Entanglement entropy of Pais-Uhlenbeck oscillators and excited holographic states

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Outline

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- The PU oscillator appears in the context of AdS/CFT correspondence, for instance in the Pilch-Warner supergravity solution!
- Ostrogradsky's Hamiltonian is unbounded from below, hence ghost problem in quantum theory.
- Nevertheless, several alternative Hamiltonian formulations exist!
- The PU oscillator is conformally invariant for frequencies $\omega_k = (2k+1)\omega_0$.
- Stable coherent states, which have constant dispersions and a modified Heisenberg uncertainty relation.

An alternative Hamiltonian

The EoM of the PU oscillator of order $2n$ can be obtained by varying the action

$$S = \frac{1}{2} \int dt \, x_i \prod_{k=0}^{n-1} \left(\frac{d^2}{dt^2} + \omega_k^2 \right) x_i .$$

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Ansatz for an alternative Hamiltonian as a linear combination of integrals of motion according to Noether's theorem ($\alpha_k \neq 0$):

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H_n can play the role of Hamiltonian for the compatible Poisson structure

$$\{A,B\} = \sum_{s,m=0}^{2n-1} w_{sm} \frac{\partial A}{\partial x_i^{(s)}} \frac{\partial B}{\partial x_i^{(m)}}.$$

The variables, which satisfy the structure relations $\{x_i^k, p_j^m\} = \delta_{ij}\delta_{km}$, are

$$x_i^k = \sqrt{|\alpha_k|\rho_k} \prod_{\substack{m=0\\ m\neq k}}^{n-1}\left(\frac{d^2}{dt^2}+\omega_m^2\right) x_i, \qquad p_i^k = \mathrm{sgn}(\alpha_k)\frac{dx_i^k}{dt}.$$

Set-up: ring of PU oscillators

$$H_N = \frac{1}{2}\sum_{\mu=1}^{N}\sum_{k=0}^{1}\mathrm{sgn}(\alpha_{\mu,k})\left(p_\mu^k p_\mu^k + \omega_{\mu,k}^2 x_\mu^k x_\mu^k\right) + \frac{1}{2}\sum_{\langle\mu,\nu\rangle=1}^{N}\sum_{k,l=0}^{1}\tilde{c}_{\mu\nu}^{kl} x_\mu^k x_\nu^l$$

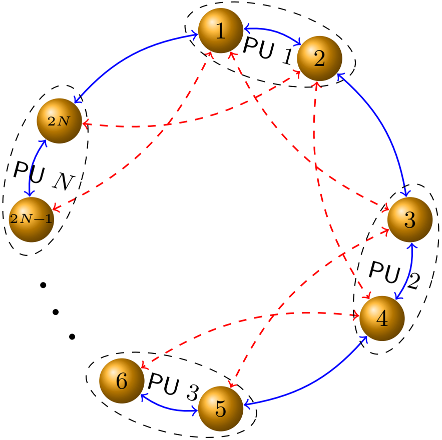

Figure: Closed chain of N identical PU oscillators.

Diagonalisation

The Hamiltonian can be written in matrix form as

$$H_N = \frac{1}{2}\eta^{\mathrm{T}} \begin{pmatrix} \Omega & 0 \\ 0 & \mathbb{1}_{2N} \end{pmatrix} \eta, \qquad \Omega = \begin{pmatrix} W & C & 0 & \cdots & C \\ C & W & C & \cdots & 0 \\ 0 & C & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ C & 0 & \cdots & C & W \end{pmatrix}.$$

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Symmetric block circulant matrices with symmetric blocks are diagonalised by Discrete Fourier Transform (DFT):

$$\widehat{\Omega} = U^{-1}\Omega U = \mathrm{diag}[D_1, D_2, \ldots, D_N],$$

$$U_{kl} = \frac{1}{\sqrt{N}} e^{2\pi i k l/N} \mathbb{1}_2, \quad k, l = 0, 1, \ldots, N-1,$$

where the eigenvalues of Ω are given by

$$D_{k+1} = W + 2\cos\left(2\pi k/N\right) C.$$

Building the Fock space

The creation and annihilation operators are defined by

$$\mathfrak{a}_j = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\lambda_j}\widehat{x}_j + \frac{i}{\sqrt{\lambda_j}}\widehat{p}_j \right), \qquad \mathfrak{a}_j^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\lambda_j}\widehat{x}_j - \frac{i}{\sqrt{\lambda_j}}\widehat{p}_j \right).$$

The diagonalised Hamiltonian is equivalent to that of $2N$ harmonic oscillators

$$H_N = \sum_{j=1}^{2N} \hbar\lambda_j \left(\mathfrak{a}_j^\dagger \mathfrak{a}_j + \frac{1}{2} \right).$$

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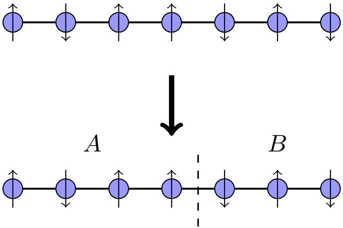
The Fock space is built up from the vacuum

$$|\{n_j\}\rangle = \prod_{j=1}^{2N} \frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}} |\{0\}\rangle, \qquad |\{n_j\}\rangle = |n_1\rangle \otimes \cdots \otimes |n_{2N}\rangle.$$

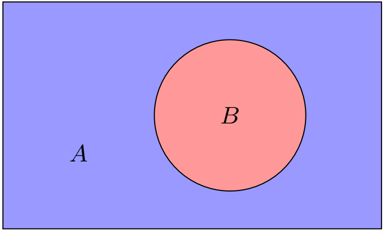
The excited states are orthonormal, $\langle\{m_j\}|\{n_j\}\rangle = \delta_{\{m_j\},\{n_j\}}$.

Entanglement entropy — an idea and definitions

Discrete system (spin chain)



Quantum field theory



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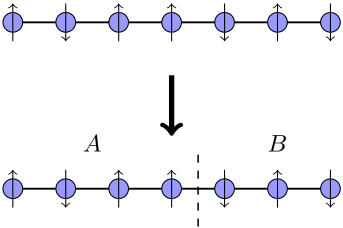

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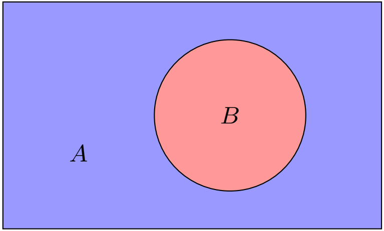

Density matrix (pure state):

$$\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|.$$

Reduced density matrix:

$$\rho_A = \text{Tr}_B \rho_{\text{tot}}.$$

Von Neumann entropy:

$$S_A = -\text{Tr}_A \rho_A \log \rho_A.$$

EE at finite temperature $T = \beta^{-1}$:

$$\rho_{\text{thermal}} = e^{-\beta H}.$$

Holographic entanglement entropy

Black hole entropy [Bekenstein '73, Hawking '75]:

$$S_{\mathrm{BH}} = \frac{\mathrm{Area}(\Sigma)}{4G_{\mathrm{N}}}.$$

Holographic entanglement entropy [Ryu-Takayanagi '06]:

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_{\mathrm{N}}^{(d+2)}}.$$

EE in quantum field theories:

$$S_A = \gamma \frac{\mathrm{Area}(\partial A)}{a^{(d-1)}} + \text{subleading terms},$$

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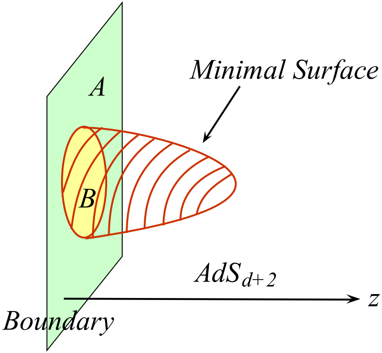

Thermo-field dynamics (TFD)

Having the Hamiltonian diagonalised, the standard density matrix is:

$$\rho_{\text{eq}}(K_j) = \frac{1}{Z(K_j)} e^{-\beta H_N}, \qquad Z(K_j) := \mathrm{Tr}_{\{j\}} e^{-\beta H_N}.$$

TFD explores double Hilbert space with basis $\{|n\rangle \otimes |\tilde{n}\rangle\} \equiv \{|n, \tilde{n}\rangle\}$.

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- Extended entanglement entropy:

$$\widehat{S}_{1,2} = -k_{\text{B}} \mathrm{Tr}_{1,2} \left[\widehat{\rho}_{1,2} \log \widehat{\rho}_{1,2}\right].$$

EE of PU oscillators

Entanglement entropy:

$$\widehat{S}_{1,2}(K_1, K_2) = \frac{k_{\mathrm{B}}}{2} \coth\frac{K_1}{4} \coth\frac{K_2}{4} \left[K_1 \left(1 + \coth\frac{K_1}{4}\right) + K_2 \left(1 + \coth\frac{K_2}{4}\right) \right. \\ \left. -2\log\left[\left(e^{K_1} - 1\right)\left(e^{K_2} - 1\right)\right]\right].$$

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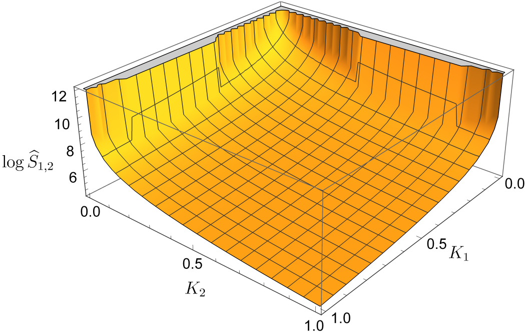

Figure: The entanglement entropy as function of K_1 and K_2 in units $k_{\mathrm{B}}=1$. Evidently, the Nernst heat theorem is satisfied.

Information space metric

Riemannian metric defined on a smooth statistical manifold whose point are probability measures.

- Infinitesimal form of the relative entropy (Hessian of the Kullback-Leibler divergence).
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Fisher metric $g_{\mu\nu}(K_1, K_2) = \partial_\mu \partial_\nu S(K_1, K_2)$:

$$g_{11} = \frac{1}{64} k_{\rm B} \coth\frac{K_2}{4} \operatorname{csch}^2\frac{K_1}{4} \Bigg[K_1 \left(3 + 5\coth^2\frac{K_1}{4} + 7\operatorname{csch}^2\frac{K_1}{4}\right) + 4\tanh\frac{K_1}{4}$$
$$+4\coth\frac{K_1}{4}\left(K_1 + K_2 - 5 + K_2\coth\frac{K_2}{4} - 2\log\left[\left(e^{K_1}-1\right)\left(e^{K_2}-1\right)\right]\right)\Bigg],$$
$$g_{12} = \frac{1}{32} k_{\rm B} \operatorname{csch}^2\frac{K_1}{4} \operatorname{csch}^2\frac{K_2}{4} \Bigg[K_1 \left(1 + 2\coth\frac{K_1}{4}\right) + K_2\left(1 + 2\coth\frac{K_2}{4}\right) - 4$$
$$-2\log\left[\left(e^{K_1}-1\right)\left(e^{K_2}-1\right)\right]\Bigg].$$

Penrose limit of the PW solution

Consider a null geodesic corresponding to the moduli space of a D3-brane probe in the IR point of PW background. The bosonic part of the pp-wave action is

$$S_B = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma \left[\sqrt{-g} g^{\alpha\beta}\left(2\partial_\alpha U \partial_\beta V + A_{ij} X^i X^j \partial_\alpha U \partial_\beta U + \partial_\alpha X^i \partial_\beta X^i\right) \right.$$
$$\left. -2\sqrt{3} E \epsilon^{\alpha\beta}\left(X^1 \partial_\alpha U \partial_\beta X^3 - X^2 \partial_\alpha U \partial_\beta X^4\right)\right],$$

where $A_{ij} = \mathrm{diag}[1, 1, 4]$ is the pp-wave spectrum matrix.

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where $A_{ij} = \mathrm{diag}[1,1,4]$ is the pp-wave spectrum matrix.

The ansatz $X^i(\tau,\sigma) = e^{i\sigma}x_i(\tau)$ decouples the system of second-order partial differential EoM to the system of ODEs:

$$x_q^{(4)} + \left(5M^2+2\right)x_q^{(2)} + \left(4M^4+2M^2+1\right)x_q = 0,$$
$$x_p^{(2)} + \left(M^2+1\right)x_p = 0,$$

where $M = E\alpha' p^+$, $q = 1,2$ and the directions $p = 5,6,7,8$ stay unaffected by the B-field.

Quadratic fluctuations around classical solutions

The Lagrangian describing quadratic fluctuations around the classical solutions of rotating strings in the PW geometry (the five-sphere part wherein the B-field is turned on) is

$$\mathcal{L}^{\text{tot}}_{S^5} = \partial_\alpha \tilde{\zeta}^A \partial^\alpha \tilde{\zeta}^A + \tilde{M}^2 \left(\tilde{\zeta}^2_{\tilde{2}} + \tilde{\zeta}^2_{\tilde{4}} \right) + \left(4\tilde{M}^2 + \frac{3}{2} \bar{\rho}'^2 \right) \left(\tilde{\zeta}^2_{\tilde{3}} + \tilde{\zeta}^2_{\tilde{5}} \right) \\ + 2\sqrt{3}\tilde{M} \left(\tilde{\zeta}_{\tilde{4}} \partial_1 \tilde{\zeta}_{\tilde{5}} - \tilde{\zeta}_{\tilde{2}} \partial_1 \tilde{\zeta}_{\tilde{3}} \right),$$

where $\tilde{M}^2 = \frac{4}{9}(c_\beta + c_\gamma + c_\phi)^2$ depends on the angular velocities c_β, c_γ, and c_ϕ of the rotating string.

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where $\tilde{M}^2 = \frac{4}{9}(c_\beta + c_\gamma + c_\phi)^2$ depends on the angular velocities c_β, c_γ, and c_ϕ of the rotating string.

Using the ansatz $\tilde{\zeta}_{\tilde{i}} = e^{i\sigma} y_i(\tau)$ in the limiting cases of short and long strings ($\bar{\rho}'^2 = \mathrm{const}$), the EoM take the form:

$$y_p^{(4)} + \left[5\tilde{M}^2 + 2 + \frac{3}{2}\bar{\rho}'^2\right] y_p^{(2)} + \left[4\tilde{M}^4 + 2\tilde{M}^2 + 1 + \left(\tilde{M}^2 + 1\right)\frac{3}{2}\bar{\rho}'^2\right] y_p = 0,$$

$$y_1^{(2)} + y_1 = 0, \qquad p = 2, 4.$$

Instabilities and phase transitions

The dynamics of two free PU oscillators each of fourth order is driven by the following two ODEs:

$$x_1^{(4)} + \left(\lambda_{0+}^2 + \lambda_{0-}^2\right) x_1^{(2)} + \lambda_{0+}^2 \lambda_{0-}^2 x_1 = 0,$$
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We have shown that this system is a particular representation of a system of two minimally interacting PU oscillators of fourth order. The frequencies are determined by the diagonalisation of the Hamiltonian:

$$\lambda_{0\pm} = \frac{1}{2}\left[\omega_0 + \omega_1 + (c_0 + c_1)\right] \pm \frac{1}{2}\sqrt{2c_2^2 + \left[c_0 - c_1 + (\omega_0 - \omega_1)\right]^2},$$
$$\lambda_{1\pm} = \frac{1}{2}\left[\omega_0 + \omega_1 - (c_0 + c_1)\right] \pm \frac{1}{2}\sqrt{2c_2^2 + \left[c_0 - c_1 - (\omega_0 - \omega_1)\right]^2}.$$

Instabilities appear when the frequencies become imaginary:

$$2c_2^2 \leq \left[c_0 - c_1 \pm (\omega_0 - \omega_1)\right]^2.$$

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The instabilities due to occurrence of critical point of the B-field in the PW background are related to instabilities in the system of two interacting PUO!

Excited holographic states

According to the holographic conjecture, the states that capture the gravitational sector of the dual theory are:

$$\mathrm{Id} \sim \mathbb{1}, T, \partial^k T, T^2, T\partial^l T, \dots .$$

Consider conformal transformation $z \mapsto w = f(z)$ realised by an unitary operator U_f. The primary operators $\Phi_\pm$ transform as:

$$\langle \Phi_+(z_1)\Phi_-(z_2)\rangle = \left(\frac{\partial f(z_1)}{\partial z_1}\right)^{h_n} \left(\frac{\partial f(z_2)}{\partial z_2}\right)^{h_n} \langle \Phi_+(f(z_1))\Phi_-(f(z_2))\rangle \, .$$

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The Rényi entropy can be calculated using replica trick:

$$\exp\left((1-n)S^{(n)}_{\mathrm{vac}}\right) = \langle \Phi_+(z_1)\Phi_-(z_2)\rangle = \frac{1}{(z_1-z_2)^{2h_n}}.$$

The vacuum entanglement entropy is obtained in the limit $n \to 1$:

$$S_{\mathrm{vac}} = \lim_{n\to 1} S^{(n)}_{\mathrm{vac}} = \frac{c}{12}\log\frac{z_1-z_2}{\delta^2}.$$

Aharonov invariants

In the same manner, entanglement entropy of excited states $\exp\left((1-n)S_{\text{ex}}^{(n)}\right) = \langle f|\Phi_+(z_1)\Phi_-(z_2)|f\rangle$ is given by:

$$S_{\text{ex}} = \lim_{n\to 1} S_{\text{ex}}^{(n)} = \frac{c}{12}\log\left|\frac{f'(z_1)f'(z_2)}{\delta^2\left(f(z_1)-f(z_2)\right)^2}\right|.$$

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For $f(z)$ being non-constant meromorphic function on a domain $D\subset\mathbb{C}$, the quantity $G(\zeta,z)=\frac{f'(z)}{f(\zeta)-f(z)}$ has the expansion ($f(z)\neq\infty$, $f'(z)\neq 0$):

$$G(\zeta,z) = \frac{1}{\zeta - z} - \sum_{n=1}^{\infty}\psi_n[f](z)(\zeta-z)^{n-1}.$$

Evaluating $\partial G/\partial\zeta$, one can easily express the contributions of excited states to the vacuum EE in terms of the Aharonov invariants $\psi_n[f](z)$.

Contributions of excited states in terms of Aharonov invariants:

$$S_{\text{ex}} - S_{\text{vac}} = \frac{c}{12}\log\left(1+\sum_{n=2}^{\infty}(n-1)\psi_n[f](z)(\zeta-z)^n\right).$$

Faber polynomials & Grunsky coefficients

The following two sets of functions are correspondingly univalent in neighbourhoods of 0 and ∞:

$$\left\{ f(z) = \sum_{n=1}^{\infty} a_n z^n,\ a_1 \neq 0 \right\} \quad \text{and} \quad \left\{ g(z) = bz + \sum_{n=0}^{\infty} b_n z^{-n} \right\}.$$

The Faber polynomials are defined via:

$$\frac{g'(z)}{g(z) - w} = \sum_{n=0}^{\infty} \Phi_n(w) z^{-n-1}, \quad \Phi_n(w) = \sum_{m=0}^{n} b_{n,m} w^m.$$

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On the other hand, by taking second derivative $\frac{\partial^2}{\partial z \partial \zeta}$ on both sides of

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{n,m=0}^{\infty} b_{-n,-m} z^n \zeta^m,$$

one can readily find the EE in terms of the Grunsky coefficients $b_{-n,-m}$.

Contributions of excited states in terms of Grunsky coefficients:

$$S_{\text{ex}} - S_{\text{vac}} = \frac{c}{12} \log \left(1 - (\zeta - z)^2 \sum_{n,m=1}^{\infty} n\, m\, b_{-n,-m} z^{n-1} \zeta^{m-1} \right).$$

Conclusion and perspectives

- Generalisations to odd/higher-order PUOs and more complex interactions.
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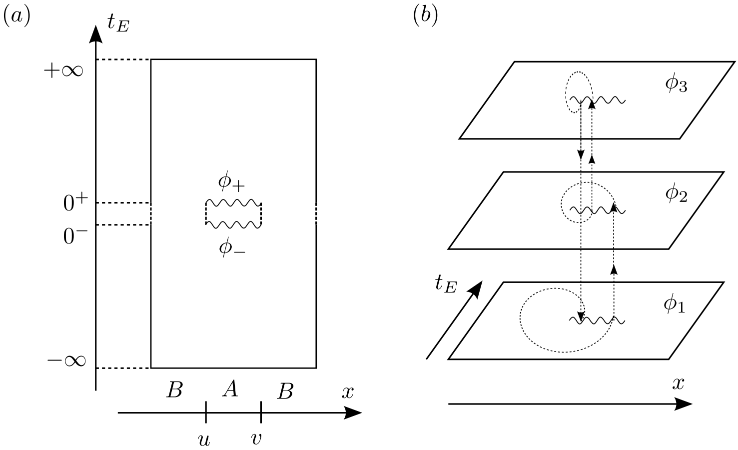
Thank you for your attention!

Replica trick



$$S_A = -\frac{\partial}{\partial n} \mathrm{Tr}_A \rho_A^n|_{n=1} = -\frac{\partial}{\partial n} \log \mathrm{Tr}_A \rho_A^n|_{n=1}$$