
- g has no pole and by the above f has a minimum at 0.

- Witten's model is case 1, but we have seen that regularising the pole brings it to a form satisfying case 2 with $k = 2, j = 0$. These numbers could be higher in accidental cases with specially constructed functions but generically f, g have full set of non-zero coefficients down to minimal degree, i.e. $k = 2, j = 0$ should be the most common case.
- Klebanov-Strassler is case 2 but with a cone over an S_2. This requires one extra Kruskal coordinate for the transformation but proceeds formally the same up to numerical constants, which anyway should be set *post-facto* to match data.

5. The n-point amplitude

- The Regge function can be extracted from higher n-point amplitude in the *multi-Regge* limit. We should check that the computed effect is identical in those cases. Writing $s_{ij} = k_i.k_j$, we take

$$s_{1,i} = \text{cst.}\ ,\ \{s_{i,i+1}\} \gg \{s_{1,i}\}\ ,\ \frac{s_{i,i+1}s_{i+1,i+2}}{s_{i,i+2}} = \lambda_{i,i+2} = \text{cst.} \tag{15}$$

- The leading behaviour of open string scattering in this limit can be then proven to be

$$\mathcal{A}(p_i) = \left(\prod_i s_{i,i+1}^{-s_{1,i}}\right) \times G(s_{1,i}, \lambda_{i,j}) \tag{16}$$

  By crossing symmetry, the correction acts as $\mathcal{A} \to \left(1 - \rho \sum_{i,j} \frac{\partial^3}{\partial s_{i,j}^3}\right) \mathcal{A}$, the lead contribution will as before come from $\frac{\partial^3}{\partial s_{1,i}^3}$.

- This verifies that in every channel of the multi-Regge limit of higher point amplitudes the correction applies itself in much the same way as in the 4-point case, which is encouraging.

Summary

- Can map the 4-point meson amplitude problem into a Wilson loop computation and thus into String Theory,
- Certain limits of confining string duals flatten out the Wilson loops, recovering the Veneziano amplitude,
- These limits can be softened, creating a worldsheet perturbative QFT, whose form is generically predictable,
- The first few loop corrections seem to correct the amplitude in physically relevant ways.

Future work

- Adding quark masses to the computation (simple cases to start with)
- Refining the computation: many approximations in the derivation, e.g. what of edge effects?
- (Eventually) compare with spectrum data once a better handle on the process is achieved.
- Lattice verification: strong coupling expansion + hopping expansion would subsume to a very similar computation, would allow for a numerical construction of the Regge function.