Poincaré Symmetry shapes the Massive 3-point Amplitude

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Motivations

- On-shell recursion relations for scattering amplitudes *e.g.:* Parke-Taylor formula for MHV gluon tree-level amplitudes

$$M_n(\ldots, i^-, \ldots, j^-, \ldots) = \frac{\langle i, j\rangle}{\langle 1,2\rangle\langle 2,3\rangle\cdots\langle n-1,n\rangle\langle n,1\rangle}$$

proved by induction with BCFW recursion relations for any n, while increasingly painful for increasing n with Feynman graphs...

n	4	5	6	$\cdots$	8
#	4	25	220	$\cdots$	10525900

Motivations

- On-shell recursion relations for scattering amplitudes
- Non perturbative results for scattering amplitudes

Main result

For *massless* complex external momenta, the Poincaré invariant 3-point amplitude is fixed up to a constant (coupling).

[Benincasa-Cachazo '07]

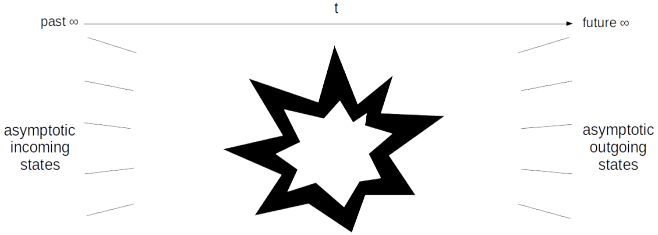
Poincaré invariance determines the 3-point amplitude also in the case where external states can be massive, up to *some* constants.

[E. Conde, AM arxiv/1601.08113]

Philosophy

Reconstructing the amplitude with the minimal amount of information.

Amplitude as an asymptotic object



The states of the Hilbert space (particles) are identified by the symmetry of space-time (Wigner classification)

Outline

Poincaré in 4 dim

Casimir operators

P^2 square of translation generator $\longrightarrow$ *mass*

W^2 square of Pauli-Lubanski operator $\longrightarrow$ *spin*

$$W_\lambda = \epsilon_{\lambda\mu\nu\rho} M^{\mu\nu} P^\rho$$ generator of the Little Group

$$\mathrm{LG}_p = \left\{ \Lambda_p \in L_+^\uparrow \Big/ \Lambda_p p = p \right\}$$

LG for massless representations

$$p \xrightarrow{L_p} k = (E, 0, 0, E)$$

$\Longrightarrow \; \mathrm{LG}_k \equiv \mathrm{ISO}_2$: Isometries in 2 dim. euclidean space

$$\Lambda_k |k; a\rangle = e^{i\alpha A} e^{i\beta B} e^{i\theta J_3} |k; a\rangle$$

If $\alpha, \beta \neq 0 \Rightarrow$ continuous spin

J_3 admits for discrete eigenvalues: $\pm h \;\longrightarrow\;$ *helicity*

$$J_3 \, |p; h\rangle = h \, |p; h\rangle$$

LG for massive representations

$$P \xrightarrow{L_P} K = (m, 0, 0, 0)$$

$\Longrightarrow \ \mathrm{LG}_K \equiv \mathrm{SO}(3)$: 3-dim. spatial rotations

$$J_0 \left|P; s, \sigma\right\rangle = \sigma \left|P; s, \sigma\right\rangle$$

$$J_{\pm}|P; s, \sigma\rangle = \sigma_{\pm}|P; s, \sigma \pm 1\rangle$$

$$\sigma \in \{-s, \ldots, +s\}$$

$$\sigma_{\pm} = \sqrt{(s \mp \sigma)(s \pm \sigma + 1)}$$

How is this story helpful to constrain the amplitude?

$$|p; a\rangle \quad \longrightarrow \quad M_n \sim \bigotimes_{i=1}^{n} |p_i; a_i\rangle$$

P	$\longrightarrow$	momentum conservation	$\longrightarrow$	$M_n \propto \delta\big(\sum_i p_i\big)$
W	$\longrightarrow$	little group scaling "spin conservation"	$\longrightarrow$	LG equations

LG equations in the massless case

From the LG action on the states descends the LG action on the amplitude

$$e^{i\theta J_3}|p;h\rangle = e^{i\theta h}|p;h\rangle$$
$$\Downarrow$$
$$e^{i\theta J_3^j}M_n(\{p_i,h_i\}) = e^{i\theta h_j}M_n(\{p_i,h_i\})$$

The infinitesimal version of this equation,

$$J_3^j M_n(\{p_i,h_i\}) = h_j M_n(\{p_i,h_i\})$$

yields strong constraints on the amplitude, and it is actually enough to fully fix the 3-point one.

Spinor-Helicity formalism...

$$L_+^{\uparrow}(\mathbb{R}) \xrightarrow[\text{1 to 2}]{\text{homomorph.}} SL(2,\mathbb{C})$$

$$L_+(\mathbb{C}) \xrightarrow[\text{1 to 2}]{\text{homomorph.}} SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$$

$$p_\mu \longrightarrow p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu$$

$$\Lambda_\mu{}^\nu p_\nu \longrightarrow \zeta_a{}^b \, p_{b\dot{b}} \, \eta^{\dot{b}}{}_{\dot{a}}$$

$$\sigma^\mu = (\mathbb{I}, \vec{\sigma})$$

$$\boxed{p_\mu p^\mu = \det |p_{a\dot{a}}|}$$

... FOR MASSLESS PARTICLES

$$\det |p_{a\dot{a}}| = p_\mu p^\mu = 0$$

$$\Downarrow$$

$$\boxed{p_{a\dot{a}} = \lambda_a \otimes \tilde{\lambda}_{\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}}$$

reality condition: $\tilde{\lambda}_{\dot{a}} \equiv (\lambda_a)^*$, $\eta \equiv \zeta^\dagger$.

We define:
$$\langle \lambda, \mu \rangle = \epsilon^{ab} \lambda_b \mu_a$$
$$[\tilde{\lambda}, \tilde{\mu}] = \tilde{\lambda}_{\dot{a}} \epsilon^{\dot{a}\dot{b}} \tilde{\mu}_{\dot{b}}$$

$$\langle i, j \rangle [i, j] \equiv \langle \lambda_i, \lambda_j \rangle = 2\, p_i \cdot p_j$$

LG scaling

$$e^{i\theta h_i} M_n(\lambda_j \tilde{\lambda}_j; h_j) \quad \longrightarrow \quad t^{-2h_i} M_n(\lambda_j \tilde{\lambda}_j; h_j) \qquad t \in \mathbb{C}$$

$$\left. \begin{array}{l} \lambda \longrightarrow t\,\lambda \\ \tilde{\lambda} \longrightarrow t^{-1} \tilde{\lambda} \end{array} \right\} \Rightarrow \lambda\tilde{\lambda} \longrightarrow \lambda\tilde{\lambda}$$

LG scaling

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LG differential equation:

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}\right) M_n(\lambda_j \tilde{\lambda}_j; h_j) = -2h_i\, M_n(\lambda_j \tilde{\lambda}_j; h_j)$$

3-point massless amplitude

[Benincasa-Cachazo '07]

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}\right) M_3\left(\lambda_j \tilde{\lambda}_j; h_j\right) = -2h_i \, M_3\left(\lambda_j \tilde{\lambda}_j; h_j\right)$$

3 equations for 6 variables

$$\begin{aligned} x_1 = \langle 2, 3\rangle \;, \; x_2 = \langle 3, 1\rangle \;, \; x_3 = \langle 1, 2\rangle \\ y_1 = [2, 3] \;, \; y_2 = [3, 1] \;, \; y_3 = [1, 2] \end{aligned}$$

$$\begin{aligned} M_3^{h_1, h_2, h_3} &= x_1^{h_1 - h_2 - h_3} x_2^{h_2 - h_3 - h_1} x_3^{h_3 - h_1 - h_2} f\left(x_1 y_1, x_2 y_2, x_3 y_3\right) \\ &= y_1^{h_2 + h_3 - h_1} y_2^{h_3 + h_1 - h_2} y_3^{h_1 + h_2 - h_3} \tilde{f}\left(x_1 y_1, x_2 y_2, x_3 y_3\right) \end{aligned}$$

... but then we have to impose momentum conservation:

$$0 = p_1^2 = (-p_2 - p_3)^2 = 2p_2 \cdot p_3 = \langle 2, 3 \rangle [2, 3] \ \Rightarrow x_1 = 0 \,, \text{ or } \ y_1 = 0$$

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Let's say $x_1 \equiv \langle 2,3\rangle = 0 \;\Rightarrow\; \lambda_2 \propto \lambda_3$. But in a 2-dim. vector space three vectors cannot be linearly independent, so

$$\lambda_1 = \alpha\lambda_2 + \beta\lambda_3 \;\Rightarrow\; \lambda_1 \propto \lambda_2 \propto \lambda_3 \;\Rightarrow\; x_i = 0 \;\forall i$$

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Or all x_i are zero, or all y_i are zero.

For real kinematics ($x_i = 0 = y_i$) a 3p amplitude for massless particles is zero, so the complex 3p amplitude had better go to zero in this limit, rather than exploding. This selects

$$M^{\{h_j\}} = g_{\rm H}\, x_1^{h_1-h_2-h_3} x_2^{h_2-h_3-h_1} x_3^{h_3-h_1-h_2} \quad \text{for} \quad h_1+h_2+h_3 < 0$$

$$M^{\{h_j\}} = g_{\rm A}\, y_1^{h_2+h_3-h_1} y_2^{h_3+h_1-h_2} y_3^{h_1+h_2-h_3} \quad \text{for} \quad h_1+h_2+h_3 > 0$$

For $h_1+h_2+h_3 = 0$ the answer is left undetermined (there are claims that such interactions cannot exist).

Let's now extend the same successful strategy to massive particles.

To-Do list

- LG scaling for massive particles

To-Do list

- LG scaling for massive particles
- Spinor-Helicity formalism for massive momenta

Spinor formalism for massive momenta

A time-like momentum can be always decomposed into two light-like ones

$$P = \lambda\tilde{\lambda} + \mu\tilde{\mu}$$

$$\text{with} \quad P^2 = -m^2 = \langle\lambda,\mu\rangle[\tilde{\lambda},\tilde{\mu}]$$

Crucial disadvantages with respect to the massless case:

- The on-shell condition was built-in in the spinor formalism for massless particles, here we have to impose it

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- The decomposition is not unique, so we are introducing some non-physical redundancy

But we can still keep the advantage of having LG differential equations in a simple and effective form!

LG equations for massive particles

$$J_0^I\, M_n(\lambda_j\tilde{\lambda}_j; a_k) = \sigma_I\, M_n(\lambda_j\tilde{\lambda}_j; a_k)$$ equivalent of helicity eq.

$$J_\pm^I\, M_n(\lambda_j\tilde{\lambda}_j; ..., \sigma_I, ...) = \sigma_I^\pm\, M_n(\lambda_j\tilde{\lambda}_j; ..., \sigma_I \pm 1, ...)$$

$j = 1, \ldots, n + \#$ of massive particles

The latter equations relate different amplitudes! What we wish is to have a system with a maximal number of equations acting on a unique function...

Equations for the "lowest-spin" amplitude

Solution: let's take $\sigma_I = -s_I$ for every massive particle. (helicities of possible massless legs are still free to vary)
Then

$$J_-^I\, M_n = 0$$

$$J_0^I\, M_n = -s_I\, M_n$$

$$(J_+^I)^{2s_I+1} M_n = 0$$

The third equation is not as simple as the others, let's keep it for the end. So

2 eq.s for every massive leg + 1 eq. for every massless leg

Massive LG equations in spinor formalism

If we take the transformation

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \to U \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \qquad \begin{pmatrix} \tilde{\lambda} & \tilde{\mu} \end{pmatrix} \to \begin{pmatrix} \tilde{\lambda} & \tilde{\mu} \end{pmatrix} U^{\dagger} \qquad U \in \mathrm{U}(2)$$

under which the massive momentum is invariant (LG), then

$$J_{+} = -\mu \frac{\partial}{\partial \lambda} + \tilde{\lambda} \frac{\partial}{\partial \tilde{\mu}}$$
$$J_{0} = -\frac{1}{2} \left(\lambda \frac{\partial}{\partial \lambda} - \tilde{\lambda} \frac{\partial}{\partial \tilde{\lambda}} - \mu \frac{\partial}{\partial \mu} + \tilde{\mu} \frac{\partial}{\partial \tilde{\mu}} \right)$$
$$J_{-} = -\lambda \frac{\partial}{\partial \mu} + \tilde{\mu} \frac{\partial}{\partial \tilde{\lambda}}$$

Massive LG equations in spinor formalism

and so

$$\left(\lambda_I \frac{\partial}{\partial \lambda_I} - \tilde{\lambda}_I \frac{\partial}{\partial \tilde{\lambda}_I} - \mu_I \frac{\partial}{\partial \mu_I} + \tilde{\mu}_I \frac{\partial}{\partial \tilde{\mu}_I}\right) M_n = 2 s_I \, M_n$$

$$\left(\lambda_I \frac{\partial}{\partial \mu_I} - \tilde{\mu}_I \frac{\partial}{\partial \tilde{\lambda}_I}\right) M_n = 0$$

1-massive 2-massless 3-point amplitude

$$p_1 = \lambda_1 \tilde{\lambda}_1 \quad p_2 = \lambda_2 \tilde{\lambda}_2 \quad P_3 = \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4$$

$$\langle 3, 4 \rangle [3, 4] = -{m_3}^2$$

$$1 + 1 + 2 \; = \; 4 \text{ eq.s}$$

$$\frac{4 \cdot 3}{2} \text{ angle prod.s} + \frac{4 \cdot 3}{2} \text{ square prod.s} = 12 \text{ spinor prod.s}$$

Kinematic Constraints

$$\sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i = 0 \qquad (n = 3 + \# \text{ of mass. particles})$$

Schouten identity (linear dependency in 2 dim. vector sp.):

$$\langle j, k \rangle \lambda_i + \langle k, i \rangle \lambda_j + \langle i, j \rangle \lambda_k = 0$$

Choose λ_1 and λ_2, and express all the spinor products in term of

$$\langle 1, 2 \rangle \,,\; \langle 1, i \rangle \,,\; \langle 2, i \rangle \,, \qquad \text{with } i = 3, \ldots, n$$

and then we use momentum conservation for the tilded spinors

$$\tilde{\lambda}_1 = -\sum_{i=3}^{n} \frac{\langle i, 2 \rangle}{\langle 1, 2 \rangle} \tilde{\lambda}_i \qquad \tilde{\lambda}_2 = -\sum_{i=3}^{n} \frac{\langle 1, i \rangle}{\langle 1, 2 \rangle} \tilde{\lambda}_i$$

Kinematic Constraints

If $n > 5$ there is still room for using Schouten on tilded variables as well.

So eventually the total number of independent variables is

$$\begin{cases} 2n-3+\frac{1}{2}(n-2)(n-3) = \frac{1}{2}n\,(n-1) & \text{if } n \leq 5 \\ 2n-3+\qquad 2n-7 \qquad = 2\,(2n-5) & \text{if } n > 5 \end{cases}$$

1-massive 2-massless 3-point amplitude

$$p_1 = \lambda_1 \tilde{\lambda}_1 \quad p_2 = \lambda_2 \tilde{\lambda}_2 \quad P_3 = \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4$$

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momentum conservation: $12 \longrightarrow 6$

mass on-shell condition: $6 \longrightarrow 5$

1-massive 2-massless 3-point amplitude

$$M^{h_1,h_2,-s_3} = \langle 1,2\rangle^{-s_3-h_1-h_2}\langle 2,3\rangle^{h_1-h_2+s_3}\langle 3,1\rangle^{h_2-h_1+s_3}\ f_1\big(\langle 3,4\rangle\big)$$

all angle products!

1-massive 2-massless 3-point amplitude

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$\langle 3,4\rangle$ on the real-momenta limit is basically the mass, so f_1, by matching the right dimensions, can be reduced to dimensionless constant

$$f_1(\langle 3,4\rangle) = g\, m_3^{1+h_1+h_2-s_3-[g]}\,\tilde{f}_1\left(\frac{\langle 3,4\rangle}{m_3}\right)$$

1-massive 2-massless 3-point amplitude

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1-massive 2-massless 3-point amplitude fixed up to 1 constant

1-massive 2-massless 3-point amplitude

Why $f_1(\langle 3,4\rangle)$ should be a constant?

Redundancy in our description of time-like momentum:

$$\lambda_3\tilde{\lambda}_3 + \lambda_4\tilde{\lambda}_4$$

After we have fixed a frame by a LG transformation, we have still the freedom to rotate $\lambda_4\tilde{\lambda}_4$ independently of $\lambda_3\tilde{\lambda}_3$

$$\lambda_4 \to t\,\lambda_4 \qquad \tilde{\lambda}_4 \to t^{-1}\tilde{\lambda}_4$$

Such transformation is not physical, so the amplitude must be invariant under it $\quad \Rightarrow \; f^3$ constant

1-massive 2-massless 3-point amplitude

$$M^{h_1, h_2, -s_3} = \langle 1,2 \rangle^{-s_3-h_1-h_2} \langle 2,3 \rangle^{h_1-h_2+s_3} \langle 3,1 \rangle^{h_2-h_1+s_3} f_1$$

This amplitude is physically allowed for real momenta!

$\Rightarrow$ full non-perturbative result

1-massive 2-massless 3-point amplitude

$$M^{h_1,h_2,-s_3} = \langle 1,2\rangle^{-s_3-h_1-h_2}\,\langle 2,3\rangle^{h_1-h_2+s_3}\,\langle 3,1\rangle^{h_2-h_1+s_3}\,f_1$$

This amplitude is physically allowed for real momenta!
$\Rightarrow$ full non-perturbative result

If we apply the third LG equation

$$(J_+^3)^{2s_3+1} M^{h_1,h_2,-s_3} = 0$$

we get the following condition on the allowed helicities

$$|h_1 - h_2| \leq s_3$$

1-massive 2-massless 3-point amplitude

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"conservation of the spin"!

2-massive 1-massless 3-point amplitude

$$P_1 = \lambda_1\tilde{\lambda}_1 + \lambda_5\tilde{\lambda}_5 \quad P_2 = \lambda_2\tilde{\lambda}_2 + \lambda_4\tilde{\lambda}_4 \quad p_3 = \lambda_3\tilde{\lambda}_3$$

$$\langle 1,5\rangle[1,5] = -{m_1}^2 \quad \langle 2,4\rangle[2,4] = -{m_2}^2$$

$2+2+1 \;=\; 5$ eq.s

$5\cdot 4 \;=\; 20$ spinor prod.s

momentum conservation: $20 \longrightarrow 10$

mass on-shell conditions: $10 \longrightarrow 8$

2-massive 1-massless 3-point amplitude

$$M^{-s_1,-s_2,h_3} =$$

$$\langle 1,2\rangle^{s_1+s_2+h_3}\,\langle 3,1\rangle^{s_1-s_2-h_3}\,\langle 2,3\rangle^{s_2-s_1-h_3}\,f_2\Big(\langle 1,5\rangle,\langle 2,4\rangle,\frac{[4,5]}{\langle 1,2\rangle}\Big)$$

Again from dimensional considerations

$$f_2 = g\,m_1^{1-s_1-s_2+h_3-[g]}\,\tilde{f}_2\left(\frac{\langle 1,5\rangle}{m_1},\frac{\langle 2,4\rangle}{m_2},\frac{[4,5]}{\langle 1,2\rangle}\right)$$

2-massive 1-massless 3-point amplitude

$$M^{-s_1,-s_2,h_3} =$$

$$\langle 1,2\rangle^{s_1+s_2+h_3}\,\langle 3,1\rangle^{s_1-s_2-h_3}\,\langle 2,3\rangle^{s_2-s_1-h_3}\;f_2\Big(\langle 1,5\rangle,\langle 2,4\rangle,\frac{[4,5]}{\langle 1,2\rangle}\Big)$$

Using the third LG equation: $(J_+^I)^{2s_I+1}M_n = 0$ for $I = 1,2$

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k\left(\frac{\langle 1,5\rangle}{m_1},\frac{\langle 2,4\rangle}{m_2}\right)\left(\frac{m_2}{m_1}+\frac{\langle 1,5\rangle}{m_1}\frac{\langle 2,4\rangle}{m_2}\frac{[4,5]}{\langle 1,2\rangle}\right)^{s_1+s_2+h_3-k}$$

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2-massive 1-massless 3-point amplitude

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k \left(\frac{\langle 1,5 \rangle}{m_1}, \frac{\langle 2,4 \rangle}{m_2} \right) \left(\frac{m_2}{m_1} + \frac{\langle 1,5 \rangle}{m_1} \frac{\langle 2,4 \rangle}{m_2} \frac{[4,5]}{\langle 1,2 \rangle} \right)^{s_1+s_2+h_3-k}$$

$$\tilde{f}_2 = \sum_{k=0}^{2s_2} b_k \left(\frac{\langle 1,5 \rangle}{m_1}, \frac{\langle 2,4 \rangle}{m_2} \right) \left(\frac{m_1}{m_2} + \frac{\langle 1,5 \rangle}{m_1} \frac{\langle 2,4 \rangle}{m_2} \frac{[4,5]}{\langle 1,2 \rangle} \right)^{s_1+s_2+h_3-k}$$

If $s_1 \neq s_2$, requiring the two different expression to be consistent, we get the following condition on the spins/helicities

$$|h_3| \leq s_1 + s_2$$

2-massive 1-massless 3-point amplitude

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k \left(\frac{\langle 1,5\rangle}{m_1}, \frac{\langle 2,4\rangle}{m_2}\right) \left(\frac{m_2}{m_1} + \frac{\langle 1,5\rangle}{m_1}\frac{\langle 2,4\rangle}{m_2}\frac{[4,5]}{\langle 1,2\rangle}\right)^{s_1+s_2+h_3-k}$$

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$$|h_3| \leq s_1 + s_2$$

2-mass. 1-massless 3p ampl. fixed up to max. $2s_{min}+1$ const.s

Remark:

If the two massive particles are the same

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k \left(\frac{\langle 1,5\rangle}{m_1}, \frac{\langle 2,4\rangle}{m_2}\right) \left(\frac{m_2}{m_1} + \frac{\langle 1,5\rangle}{m_1}\frac{\langle 2,4\rangle}{m_2}\frac{[4,5]}{\langle 1,2\rangle}\right)^{s_1+s_2+h_3-k}$$

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If the two massive particle are the same the two expressions are the same

$$\tilde{f}_2 = \sum_{k=0}^{2s} a_k \left(\frac{\langle 1,5 \rangle}{m}, \frac{\langle 2,4 \rangle}{m} \right) \left(1 + \frac{\langle 1,5 \rangle}{m} \frac{\langle 2,4 \rangle}{m} \frac{[4,5]}{\langle 1,2 \rangle} \right)^{2s+h_3-k}$$

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So we cannot match, and we cannot derive any condition on spins...

But this amplitude is zero for real momenta!
So analogously to the massless case there are no constraints on spins and helicities

3-massive 3-point amplitude

$$P_1 = \lambda_1\tilde{\lambda}_1 + \lambda_3\tilde{\lambda}_3 \quad P_2 = \lambda_2\tilde{\lambda}_2 + \lambda_4\tilde{\lambda}_4 \quad P_3 = \lambda_3\tilde{\lambda}_3 + \lambda_6\tilde{\lambda}_6$$

$$\langle 1,4\rangle[1,4] = -{m_1}^2 \quad \langle 2,4\rangle[2,4] = -{m_2}^2 \quad \langle 3,6\rangle[3,6] = -{m_3}^2$$

$$2+2+2 \;=\; 6 \text{ eq.s}$$

$$6\cdot 5 \;=\; 30 \text{ spinor prod.s}$$

momentum conservation: $30 \longrightarrow 14$

mass on-shell conditions: $14 \longrightarrow 11$

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$$M^{-s_1,-s_2,-s_3} = \langle 1,2\rangle^{s_1+s_2-s_3}\,\langle 3,1\rangle^{s_3+s_1-s_2}\,\langle 2,3\rangle^{s_2+s_3-s_1}\times \times f_3\left(\langle 1,4\rangle,\langle 2,5\rangle,\langle 3,6\rangle;\frac{[4,5]}{\langle 1,2\rangle},\frac{[6,4]}{\langle 3,1\rangle}\right)$$

Again from dimensional considerations

$$f_3 = g\,m_1^{1-s_1-s_2-s_3-[g]}\,\tilde{f}_3\left(\frac{\langle 1,4\rangle}{m_1},\frac{\langle 2,5\rangle}{m_2},\frac{\langle 3,6\rangle}{m_3};\frac{[4,5]}{\langle 1,2\rangle},\frac{[6,4]}{\langle 3,1\rangle}\right)$$

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Here the third equations are more involved, since f_3 depends on two scaling variables. J_+^2 acts only on $\frac{[4,5]}{\langle 1,2\rangle}$, J_+^3 acts only on $\frac{[6,4]}{\langle 3,1\rangle}$, while J_+^1 acts on both.

3-massive 3-point amplitude

From $(J_+^I)^{2s_I+1} M_n = 0$ for $I = 2, 3$

$$f_3\big(\ldots;\xi_2,\xi_3\big) = x^{s_1-s_2-s_3}\,\sum_{k=0}^{2s_I} c_k^{(I)}\big(\ldots;\xi_{\bar{I}}\big)\,x^k$$

with $\xi_2 = \dfrac{[4,5]}{\langle 1,2\rangle}\,,\quad \xi_3 = \dfrac{[6,4]}{\langle 3,1\rangle}\,,\quad x = \langle 2,5\rangle\,\xi_2 + \langle 3,6\rangle\,\xi_3 - \dfrac{{m_1}^2}{\langle 1,4\rangle}$

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... but can be worked out case by case

Looking backward...

Summary

We have determined the most general Poincaré invariant 3-point amplitude where massive particles are involved, in spinor-helicity formalism.

We have it for the lowest value of the spin projection, but

$$J_+^I \, M_3^{\ldots,-s_I,\ldots} = M_3^{\ldots,-s_I+1,\ldots}$$

Looking backward...

Summary

We have determined the most general Poincaré invariant 3-point amplitude where massive particles are involved, in spinor-helicity formalism.

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For given interactions these theoretical expressions match existing results in the literature.

... and forward

Outlook

- Massive BCFW recursion relations

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Outlook

- Massive BCFW recursion relations
- 4 particle-test to constrain the remaining undetermined constants
- Massive Higher-Spins amplitudes in 4 dimensions
  [E. Conde *et al.* 1605.07402]

¡Gracias!

Detail of the massive LG

The direct translation of the massive LG transformations in spinor language would be

$$\lambda, \mu \to t\,\lambda, \mu \qquad \tilde{\lambda}, \tilde{\mu} \to t^{-1}\,\tilde{\lambda}, \tilde{\mu}$$

while the scaling we use is

$$\lambda, \tilde{\mu} \to t\,\lambda, \mu \qquad \tilde{\lambda}, \mu \to t^{-1}\,\tilde{\lambda}, \mu$$

Nonetheless the two groups of transformations are isomorphic

$$\begin{pmatrix} R\,\lambda \\ R\,\mu \end{pmatrix} \xleftrightarrow{1-1} U \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$