On the (non-)uniqueness of the Levi-Civita solution in the Einstein-Hilbert-Palatini formalism

José Alberto Orejuela

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arXiv:1606.08756: Antonio N. Bernal, Bert Janssen, Alejandro Jimenez-Cano, J.A.O., Miguel Sanchez, Pablo Sanchez-Moreno

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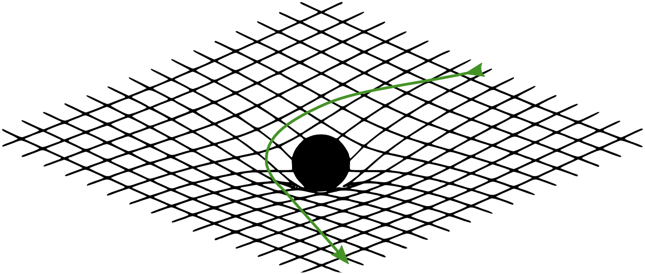
Introduction

General relativity:

- Gravity is a curvature effect.
- Free particles follow geodesics.



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Spacetime: D-dimensional time-orientable Lorentzian manifold equipped with:

- Metric $g_{\mu\nu}$.
- Levi-Civita connection:

$$\Gamma^{\rho}_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\} = \frac{1}{2} g^{\rho\lambda} \left(\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu} \right).$$

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Geodesic curves (affine and metric):

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho} \dot{x}^{\nu} \dot{x}^{\rho} = 0.$$

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- It would be perfect if there was a physical mechanism that selects Levi-Civita over other possibilities.
- If I find a variational principle that have L-C as a solution, is it unique? Which one is the most general solution?

Palatini formalism

Metric $g_{\mu\nu}$ and connection $\Gamma^{\rho}_{\mu\nu}$ independent, as in differential geometry. Action dependent on both:

$$S = S(g, \Gamma) = \int \mathrm{d}^D x \sqrt{|g|} \left[\frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu}(\Gamma) + \mathcal{L}_{\mathrm{M}}(\phi, g) \right].$$

- $\dfrac{\delta S}{\delta g} \rightarrow$ Einstein equation.
- $\dfrac{\delta S}{\delta \Gamma} \rightarrow$ Connection equation.

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What do we expect? We hope to find Levi-Civita as the unique solution.

General solution

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Equations of motion:

$$R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R = -\kappa \mathcal{T}_{\mu\nu}, \quad R = g^{\rho\lambda} R_{\rho\lambda},$$

$$\nabla_\lambda g_{\mu\nu} - T^\sigma_{\nu\lambda} g_{\sigma\mu} - \frac{1}{D-1} T^\sigma_{\sigma\lambda} g_{\mu\nu} - \frac{1}{D-1} T^\sigma_{\sigma\nu} g_{\mu\lambda} = 0.$$

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$$\bar{T}^{\rho}_{\mu\nu} = \mathcal{A}_\mu \delta^\rho_\nu - \mathcal{A}_\nu \delta^\rho_\mu , \qquad \bar{\nabla}_\rho g_{\mu\nu} = -2\mathcal{A}_\rho g_{\mu\nu} .$$

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Curvature tensors:

$$\bar{R}_{\mu\nu\rho}{}^{\lambda} = R_{\mu\nu\rho}{}^{\lambda} + \mathcal{F}_{\mu\nu}\delta^{\lambda}_{\rho}, \qquad \bar{R}_{\mu\nu} = R_{\mu\nu} + \mathcal{F}_{\mu\nu}, \qquad \bar{R} = R,$$

where

$$\mathcal{F}_{\mu\nu} = \partial_{\mu}\mathcal{A}_{\nu} - \partial_{\nu}\mathcal{A}_{\mu} = \nabla_{\mu}\mathcal{A}_{\nu} - \nabla_{\nu}\mathcal{A}_{\mu}.$$

Geometrical properties

Affine geodesic equation:

$$\dot{x}^\rho \bar{\nabla}_\rho \dot{x}^\mu = 0 \Leftrightarrow \dot{x}^\rho \nabla_\rho \dot{x}^\mu = -\mathcal{A}_\rho \dot{x}^\rho \dot{x}^\mu$$

$$\Leftrightarrow \dot{x}^\rho \nabla_\rho \dot{x}^\mu = \left(\frac{\ddot{s}}{\dot{s}}\right) \dot{x}^\mu, \; s(\lambda) = \int_0^\lambda e^{-\int_0^{\lambda'} \dot{x}^\rho \mathcal{A}_\rho \, \mathrm{d}\lambda''} \, \mathrm{d}\lambda'$$

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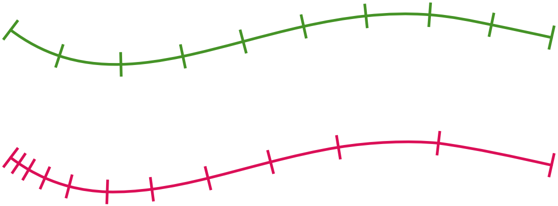
Same trajectories but with different parametrisation.



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Parallel transport:

$$\dot{x}^{\rho}\bar{\nabla}_{\rho}V^{\mu} - \dot{x}^{\rho}\nabla_{\rho}V^{\mu} = \dot{x}^{\rho}\mathcal{A}_{\rho}V^{\mu}$$

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$$\dot{x}^\rho \bar{\nabla}_\rho V^\mu - \dot{x}^\rho \nabla_\rho V^\mu = \dot{x}^\rho \mathcal{A}_\rho V^\mu \quad \Rightarrow \quad V^\mu_{\bar{\Gamma}}(\lambda) = e^{-G(\lambda)} V^\mu(\lambda),$$

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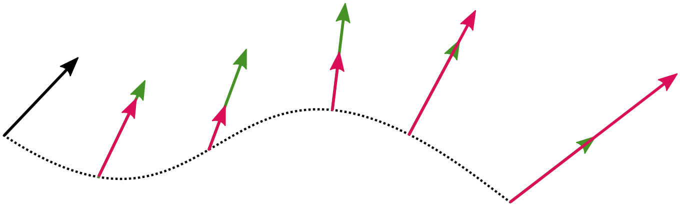
- Consequence of non metric compatibility.
- Similar to L-C transport composed with homothety.
- Uniqueness.



Physical observability

Let's summarise:

- General solution

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- Homothetic parallel transport.

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Same rough physics:

- Same solutions to Einstein equation:

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  - Same tidal forces (geodesic deviation).

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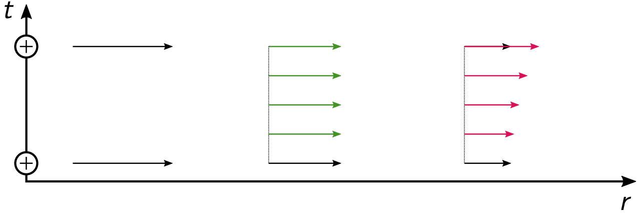
- Parallel transport with homothety $\longleftrightarrow$ Staticity:



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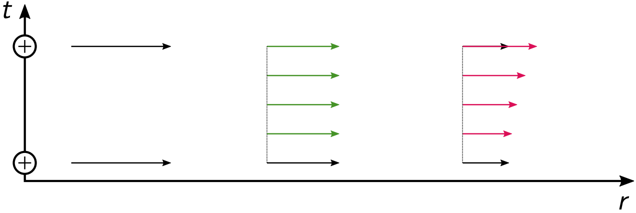

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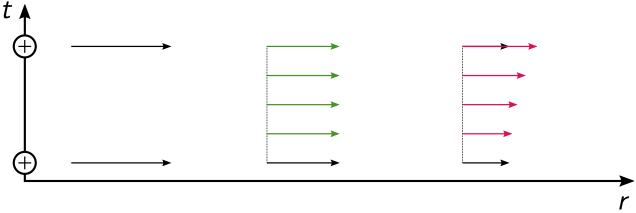

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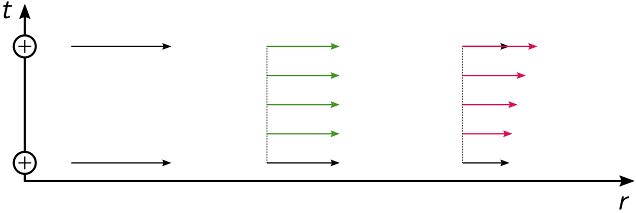

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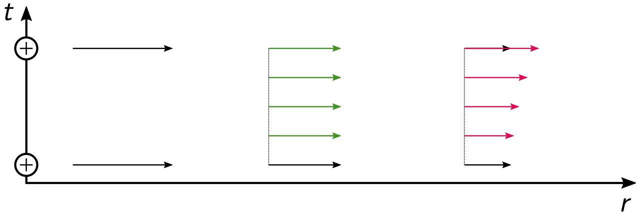

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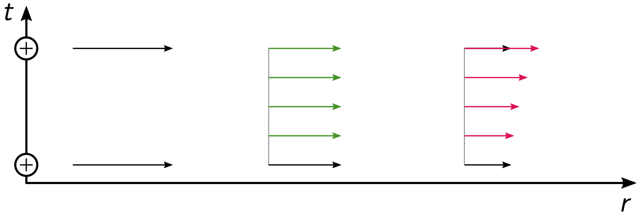

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Moral: Compare directions with the connection and norms with the metric.

Future work

Next step: Lovelock Gravities:

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In particular, Gauss-Bonnet,

$$S = \int \mathrm{d}^D x \sqrt{|g|} \left[R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right].$$

We have already obtained the variations of the action and have seen that Palatini connections are solutions.

Conclusions

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Thanks for your attention!