Perturbative running of the twisted Yang-Mills coupling in the gradient flow scheme

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EXCELENCIA SEVERO OCHOA

Outline

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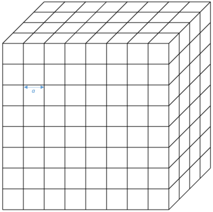
Introduction

- Lattice gauge theory was introduced by Kenneth Wilson in 1974. While it was interesting and useful from the beginning, it was with the increase in computing power that it matured into an extremely useful tool, as well as a crucial theoretical framework with a deep connection to statistical mechanics.
- Working on the lattice provides a framework allowing for both perturbative and non-perturbative approaches, which is extremely important to understand phenomena such as confinement and asymptotic freedom in non-abelian gauge theories. Moreover, it is the only way to properly define a quantum field theory in a non-perturbative manner.
- Although we have not used lattice simulations in our work yet, an introduction to it is necessary to properly explain our model, motivation and and goals.

Quantum field theory on the lattice

- Lattice gauge theory works by discretising space-time: one defines a d-dimensional hypercubic lattice of spacing a, onto which the quantum fields are defined.



- Quantisation comes through the use of the euclidean path integral formalism:

$$Z = \int [D\phi]\, e^{-S_E}$$

  Where we used $t \to -it$ to go to euclidean time, and $[D\phi]$ denotes the integral over all field configurations. The continuum action must be recovered for $a \to 0$.

- Several equivalent actions can be chosen as long as they have the correct continuum limit. This choice is not trivial: some actions behave much better than others, and some have critical issues (e.g. doublers for naive lattice fermions).

Gauge fields on the lattice I

- We will only focus in pure gauge Yang-Mills theory:

$$S_{YM} = \frac{1}{2g_0^2} \int d^4x \ \mathrm{Tr} \ F_{\mu\nu} F^{\mu\nu}$$

- While one could try to directly do the naive, trivial discretisation that is often used for fermions and scalars:

$$x_\mu = a n_\mu, \quad n_\mu \in \mathbb{Z}; \qquad \int d^4x \to a^4 \sum_{x_\mu}$$

$$\partial_\mu O(x) = \tfrac{1}{a}\left(O(x + a\hat{\mu}) - O(x)\right)$$

  This is very problematic for gauge fields, as it breaks gauge invariance.

- Instead, one works with the parallel transporters between two neighbouring points, given by $N \times N$ unitary matrices:

$$U_\mu(x) = T \exp\left(-iaA_\mu(x)\right)$$

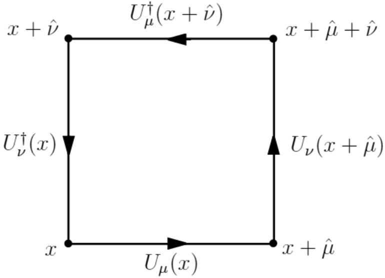
Gauge fields on the lattice II

- These matrices live on the links of the lattice, and transform under a gauge transformation $\Omega(x)$ as:

$$U'_\mu(x) = \Omega(x) U_\mu(x) \Omega^\dagger(x+\mu)$$

- The only possible invariants built from U_μ matrices are traces over closed paths. The simplest ones among them are the plaquettes:

$$P_{\mu\nu}(x) = \mathrm{Tr}\left(U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x) \right)$$



- Using these plaquettes, one can define the gauge-invariant Wilson action:

$$S_W = \frac{1}{g_0^2}\left(N - \text{Tr}\sum_{x,\mu,\nu} U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x) + \text{c.c.}\right)$$

- One can expand the U matrices in powers of a to determine the leading order terms. This yields:

$$S_W = S_{YM} + \mathcal{O}(a^2)$$

Meaning that the Wilson action is a lattice implementation of Yang-Mills theory, which is recovered in the continuum limit $a \to 0$.

The lattice as a regulator

- Discretising spacetime can be seen as a way of regularising field theories. The cutoff appears when looking at the Fourier transform of the field: $A_\mu(p) = a^4 \sum_{x=an} e^{iapn} A_\mu(x)$.
- The periodicity of the function allows us to identify the momenta $p_\mu \sim p_\mu + \frac{2\pi k}{a}$, setting up a cutoff $|p_\mu| \leq \frac{\pi}{a} \equiv \Lambda$.
- For numerical simulations, finite lattices are required. To avoid breaking translation invariance, the usual approach implies using periodic boundary conditions of period l:

$$U_\mu\left(x + l\hat{\nu}\right) = U_\mu\left(x\right).$$

- These boundary conditions imply a quantisation of momenta:

$$p_\mu = \frac{2\pi}{a}\frac{m_\mu}{l} \qquad m_\mu \in \mathbb{Z}$$

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Twisted boundary conditions

- The boundary conditions from the previous slide, while quite common in literature, are not the most general choice. As physical observables are gauge-independent, generality implies setting up periodic boundary conditions *up to arbitrary gauge transformations*.
- This idea, known as twisted boundary conditions, was introduced by 't Hooft in the seventies for the continuum. In our case the following set of twisted boundary conditions was considered:

$$U_\mu \left(x + l\hat{\nu}\right) = \Omega_\nu \left(x\right) U_\mu \left(x\right) \Omega_\nu^\dagger \left(x + \hat{\mu}\right)$$

- A consistency condition is then required for the corner plaquettes:

$$\Omega_\mu \left(x + l\hat{\nu}\right) \Omega_\nu \left(x\right) = z_{\mu\nu} \Omega_\nu \left(x + l\hat{\mu}\right) \Omega_\mu \left(x\right)$$

- The factor $z_{\mu\nu}$ is known as the twist of the theory:

$$z_{\mu\nu} = \exp\left(2\pi i \frac{n_{\mu\nu}}{N}\right), \qquad n_{\mu\nu} \in \mathbb{Z}$$

The twisted Eguchi-Kawai model

- Implementing twisted boundary conditions to the Wilson action results in added twist factors in the corners:

$$S = \frac{1}{g_0^2}\left(N - \mathrm{Tr}\sum_{x\mu\nu} z_{\mu\nu} U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x) + \text{c.c.}\right)$$

- Under certain conditions, in the $N \to \infty$ limit the lattice version of the Schwinger-Dyson equations does not depend on the size of the lattice. The theory can be reduced to a single-point lattice:

$$S_{\mathrm{TEK}} = \frac{N}{\lambda_0}\left(N - \mathrm{Tr}\sum_{\mu\nu} z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \text{c.c.}\right); \quad \lambda_0 = g_0^2 N$$

- This is an example of reduction: somehow, gauge and spacetime DOFs are redundant in the large N limit.

Volume independence

- To prevent reduction from breaking down, the centre $\mathbb{Z}^d(N)$ symmetry of the action must be preserved, which requires the use of the so-called symmetric twist:

$$n_{\mu\nu} = \epsilon_{\mu\nu} k l_g; \qquad \begin{matrix} \epsilon_{\mu\nu} = \theta(\nu - \mu) - \theta(\mu - \nu) \\ l_g = N^{\frac{2}{d_t}}, \qquad k \in \mathbb{Z} \end{matrix}$$

- This simple model can be generalised to finite N, leading to the hypothesis of volume independence in $SU(N)$ twisted gauge theories:

  In finite volume twisted Yang-Mills theory, volume and colour effects are intertwined, with the torus length and number of colours appearing combined into an effective length $\tilde{l} = l_g l$.

- One of our main goals is to check the validity of this hypothesis, both in perturbation theory (in the continuum) and nonperturbatively (on the lattice).

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Twisted gradient flow

- We adapted A. Ramos' twisted gradient flow scheme to obtain the running of the renormalised 't Hooft coupling $\lambda = g^2 N$
- This scheme works by introducing an additional time dimension t and a flow field $B_\mu(x,t)$ following the flow equations:

$$\partial_t B_\nu(x,t) = D_\mu G_{\mu\nu}(x,t) \qquad B_\mu(x,0) = A_\mu(x)$$

- The flow equations drive the fields towards the Yang-Mills stationary points, averaging them over a spherical volume of mean-square radius $\sqrt{8t}$ in four dimensions.
- The action density in this scheme is a renormalised quantity allowing to define a renormalised coupling $\lambda_{\rm TGF}$ using $\tilde{l}$ as a scale:

$$\lambda_{\rm TGF}(\tilde{l}) = \mathcal{N}^{-1}(c)\, \frac{t^2 \langle E(t)\rangle}{N}\bigg|_{t=\frac{c^2\tilde{l}^2}{8}} \qquad E(t) = \frac{1}{2}\mathrm{Tr}\; G_{\mu\nu}(x,t) G^{\mu\nu}(x,t)$$

Perturbation theory in the twisted box I

- As a first step towards checking the validity of the volume independence hypothesis, we are in the process of computing the running of the twisted gradient flow coupling in perturbation theory.
- We expanded the fields in powers of the coupling:

$$B_\mu(x,t) = \sum_k g_0^k(t) B_\mu^{(k)}(x,t)$$

- And then went to momentum space in a traceless, momentum-dependent basis:

$$B_\mu^{(k)}(x,t) = \tilde{l}^{-\frac{d}{2}} \sum_q{}' e^{iqx} B_\mu^{(k)}(q,t)\hat{\Gamma}(q)$$

Perturbation theory in the twisted box II

- The momenta are now quantised in terms of the effective length $\tilde{l}$:

$$p_\mu = \frac{2\pi m_\mu}{\tilde{l}} \qquad m_\mu \in \mathbb{Z}$$

- And the basis $\hat{\Gamma}(q)$ (whose explicit expression will be omitted) satisfies by construction:

$$\left[\hat{\Gamma}(p), \hat{\Gamma}(q)\right] = iF(p, q, -p-q)\hat{\Gamma}(p+q)$$

- With the structure constants:

$$F(p, q, -p-q) = -\sqrt{\frac{2}{N}} \sin\left(\frac{1}{2}\theta_{\mu\nu} p_\mu q_\nu\right); \quad 2\pi\theta_{\mu\nu} = \tilde{\theta}\tilde{l}^2 \tilde{\epsilon}_{\mu\nu}$$

- We defined a few auxiliary variables given by $\tilde{\epsilon}_{\mu\nu}\epsilon_{\nu\lambda} = \delta_{\mu\lambda}$ and $\tilde{\theta} = \bar{k}/l_g$, with $\bar{k}$ given by the twist: $k\bar{k} = 1 \,(\mathrm{mod}\ l_g)$.

Solving the gradient flow equations

- The flow equation can be solved in momentum space, order by order in perturbation theory:

$$\partial_t B_\nu(x,t) = D_\mu G_{\mu\nu}(x,t)\,; \qquad B_\mu(x,0) = A_\mu(x)$$

- This was done up to order g_0^4:

$$B_\mu^{(1)}(q,t) = e^{-q^2 t} A_\mu(q)$$
$$B_\mu^{(2)}(q,t) = i\tilde{l}^{-\frac{d}{2}} e^{-q^2 t} \sum_p{}' F(p,q,p-q)$$
$$\times \int_0^t ds e^{2(p\cdot q - p^2)s} A_\nu(q-p)(2p_\nu A_\mu(p) - p_\mu A_\nu(p))$$
$$B_\mu^{(3)}(q,t) = \ldots$$

Integral form of the coupling constant

- After some of algebra, we are left with thirteen different contributions to the running coupling, which combine multiple integrals over flow time and sums over momenta:

$$\lambda_{\rm TGF}(\tilde{l}, c) = \lambda_0 \mathcal{N}^{-1}(c)(1 + \lambda_0 \sum_{i=1}^{13} c_i I_i)$$

- For example, one of such integrals is, setting $q = p + r$:

$$I(t) = \tilde{l}^{-2d} \int_0^t dx \int_0^\infty dz \sum_{r,p} NF^2(r, p, -q) e^{-(2t-x)q^2 - x(r^2+p^2) - zr^2}$$

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Siegel theta form of the argument

- The integrals can be rewritten in terms of Siegel theta functions:

$$\Theta\left(0|iA(s,u,v,\tilde{\theta})\right) = \sum_{M\in\mathbb{Z}^{2d}} \exp\left\{-\pi M^t A\left(s,u,v,\tilde{\theta}\right) M\right\}$$

- Defining $M^t = (m,n)$ and:

$$A\left(s,u,v,\tilde{\theta}\right) = \begin{pmatrix} s\hat{c}\,\mathbb{I}_d & v\hat{c}\,\mathbb{I}_d + i\tilde{\theta}\tilde{\epsilon} \\ v\hat{c}\,\mathbb{I}_d - i\tilde{\theta}\tilde{\epsilon} & u\hat{c}\,\mathbb{I}_d \end{pmatrix}$$

- Then, defining:

$$F_c\left(s,u,v,\tilde{\theta}\right) = \frac{\hat{c}^2}{16\pi^2\tilde{l}^{2d-4}}\mathrm{Re}\,\left(\Theta\left(s,u,v,0\right) - \Theta\left(s,u,v,\tilde{\theta}\right)\right)$$

- All integrals can be rewritten in this manner. For instance, the previous example can be rewritten as:

$$I(t) = \int_0^1 dx \int_0^\infty dz\, F_c\left(2, z+2x, x, \tilde{\theta}\right)$$

Regularisation

- Several terms in the integrals are divergent, and require regularisation. We chose to use dimensional regularisation, by setting $d = 4 - \epsilon$
- We then identified the divergent terms by inspection, and substracted them manually from our integrals:

$$H^{\text{div}}(s,u,v) = \frac{\hat{c}^2}{16\pi^2 \tilde{l}^{2d-4}} (\hat{c}u)^{-\frac{d}{2}} \sum_m{}' \exp\left\{-\pi\hat{c}\frac{su - v^2}{u} m^2\right\}$$

- As a consistency check, we reproduced the universal coefficient of the $\frac{1}{\epsilon}$ term, by noticing that our integrals can be related to the infinite volume ones:

$$\left\langle \frac{E(t_0)}{N} \right\rangle = \frac{\lambda_0(d-1)}{2\tilde{l}^d} \left(\sum_m{}' e^{-2\pi m^2 \hat{c}}\right) \left\{1 + \left(\frac{11}{48\pi^2\epsilon} + \alpha\right)\lambda_0 + \ldots\right\}$$

Numerical computations

- We are currently computing the finite terms numerically in order to obtain the Λ coefficients between this scheme and the $\overline{\mathrm{MS}}$ one.
- The computations are being done through a combination of Mathematica programs for the simpler terms, and C++ programs which use trapezoid integration to obtain more complicated terms which Mathematica cannot compute. These computations are still ongoing.
- Once these are done, we will study whether the hypothesis of volume independence holds, or if finite N corrections appear, and, if they do, their relative magnitude.
- If the hypothesis holds in perturbation theory, we will test if it does so as well in nonperturbative computations, through the use of lattice simulations.

Thank you.