On the SO(3)-gauged maximal d=8 supergravities

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Overview

Cremmer and Julia (1978)
The bosonic fields of $N = 1, d = 11$ supergravity are:

$$\left\{ \hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}} \right\} .$$

The field strength of the 3-form is

$$\hat{\hat{G}} = 4\partial\,\hat{\hat{C}} ,$$

and is obviously invariant under the gauge transformations

$$\delta\,\hat{\hat{C}} = 3\partial\,\hat{\hat{\chi}} ,$$

where $\hat{\hat{\chi}}$ is a 2-form.
The action for these bosonic fields is

$$\hat{\hat{S}} = \int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\left[\hat{\hat{R}} - \frac{1}{2\cdot 4!}\,\hat{\hat{G}}^2 - \frac{1}{6^4}\frac{1}{\sqrt{|\hat{\hat{g}}|}}\,\hat{\hat{\varepsilon}}\partial\,\hat{\hat{C}}\partial\,\hat{\hat{C}}\,\hat{\hat{C}}\right] .$$

.

Alonso-Alberca, Messen and Ortín (2000)
After compactification on T^3 we get:

Bosonic fields of $8-$*dimensional* supergravity

$$\{g_{\mu\nu}, C, B_m, A^{1\,m}, A^{2\,m}, a, \varphi, \mathcal{M}_{mn}\},$$

The action:

$$\begin{aligned} S \;=\; & \int d^8x\sqrt{|g_E|}\,\Big\{R_E + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{W}\mathcal{W}^{-1}\right)^2 \\ & -\tfrac{1}{4}F^{i\,m}\mathcal{M}_{mn}\mathcal{W}_{ij}F^{j\,n} + \tfrac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n - \tfrac{1}{2\cdot 4!}e^{-\varphi}G^2\,, \\ & -\tfrac{1}{6^3\cdot 2^4}\,\tfrac{1}{\sqrt{|g_E|}}\,\varepsilon\left[GGa - 8GH_mA^{2\,m} + 12G(F^{2\,m}+aF^{1\,m})B_m\right. \\ & \left.-8\varepsilon^{mnp}H_mH_nB_p - 8G\partial aC - 16H_m(F^{2\,m}+aF^{1\,m})C\right]\Big\}\,. \end{aligned}$$

With Field Strengths:

$$G = 4\partial C + 6F^{1\,m}B_m\,,$$

$$H_m = 3\partial B_m + 3\varepsilon_{mnp}F^{1\,n}A^{2\,p}\,,$$

$$F^{2\,m} = 2\partial A^{2\,m}\,,$$

leading to the following non-trivial Bianchi identities:

$$\partial G = 2F^{1\,m}H_m\,,$$

$$\partial H_m = \tfrac{3}{2}\varepsilon_{mnp}F^{1\,n}F^{2\,p}\,,$$

Goals

- The construction of a generic (up to second order in derivatives) 8-dimensional theories with Abelian gauge symmetry and non-trivial Chern-Simons terms compatible with the existence of a group of electric-magnetic duality rotations of the equations of motion(in 8 dimensions it must be a subgroup of the symplectic group). There are previous works in different dimensions,d=3,d=4,d=5,d=6,d=9. Bergshoeff, Hartong, Hohm, Hubscher, Ortín, 2009. Hartong, Hohm, Hubscher, Ortín, 2009. Hubscher, Ortín, Shahbazi, 2014. Hubscher, Ortín, Shahbazi, 2014. Fernandez-Melgarejo,Ortín, Torrente-Lujan, 2012.

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- The general gauging of the global symmetry group using the embedding-tensor fomalism including the possibility of adding Stückelberg couplings consistent with the above-mentioned electric-magnetic duality.

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- A simplification/sitematization of the construction of maximal 8-dimensional supergravities with SO(3) gaugings. Salam and Sezgin (1985), Alonso-Alberca,Messen, Ortín (2000), Alonso-Alberca,Bergshoeff, Gran, Linares, Ortín (2003)

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Ungauged $d = 8$ theories

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- 3-form fields $C^a = \frac{1}{3!} C^a{}_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$.

The way

Which is the simplest theory one can construct with these fields?.

The most simple action with its field strengths

The simplest field strengths are the exterior derivatives:

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The most general gauge-invariant action

$$\begin{aligned} S \quad = \quad & \int \{\star 1 R + \tfrac{1}{2}\mathscr{G}_{xy} d\phi^x \wedge \star d\phi^y - \tfrac{1}{2}\mathscr{M}_{IJ} F^I \wedge \star F^J + \tfrac{1}{2}\mathscr{M}^{mn} H_m \wedge \star H_n \\ & -\tfrac{1}{2}\mathfrak{Im}\mathscr{N}_{ab} G^a \wedge \star G^b - \tfrac{1}{2}\mathfrak{Re}\mathscr{N}_{ab} G^a \wedge G^b\} \,, \end{aligned}$$

where the kinetic matrices $\mathscr{G}_{xy}, \mathscr{M}_{IJ}, \mathscr{M}^{mn}, \mathfrak{Im}\mathscr{N}_{ab}$ as well as the matrix $\mathfrak{Re}\mathscr{N}_{ab}$ are scalar-dependent.

Equations of motion

The equations of motion of the 3-forms C^a are

$$\frac{\delta S}{\delta C^a} = -d\frac{\delta S}{\delta G^a} = 0\,, \qquad \frac{\delta S}{\delta G^a} = R_a \equiv -\mathfrak{Re}\mathcal{N}_{ab}G^b - \mathfrak{Im}\mathcal{N}_{ab}\star G^b\,.$$

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These equations can be solved locally by introducing a set of dual 3-forms C_a.

$$dC_a \equiv R_a\,.$$

Moreover, we can built a vector containing the fundamental and dual 3-forms:

$$(C^i) \equiv \begin{pmatrix} C^a \\ C_a \end{pmatrix}\,, \qquad G^i \equiv dC^i\,,$$

so that the equations of motion and the Bianchi identities for the fundamental field strengths take the simple form

$$dG^i = 0\,.$$

First abelian deformation.

$$G^a = dC^a + d^a{}_l{}^m F^l B_m \,,$$

$$\delta_\sigma A^l = d\sigma^l \,, \qquad \delta_\sigma B_m = d\sigma_m \,, \qquad \delta_\sigma C^a = d\sigma^a - d^a{}_l{}^m F^l \sigma_m \,.$$

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Problems!

- The action remains gauge-invariant but the formal symplectic invariance is broken: if we do not modify the action, the dual 4-form field strengths are just $G_a = dC_a$ and $\mathrm{Sp}(2n_3, \mathbb{R})$ cannot rotate these into G^a
- Furthermore, the 1-form and 2-form equations of motion do not have a symplectic-invariant form.

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- Furthermore, the 1-form and 2-form equations of motion do not have a symplectic-invariant form.

Solution:Add a CS term to the action

$$S_{CS} = \int \{-d_{al}{}^m dC^a F^l B_m\} \, ,$$

Many things change

New equation of motion:

$$-d\frac{\delta S}{\delta dC^a} = 0\,, \qquad \frac{\delta S}{\delta dC^a} = R_a - d_{al}{}^m dC^a F^l B_m\,.$$

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The local solution is now

$$dC_a \equiv R_a - d_{al}{}^m dC^a F^l B_m\,,$$

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The local solution is now

$$dC_a\equiv R_a-d_{al}{}^m dC^a F^l B_m\,,$$

The dual, gauge-invariant, field strength now is:

$$R_a=dC_a+d_{al}{}^m dC^a F^l B_m\equiv G_a\,.$$

$(C^i)=\begin{pmatrix} C^a \\ C_a \end{pmatrix}$ transforms linearly as a symplectic vector if $(d^i{}_l{}^m)\equiv\begin{pmatrix} d^a{}_l{}^m \\ d_{al}{}^m \end{pmatrix}$ also does.

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invariant under the deformed gauge transformations

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Problem!

The deformed gauge transformations do not leave invariant the CS term.

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Problem!

The deformed gauge transformations do not leave invariant the CS term.

Solution

Add another term of the form

$$S_{CS} = \int \{ -d_{aI}{}^m dC^a F^I B_m - \tfrac{1}{2} d_{aI}{}^m d^a{}_J{}^m F^{IJ} B_{mn} \} \, ,$$

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Constraints

$$d_{a(I}{}^{[m} d^a{}_{J)}{}^{m]} = 0\,, \text{ so } d_{i(I}{}^{(m} d^i{}_{J)}{}^{m)} = 0\,.$$

Checking the symplectic invariance

Using the duality relation $R_a = G_a$ the equations of motion of the 1-forms can be written in the form

$$\frac{\delta S}{\delta A^I} = d\left\{\mathscr{M}_{IJ}\star F^J + d_{iI}{}^m G^i B_m + \tfrac{1}{2} d_{iI}{}^m d^i{}_J{}^m F^J B_{mn}\right\} = 0\,,$$

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The solution

$$\begin{aligned}
\tilde{F}_I &\equiv d\tilde{A}_I + d_{iI}{}^m G^i B_m + \tfrac{1}{2} d_{iI}{}^m d^i{}_J{}^m F^J B_{mn}\,,\\
\tilde{F}_I &= -\mathscr{M}_{IJ}\star F^J\,,\\
d\tilde{F}_I &= d_{iI}{}^m G^i H_m\,,
\end{aligned}$$

where $\tilde{A}_I$ is a set of 5-forms.

EOM for the 1-forms

The equations of motion are of the 1-forms given by the Bianchi identities of the dual 6-form field strengths up to duality relations:

$$\frac{\delta S}{\delta A^I} = -\left\{ d\tilde{F}_I - d_{iI}{}^m G^i H_m \right\}.$$

The Solution

Using the duality relation $R_a = G_a$ and following the same steps for the 2-forms , we find

$$\begin{aligned} \tilde{H}^m &= d\tilde{B}^m + d^i{}_I{}^m F^I C_i\,, \\ \tilde{H}^m &= \mathscr{M}^{mn} \star H_n\,, \\ d\tilde{H}^m &= -d_{iI}{}^m G^i F^I\,, \end{aligned}$$

The equations of motion of the 2-forms are given by the Bianchi identities of the dual 5-form field strengths up to duality relations:

$$\frac{\delta S}{\delta B_m} = -\left\{ d\tilde{H}^m + d_{iI}{}^m G^i F^I \right\}.$$

The equations of motion of the 2-forms are given by the Bianchi identities of the dual 5-form field strengths up to duality relations:

$$\frac{\delta S}{\delta B_m} = -\left\{ d\tilde{H}^m + d_{iI}{}^m G^i F^I \right\}.$$

This completes the first abelian deformation

Deforming H_m

$$H_m = dB_m - d_{mIJ} F^I A^J \,,$$

which is invariant under the gauge transformations

$$\delta_\sigma A^I = d\sigma^I \,, \qquad \delta_\sigma B_m = d\sigma_m + d_{mIJ} F^I \sigma^J \,,$$

and satisfies the Bianchi identity

$$dH_m = -d_{mIJ} F^{IJ} \,.$$

The deformed G^i

$$\begin{aligned} G^i &= dC^i + d^i{}_I{}^m F^I B_m - \tfrac{1}{3} d^i{}_I{}^m d_{mJK} A^I F^J A^K \,, \\ \delta_\sigma C^i &= d\sigma^i - d^i{}_I{}^m F^I \sigma_m + \tfrac{1}{3} d^i{}_I{}^m d_{mJK} (\sigma^I F^J A^K - A^I F^J \sigma^K) \,, \\ dG^i &= d^i{}_I{}^m F^I H_m \,, \end{aligned}$$

The 6-form potentials and their 7-form field strengths

Bandos,Ortín 2016,....

- The 6-form potentials are expected to be the duals of the scalars. However, maintaining the manifest invariances of the theory in the dualization procedure requires the introduction of as many 6-forms D_A as generators of global transformations δ_A leaving the equations of motion (not just the action) invariant. Hence, the index A labels the adjoint representation of the duality group.

- The 7-form field strengths K_A are the Hodge duals of the piece $j_A^{(\sigma)}(\phi)$ of the Noether–Gaillard–Zumino (NGZ) conserved 1-form currents $j_A = j_A^{(\sigma)}(\phi) + \Delta j_A$ associated to those symmetries (or, better, dualities) which only depend on the scalar fields

$$K_A \equiv \star j_A^{(\sigma)},$$

- The Bianchi identities:

$$dK_A = d\star j_A^{(\sigma)} = d\star(j_A^{NGZ} - \Delta j_A) = -d\star\Delta j_A\,,$$

- We can rewrite that equation locally as the conservation of the NGZ current

$$d\star j_A^{NGZ} = 0\,, \qquad j_A^{NGZ} \equiv j_A^{(\sigma)} + \Delta j_A\,,$$

where Δj_A is a very long and complicated expression. A local solution is provided by $\star[j_A^{(\sigma)} + \Delta j_A] = -dD_A$ for the 6-form potential D_A and we get the definiton of the 7-form field strength

$$\star j_A^{(\sigma)} = -dD_A + \star\Delta j_A \equiv K_A\,.$$

- Finally, the Bianchi identity becomes

$$dK_A = -d\star j_A^{(\sigma)} = T_A{}^I{}_J F^J \tilde{F}_I + T_A{}^m{}_n \tilde{H}^n H_m - \tfrac{1}{2} T_{Aij} G^{ij}\,.$$

Field strengths

$$
\begin{aligned}
F^I &= dA^I\,. \\
H_m &= dB_m - d_{mIJ}F^I A^J\,, \\
G^i &= dC^i + d^i{}_I{}^m F^I B_m - \tfrac{1}{3} d^i{}_I{}^m d_{mJK} A^I F^J A^K\,, \\
\tilde{H}^m &= d\tilde{B}^m + d^i{}_I{}^m C_i F^I + d^{mnp} B_n (H_p + \Delta H_p) + \tfrac{1}{12} d^i{}_I{}^m d_{iJ}{}^n A^{IJ} \Delta H_n\,, \\
\tilde{F}_I &= d\tilde{A}_I + 2 d_{mIJ} A^J (\tilde{H}_m - \tfrac{1}{2}\Delta\tilde{H}_m) - \left(d^i{}_I{}^m B_m - \tfrac{1}{3} d^i{}_J{}^m d_{mIK} A^{JK}\right)(G_i - \tfrac{1}{2}\Delta G_i) \\
&\quad - \tfrac{1}{3}\left(d^i{}_I{}^m d_{mJK} - d^i{}_K{}^m d_{mIJ}\right) F^J A^K C_i - d^{mnp} d_{mIJ} A^J B_n H_p \\
&\quad + \tfrac{1}{24}\left(d^i{}_K{}^m d_{iL}{}^n d_{mIJ} + 2 d^i{}_{[I|}{}^m d_{i|K]}{}^n d_{mJL}\right) F^J A^{KL} B_n + \tfrac{1}{24} d^i{}_J{}^m d_{iK}{}^n d_{mIL} A^{JKL} dB_n \\
&\quad - \tfrac{1}{180} d^i{}_L{}^n d_{iQ}{}^m d_{mIJ} d_{nPK} A^{JKLQ} F^P\,,
\end{aligned}
$$

Bianchi identities

$$
\begin{aligned}
dF^I &= 0\,, \\
dH_m &= -d_{mIJ}F^{IJ}\,, \\
dG^i &= d^i{}_I{}^m F^I H_m\,, \\
d\tilde{H}^m &= d^i{}_I{}^m G_i F^I + d^{mnp}H_{np}\,, \\
d\tilde{F}_I &= 2d_{mIJ}F^J\tilde{H}^m + d_{iI}{}^m G^i H_m\,, \\
dK_A &= T_A{}^I{}_J F^J\tilde{F}_I + T_A{}^m{}_n\tilde{H}^n H_m - \tfrac{1}{2}T_{Aij}G^{ij}\,.
\end{aligned}
$$

$$
\begin{aligned}
\star G^{i} &= \Omega^{ij}\mathscr{W}_{jk}G^{k}\,, \\
\star \tilde{H}^{m} &= \mathscr{M}^{mn}H_{n}\,, \\
\star \tilde{F}_{I} &= \mathscr{M}_{IJ}F^{J}\,, \\
\star K_{A} &= -j^{(\sigma)}_{A}\,, \\
\star L_{\sharp} &= -\frac{\partial V}{\partial c^{\sharp}}
\end{aligned}
$$

Gauging the global symmetries of the theory

The most general possibilities can be explored using the embedding tensor formalism

Cordaro, Fré,Gualtieri, Termonia and Trigiante (1998). Nicolai and Samtleben (2001).De Wit and Samtleben (2001). De Wit, Samteblen and Trigiante (2003)

Bonus

The tensor hierarchy

De Wit and Samtleben (2005). De Wit, Nicolai and Samteblen (2008). Bergshoeff, hartong, Hohm,Huubscher and Ortín (2009). De Wit and Zalk (2009)

- A convenient tool to study gaugings of supergravity theories in a universal and general way, that does not require a case-by-case analysis.
- Formally maintains covariance with respect to the global invariance group G of the ungauged theory, even though in general G will ultimately be broken by the gauging to the subgroup that is gauged.
- It turns out that all couplings that deform an ungauged supergravity into a gauged one, can be given in terms of the embedding tensor.
- Gauged supergravities are classified by the embedding tensor, subject to a number of algebraic or group-theoretical constraints.

- The embedding tensor $\Theta_M{}^\alpha$ pairs the generators t_α of the group G with the vector fields $A_\mu{}^M$ used for the gauging. The indices $\alpha, \beta, \ldots$ label the adjoint representation of G and the indices $M, N, \ldots$ label the representation $\mathscr{R}_V$ of G, in which the vector fields that will be used for the gauging transform. Thus, the choice of $\Theta_M{}^\alpha$, which generally will not have maximal rank, determines which combinations of vectors

$$A_\mu{}^M \Theta_M{}^\alpha \, ,$$

can be seen as the gauge fields associated to (a subset of) the generators t_α of the group G, and, simultaneously, or alternatively, which combinations of group generators

$$X_M \;=\; \Theta_M{}^\alpha \, t_\alpha$$

can be seen as the generators of the gauge group. Consequently, the embedding tensor can be used to define covariant derivatives

$$D_\mu \;=\; \partial_\mu - A_\mu{}^M \, \Theta_M{}^\alpha \, t_\alpha \;=\; \partial_\mu - A_\mu{}^M \, X_M \, ,$$

which shows that the embedding tensor can also be interpreted as a set of gauge coupling constants of the theory.

The EOMs invariant under a global symmetry group with infinitesimal generators $\{T_A\}$ satisfying the algebra

$$[T_A, T_B] = f_{AB}{}^{C} T_C \,.$$

The group acts linearly on all the forms of rank ≥ 1, if $(\mathrm{C}^i) = \begin{pmatrix} C^a \\ C_a \end{pmatrix}$.

The generators are: $\{T_A{}^I{}_J\}, \{T_A{}^m{}_n\}, \{T_A{}^i{}_j\}$
The adjoint generators are: $T_A{}^B{}_C = f_{AC}{}^B$.
The matrices $T_A{}^i{}_j$ are generators of the symplectic group

We have:

$$\begin{aligned}
\delta_\alpha A^I &= \alpha^A T_A{}^I{}_J A^J \,, \\
\delta_\alpha B_m &= -\alpha^A T_A{}^n{}_m B_n \,, \\
\delta_\alpha C^i &= \alpha^A T_A{}^i{}_j C^j \,, \\
\delta_\alpha \tilde{A}_I &= -\alpha^A T_A{}^J{}_I \tilde{A}_J \,, \\
\delta_\alpha \tilde{B}^m &= \alpha^A T_A{}^m{}_n \tilde{B}^n \,,
\end{aligned}$$

$$\begin{aligned}
\delta_A \mathscr{M}_{IJ} &= -2 T_A{}^K{}_{(I} \mathscr{M}_{J)K} \,, \\
\delta_A \mathscr{M}^{mn} &= 2 T_A{}^{(m}{}_p \mathscr{M}^{n)p} \,, \\
\delta_A \mathscr{W}_{ij} &= -2 T_A{}^k{}_{(i} \mathscr{W}_{j)k} \,,
\end{aligned}$$

- The k-form field strengths will transform in the same representation as the corresponding $(k-1)$-form potential, but only if the d-tensors $d_{mIJ}, d^i{}_I{}^m, d^{mnp}$ are invariant under the global symmetry group, *i.e.* they must satisfy

$$\begin{aligned} \delta_A d_{mIJ} &= -T_A{}^n{}_m d_{nIJ} - 2T_A{}^K{}_{(I} d_{n|J)K} = 0\,, \\ \delta_A d^i{}_I{}^m &= T_A{}^i{}_j d^j{}_I{}^m - T_A{}^J{}_I d^i{}_J{}^m + T_A{}^m{}_n d^i{}_I{}^n = 0\,, \\ \delta_A d^{mnp} &= 3T_{Aq}^{[m|} d^{q|np]} = 0\,. \end{aligned}$$

- The theories we have constructed are invariant under Abelian gauge transformations with 0-, 1- and 2-form parameters $\sigma^I, \sigma_m, \sigma^i$:

$$\delta_\sigma A^I \sim d\sigma^I\,, \qquad \delta_\sigma B_m \sim d\sigma_m\,, \qquad \delta_\sigma C^i \sim d\sigma^i\,.$$

We promote the global parameters α^A to local ones $\alpha^A(x)$ and we make the identification:

The embedding tensor

$$\alpha^A \equiv \sigma^I \vartheta_I{}^A .$$

The gauge transformation for the the kinetic matrices $\mathscr{M}_{IJ}, \mathscr{M}^{mn}, \mathscr{W}_{ij}$ become:

$$\delta_\sigma \mathscr{M}_{IJ} = -2\sigma^L X_L{}^K{}_{(I}\mathscr{M}_{J)K}\,, \qquad \delta_\sigma \mathscr{M}^{mn} = 2\sigma^I T_I{}^{(m}{}_p \mathscr{M}^{n)p}\,, \qquad \delta_\sigma \mathscr{W}_{ij} = -2\sigma^I X_I{}^k{}_{(i}\mathscr{W}_{j)k}\,,$$

where:

$$X_I{}^J{}_K \equiv \vartheta_I{}^A T_A{}^J{}_K\,,\; X_I{}^m{}_n \equiv \vartheta_I{}^A T_A{}^m{}_n\,,\; X_I{}^i{}_j \equiv \vartheta_I{}^A T_A{}^i{}_j\,.$$

The embedding tensor and the $1-\mathit{forms}$

The gauge fields for these symmetries are given by

$$A^A \equiv A^I \vartheta_I{}^A \,.$$

The covariant derivative of a field Φ transforming as $\delta_A \Phi$ is given by

$$\mathscr{D}\Phi \equiv d\Phi - A^A \delta_A \Phi \,.$$

Then

$$\begin{aligned}
\mathscr{D}\mathscr{M}^{mn} &= d\mathscr{M}^{mn} - 2A^I X_I{}^{(m}{}_p \mathscr{M}^{n)p} \,, \\
\mathscr{D}\mathscr{M}_{IJ} &= d\mathscr{M}_{IJ} + 2A^L X_L{}^K{}_{(I}\mathscr{M}_{J)K} \,, \\
\mathscr{D}\mathscr{W}_{ij} &= d\mathscr{W}_{ij} + 2A^I X_I{}^k{}_{(i}\mathscr{W}_{j)k} \,.
\end{aligned}$$

The derivatives transform covariantly under gauge transformations $\delta_\sigma = \sigma^I \vartheta_I{}^A \delta_A$ provided that the embedding tensor is gauge-invariant

$$\delta_\sigma \vartheta_I{}^A = 0\,,$$

and provided that the 1-forms transform as

$$\delta_\sigma A^I = \mathscr{D}\sigma^I + \Delta A^I\,, \quad \text{where} \quad \left\{ \begin{array}{rcl} \Delta A^I \vartheta_I{}^A & = & 0\,, \\ \mathscr{D}\sigma^I & = & d\sigma^I - A^J X_J{}^I{}_K \sigma^K\,, \end{array} \right.$$

The gauge invariance of the embedding tensor leads to the so-called *quadratic constraint*

$$\vartheta_J{}^B \left[T_B{}^K{}_I \vartheta_K{}^A - f_{BC}{}^A \vartheta_I{}^C \right] = 0\,.$$

To determine ΔA^I we have to construct the gauge-covariant 2-form field strengths F^I.

2-form field strengths

Use Ricci identities

$$\mathscr{D}\mathscr{D}\mathscr{M}_{mn} = -F^I \vartheta_I{}^A \delta_A \mathscr{M}_{mn}\,,$$

$$\begin{aligned}
F^I &= dA^I - \tfrac{1}{2} X_J{}^I{}_K A^{JK} + Z^{I\,m} B_m\,, \\
\delta_\sigma F^I &= \sigma^J X_J{}^I{}_K F^K\,, \\
\delta_\sigma A^I &= \mathscr{D}\sigma^I - Z^{I\,m}\sigma_m\,, \\
\delta_\sigma B_m &= -\sigma^I X_I{}^n{}_m B_n + \mathscr{D}\sigma_m + 2d_{mJK}\left(F^J\sigma^K - \tfrac{1}{2}A^J\delta_\sigma A^K\right) + \Delta B_m\,,
\end{aligned}$$

with $Z^{I\,m}\Delta B_m = 0$.
In the ungauged limit $\vartheta_I{}^A = Z^{I\,m} = 0$ we get the Abelian gauge transformations.

Field Strengths

$$
\begin{aligned}
F^I &= dA^I - \tfrac{1}{2} X_J{}^I{}_K A^{JK} + Z^{I\,m} B_m\,,\\
H_m &= \mathscr{D} B_m - d_{mIJ} dA^I A^J + \tfrac{1}{3} X_J{}^M{}_K A^{IJK} + Z_{im} C^i\,,\\
G^i &= \mathscr{D} C^i + d^i{}_I{}^n \left[F^I B_n - \tfrac{1}{2} Z^{I\,p} B_n B_p + \tfrac{1}{3} d_{nJK} dA^J A^{KI} + \tfrac{1}{12} d_{mMJ} X_K{}^M{}_L A^{IJKL} \right] - Z_{im} \tilde{H}^m\\
\tilde{H}^m &= \mathscr{D} \tilde{B}^m - d_{iI}{}^m F^I C^i + d^{mnp} B_n \left(H_p + \Delta H_p - 2 Z_{ip} C^i \right)\\
&\quad + d^m{}_{IJK} dA^I dA^J A^K\\
&\quad + \left(\tfrac{1}{12} d_{iJ}{}^m d^j{}_K{}^n d_{nIL} - \tfrac{3}{4} d^m{}_{IJM} X_K{}^M{}_L \right) dA^I A^{JKL}\\
&\quad + \left(\tfrac{3}{20} d^m{}_{NPM} X_I{}^N{}_J - \tfrac{1}{60} d_{iM}{}^m d_I^{i\,n} d_{nPJ} \right) X_K{}^P{}_L A^{IJKLM}\\
&\quad + Z^{Im} \tilde{A}_I\,,
\end{aligned}
$$

Bianchi Identities

$$
\begin{aligned}
\mathscr{D}F^I &= Z^{I\,m}H_m\,,\\
\mathscr{D}H_m &= -d_{mIJ}F^{IJ}+Z_{im}G^i\,,\\
\mathscr{D}G^i &= d^i{}_I{}^m F^I H_m - Z^i{}_m\tilde{H}^m\,,\\
\mathscr{D}\tilde{H}^m &= -d_{iI}{}^m G^i F^I + d^{mnp}H_{np} + d^m{}_{IJK}F^{IJK} + Z^{I\,m}\tilde{F}_I\,,\\
\mathscr{D}\tilde{F}_I &= 2d_{mIJ}F^J\tilde{H}^m + d_{iI}{}^m G^i H_m - 3d^m{}_{IJK}F^{JK}H_m - \vartheta_I{}^A K_A\,,\\
\mathscr{D}K_A &= +T_A{}^K{}_J F^J\tilde{F}_K + T_A{}^m{}_n\tilde{H}^n H_m - \tfrac{1}{2}T_{A\,ij}G^{ij} + Y_{AI}{}^B L_B{}^I + Y_A{}^{Im}L_{Im} + Y_{Aim}L^{im}\,,\\
\mathscr{D}L_\sharp &= ?
\end{aligned}
$$

Constraints

- First of all we have the gauge-invariance constraints

$$\mathscr{Q}_{IJ}{}^{A}\,,\ \mathscr{Q}_{I}{}^{Jm}\,,\ \mathscr{Q}_{Iim}\,,$$

- Secondly, we have the global-invariance constraints

$$\mathscr{Q}_{AmIJ}\,,\ \mathscr{Q}_{A}{}^{i}{}_{I}{}^{m}\,,$$

- Thridly we have the orthogonality constraints between the three deformation tensors

$$\begin{aligned}
\mathscr{Q}^{mA} &\equiv -Z^{Im}\vartheta_{I}{}^{A}\,,\\
\mathscr{Q}_{i}{}^{I} &\equiv Z_{im}Z^{Im}\,,\\
\mathscr{Q}_{mn} &\equiv Z_{im}Z^{i}{}_{n}\,.
\end{aligned}$$

- Next, we have the constraints relating the gauge transformations to the d-tensors

$$
\begin{aligned}
\mathcal{Q}_I{}^J{}_K &\equiv X_{(I}{}^J{}_{K)} - Z^{Km} d_{mIJ}\,, \\
\mathcal{Q}_I{}^m{}_n &\equiv X_I{}^m{}_n + 2 d_{mIJ} Z^{Jn} + Z_{im} d^i{}_I{}^m\,, \\
\mathcal{Q}_{Iij} &\equiv -X_{Iij} - 2 Z_{(i|m} d_{|j)I}{}^m\,,
\end{aligned}
$$

- Finally, we have the constraints that related the d-tensors amongst them

$$
\begin{aligned}
\mathcal{Q}^{imn} &\equiv d^i{}_I{}^{[m|} Z^{I|n]} + Z^i{}_p d^{pmn}\,, \\
\mathcal{Q}_{IJ}{}^{mn} &\equiv \tfrac{1}{2} d^i{}_{(I|}{}^m d_{i|J)}{}^n + d^{mnp} d_{pIJ} + 3 d^{[m|}{}_{IJK} Z^{K|n]}\,, \\
\mathcal{Q}_{iIJK} &\equiv Z_{im} d^m{}_{IJK} - d_{i(I|}{}^m d_{m|JK)}\,.
\end{aligned}
$$

We get equations which guarantee the consistency of the whole construction of the tensor hierarchy that we have carried out.

$$\begin{aligned}&\frac{\partial \mathcal{Q}_{IJ}{}^{A}}{\partial \vartheta_{K}{}^{B}}\left[\mathscr{D}M^{IJ}{}_{A}+F^{I}L_{A}{}^{J}\right]+\frac{\partial \mathcal{Q}^{mA}}{\partial \vartheta_{K}{}^{B}}\left[\mathscr{D}M_{mA}+H_{m}K_{A}\right]\\ &+\frac{\partial \mathcal{Q}_{I}{}^{J}{}_{K}}{\partial \vartheta_{K}{}^{B}}\left[\mathscr{D}M^{I}{}_{J}{}^{K}+F^{IK}\tilde{F}_{J}\right]+\frac{\partial \mathcal{Q}_{I}{}^{m}{}_{n}}{\partial \vartheta_{K}{}^{B}}\left[\mathscr{D}M^{I}{}_{m}{}^{n}+F^{I}\tilde{H}^{m}H_{n}\right]\\ &+\frac{\partial \mathcal{Q}_{Iij}}{\partial \vartheta_{K}{}^{B}}\left[\mathscr{D}M^{Iij}+F^{I}G^{ij}\right] \quad = \quad 0\,.\end{aligned}$$

Noether identities.

$$-Z^{i}{}_{m}\frac{\delta S}{\delta C^{i}}+\mathscr{D}\frac{\delta S}{\delta \tilde{B}_{m}}=0$$

$$-\mathscr{Z}_{im}\frac{\delta S}{\delta B_{m}}+\mathscr{D}\frac{\delta S}{\delta C^{i}}=0$$

$$D\frac{\delta S}{\delta B_{n}}-\frac{\delta S}{\delta C^{j}}d^{i}{}_{I}{}^{n}F^{I}+Z^{In}B(\tilde{A}_{I})=0$$

$$-\frac{\delta S}{\delta C^{j}}d^{i}{}_{I}{}^{n}H_{n}+\frac{\delta S}{\delta B_{m}}2d_{mJI}F^{J}+\mathscr{D}\mathscr{B}(\tilde{A}_{I})+\upsilon_{I}^{A}\mathscr{B}(\phi^{x})=0$$

where

$$B(\tilde{A}_{I})=\frac{\delta S}{\delta A^{I}}-\frac{\delta S}{\delta B_{m}}A^{K}d_{mKI}+\frac{\delta S}{\delta C^{j}}(-d^{i}_{I}\;{}^{n}B_{n}+\frac{1}{3}d^{i}{}_{J}{}^{n}d_{nIK}A^{JK})$$

$$\upsilon_{I}^{A}\mathscr{B}(\phi^{x})=\upsilon_{I}^{A}(\frac{\delta S}{\delta \phi^{x}}k_{A}{}^{x}(\phi)-\frac{\delta S}{\delta C^{j}}T_{A}{}^{j}{}_{i}C_{j}-\frac{\delta S}{\delta B_{m}}T_{I}{}^{n}{}_{m}B_{n})$$

$$\frac{\delta S}{\delta B_m} = \mathscr{B}(B_m)$$
$$\frac{\delta S}{\delta C^i} = \mathscr{B}(C_i)$$

$$\frac{\delta S}{\delta A^I} = B(\tilde{A}_I) + \mathscr{B}(B_m)A^K d_{mKI} - \mathscr{B}(C_i)(-d^i_I{}^n B_n + \frac{1}{3}d^i{}_J{}^n d_{nIK}A^{JK})$$

$$\upsilon_I^A k_A{}^x(\phi)\frac{\delta S}{\delta \phi^x} = \upsilon_I^A(\mathscr{B}(\phi^x) + \mathscr{B}(C_i)T_A{}^j{}_i C_j + \mathscr{B}(B_m)T_I{}^n{}_m B_n)$$

where

$$\upsilon_I^A \mathscr{B}(\phi^x) = \upsilon_I^A(\mathscr{D}K_A + T_A{}^I{}_J F^J \tilde{F}_I + T_A{}^m{}_n \tilde{H}^n H_m - \frac{1}{2}T_{A_{ij}}G^{ij})$$

$$B(\tilde{A}_I) = -\mathscr{D}\tilde{F}_I + 2d_{mIJ}F^J\tilde{H}^m + d_i{}^m{}_I G^i H_m - 3d^m{}_{IJK}F^{JK}H_m + \upsilon_I^A K_A$$
$$\mathscr{B}(B_m) = -\mathscr{D}\tilde{H}^m - d_j{}^m{}_I G^j F^I + d^{mnp}H_{np} + d^m{}_{IJK}F^{IJK} + Z^{\mathrm{Im}}\tilde{F}_I$$
$$\mathscr{B}(C_i) = -\mathscr{D}G_i + d_i{}^m{}_I F^I H_m - Z_{im}\tilde{H}^m$$

- All these theories will be equivalent from an 8-dimensional point of view: they are all related by $SL(2,\mathbb{R})$ duality transformations that can be understood as a different changes of variables. Dibietto, Fernández-Melgarejo, Marquéz, Roest (2012).

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- The simplest mechanical procedure to obtain them from 11-dimensional supergravity would be to perform the standard Scherk-Schwarz reduction that gives an 8-dimensional SO(3)-gauged maximal supergravity in which the 3 Kaluza-Klein vectors play the role of gauge fields and then perform the $SL(2,\mathbb{R})$ duality transformations mentioned above. Salam and Sezgin (1985).

The 8-dimensional supergravities with SO(3) gaugings

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Problem!

Technically complicated (because of electric-magnetic duality). The Kaluza-Klein triplet of vector fields are the first component of a $SL(2,\mathbb{R})$ doublet and, after the duality transformations, the gauge fields are no longer the first component of that doublet, but a general linear combination of the first and the second.

- The 8-dimensional $SL(2,\mathbb{R})$ duality transformations have no clear 11-dimensional counterpart, though, and the SO(3)-gauged maximal supergravities in which the triplet of gauge fields are not the first component of the $SL(2,\mathbb{R})$ doublet obtained in this way cannot be uplifted to 11 dimensions.

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The only way to uplift an 8-dimensional solution of these theories would be to undo first the $SL(2,\mathbb{R})$ rotation converting the solution in a solution of the Salam-Sezgin theory.

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- One exception: the SO(3)-gauged 8-dimensional maximal supergravity in which the triplet of gauge fields are the second component of the $SL(2,\mathbb{R})$ doublet (they are, precisely, those coming from the reduction of the 11-dimensional 3-form). Alonso-Alberca, et al.(2001)

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- Maybe find more general non-covariant deformations of 11-dimensional supergravity leading to the rest of SO(3)-gauged 8-dimensional maximal supergravities.

- $\mathcal{N} = 2,\ d = 8$ supergravity can be obtained by direct dimensional reduction of 11-dimensional supergravity on T^3.

$\mathcal{N} = 2$, $d = 8$ supergravity

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- The scalars of the theory parametrize the coset spaces $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ and $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ and the U-duality group of the theory is $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$ and its fields are either invariant or transform in the fundamental representations of both groups.

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- We use the indices $i, j, k = 1, 2$ for $\mathrm{SL}(2,\mathbb{R})$ doublets and $m, n, p = 1, 2, 3$ for $\mathrm{SL}(3,\mathbb{R})$ triplets.

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- We use the indices $i,j,k=1,2$ for $\mathrm{SL}(2,\mathbb{R})$ doublets and $m,n,p=1,2,3$ for $\mathrm{SL}(3,\mathbb{R})$ triplets.

- The bosonic fields are

$$g_{\mu\nu}, C, B_m, A^{i\,m}, a, \varphi, \mathcal{M}_{mn},$$

where C is a 3-form, B_m a triplet of 2-forms, $A^{i\,m}$, a doublet of triplets of 1-forms (six in total), a and φ are the axion and dilaton fields which can be combined into the axidilaton field

$$\tau\equiv a+ie^{-\varphi},$$

or into the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ symmetric matrix

$$\left(\mathscr{W}_{ij}\right) \equiv e^{\varphi} \begin{pmatrix} |\tau|^2 & a \\ a & 1 \end{pmatrix}, \quad \text{with inverse} \quad \left(\mathscr{W}^{ij}\right) \equiv e^{\varphi} \begin{pmatrix} 1 & -a \\ -a & |\tau|^2 \end{pmatrix},$$

- $\mathscr{M}_{mn}$ is an $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ symmetric parametrized in terms of the scalars.

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- $\mathscr{M}_{mn}$ is an $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ symmetric parametrized in terms of the scalars.
- The bosonic action, Alonso-Alberca et. al., is

$$\begin{aligned} S \quad = \quad & \int d^8x\sqrt{|g|}\left\{R + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathscr{M}\mathscr{M}^{-1}\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathscr{W}\mathscr{W}^{-1}\right)^2\right. \\ & -\tfrac{1}{4}F^{i\,m}\mathscr{M}_{mn}\mathscr{W}_{ij}F^{j\,n} + \tfrac{1}{2\cdot 3!}H_m\mathscr{M}^{mn}H_n - \tfrac{1}{2\cdot 4!}e^{-\varphi}G^2, \\ & -\tfrac{1}{6^3\cdot 2^4}\frac{1}{\sqrt{|g|}}\,\varepsilon\left[GGa - 8GH_mA^{2\,m} + 12G(F^{2\,m} + aF^{1\,m})B_m\right. \\ & \left.\left. -8\varepsilon^{mnp}H_mH_nB_p - 8G\partial aC - 16H_m(F^{2\,m} + aF^{1\,m})C\right]\right\}, \end{aligned}$$

where the field strengths are given by

$$
\begin{aligned}
F^{i\,m} &= 2\partial A^{i\,m} . \\
H_m &= 3\partial B_m + 3\varepsilon_{mnp} F^{1\,n} A^{2\,p} , \\
G &= 4\partial C + 6 F^{1\,m} B_m ,
\end{aligned}
$$

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- We want to use differential-form language and a different basis of forms with better transformation properties under the duality groups (in particular, under $\mathrm{SL}(2,\mathbb{R})$)
- We redefine the potentials as

$$\begin{aligned}
B_m &\longrightarrow B_m - \tfrac{1}{2}\varepsilon_{mnp} A^{1n}\wedge A^{2p}\,, \\
C &\longrightarrow C^1 + \tfrac{1}{2}\varepsilon_{mnp} A^{1m}\wedge A^{1n}\wedge A^{2p}\,.
\end{aligned}$$

The bosonic action becomes

$$
\begin{aligned}
S \quad = \quad & \int \Big\{ -\star R + \tfrac{1}{4}\mathrm{Tr}\left(d\mathscr{M}\mathscr{M}^{-1} \wedge \star d\mathscr{M}\mathscr{M}^{-1}\right) + \tfrac{1}{4}\mathrm{Tr}\left(d\mathscr{W}\mathscr{W}^{-1} \wedge \star d\mathscr{W}\mathscr{W}^{-1}\right) \\
& + \tfrac{1}{2}\mathscr{W}_{ij}\mathscr{M}_{mn}F^{im} \wedge \star F^{jn} + \tfrac{1}{2}\mathscr{M}^{mn}H_m \wedge \star H_n + \tfrac{1}{2}e^{-\varphi}G^1 \wedge \star G^1 - \tfrac{1}{2}aG^1G^1 \\
& + \tfrac{1}{3}G^1\left[H_m A^{2m} - B_m F^{2m} + \tfrac{1}{2}\varepsilon_{mnp}F^{2m}A^{1n}A^{2p}\right] \\
& + \tfrac{1}{3}H_m F^{2m}\left[C^1 + \tfrac{1}{6}\varepsilon_{mnp}A^{1m}A^{1n}A^{2p}\right] \\
& + \tfrac{1}{3!}\varepsilon^{mnp}H_m H_n\left(B_p - \tfrac{1}{2}\varepsilon_{pqr}A^{1q}A^{2r}\right)\Big\} \,.
\end{aligned}
$$

and the field strengths

$$
\begin{aligned}
F^{im} &= dA^{im}\,, \\
H_m &= dB_m + \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{mnp}F^{in}A^{jp}\,, \\
G^1 &= dC^1 + F^{1m}B_m + \tfrac{1}{6}\varepsilon_{ij}\varepsilon_{mnp}A^{1m}F^{in}A^{jp}\,.
\end{aligned}
$$

- The only structure constants that we need to know explicitly are those of the $SO(3)$ subgroup

$$[T_m, T_n] = f_{mn}{}^p T_p = -\varepsilon_{mn}{}^p T_p\,,$$

- The SO(3) generators in the fundamental/adjoint representation are the matrices

$$T_m{}^n{}_p = \varepsilon_m{}^n{}_p = -\varepsilon_{mpn}\,.$$

- The coset space SL(3,$\mathbb{R}$)/SO(3) is a symmetric space and the structure constants with mixed indices $f_{ma}{}^b$ provide a representation of SO(3) acting on the SL(3,$\mathbb{R}$)/SO(3) indices $a, b, \cdots$:

$$T_m{}^a{}_b = f_{mb}{}^a\,.$$

- For the generators of SL(2,$\mathbb{R}$) $\sim$Sp(2,$\mathbb{R}$) in the fundamental representation $T_\alpha{}^i{}_j$ we just need to know the property

$$T_\alpha{}^k{}_{[j}\varepsilon_{i]k} \equiv T_{\alpha\,[ij]} = 0\,,$$

- The indices $I, J, \ldots$ must be replaced by composite indices im, jn etc. where $i, j, \ldots = 1, 2$ and $m, n, \ldots = 1, 2, 3$ are indices in the fundamental representations of $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(3, \mathbb{R})$, respectively.

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- The electric 3-forms carry an index a which is the upper component of a symplectic index denoted by $i, j, \ldots$. In the case at hands, a takes only one value: 1 (C^1) which will be sometimes omitted (C). The lower index 1 is equivalent to an upper index 2: $C_1 = \varepsilon_{12} C^2 = C^2$ and, therefore $(C^i) = \left(\begin{smallmatrix} C^1 \\ C_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} C^1 \\ C^2 \end{smallmatrix}\right)$. On the other hand, $C_i \equiv \varepsilon_{ij} C^j$.

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- Comparing the field strengths of this theory with those of the generic ungauged theory we get that the d-tensors can be constructed entirely in terms of the U-duality invariant tensors $\delta^i{}_j, \varepsilon_{ij}, \delta^m{}_n, \varepsilon_{mnp}$:

$$
\begin{aligned}
d_{mIJ} \quad &\rightarrow \quad d_{m\,in\,jp} = -\tfrac{1}{2} \varepsilon_{mnp} \varepsilon_{ij} \,, \\
d^i{}_I{}^m \quad &\rightarrow \quad d^i{}_{jn}{}^m = \delta^i{}_j \delta^m{}_n \,.
\end{aligned}
$$

Moreover

$$
d^i{}_{(I|}{}^m d_{i|J)}{}^n = -2 d^{mnp} d_{pIJ} \,, \quad \Rightarrow \quad d^{mnp} = +\tfrac{1}{2} \varepsilon^{mnp} \,.
$$

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- From the 8-dimensional supergravity point of view, one could use any other $\mathrm{SL}(2,\mathbb{R})$ transformed of the A^{1m} triplet as gauge fields.The corresponding embedding tensor has the form

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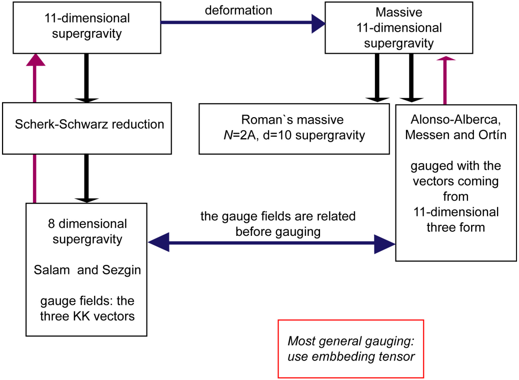
- The SO(3) gauge fields are combinations of the two triplets of vector fields

$$\vartheta_{in}{}^{m}A^{in} = v_iA^{im}\,,$$

and include, as limiting cases, the SS and the AAMO theories.

From 11-dimensional to 8-dimensional supergravity



Salam and Sezgin, (1985),
Alonso-Alberca,Messen, Ortín, (2001),
Alonso-Alberca, Bergshoeff, Gran, Linares, Ortín, Roest, (2003),
Puigdomènech, de Roo, (2008).

Solving the constraints

- The two first constraints

$$\vartheta_{im}{}^{B} Y_{B\,jn}{}^{A} = 0\,,$$

$$\vartheta_{(im|}{}^{A} T_{A}{}^{jn}{}_{|kp)} = Z^{jnq} d_{q\,im\,jp}\,.$$

- There are five constraints more relating the three deformation tensors $\vartheta_{im}{}^{A}, Z^{imn}$ and Z_{im} among themselves and to the d-tensors

$$\begin{aligned}
\vartheta_{im}{}^{A} T_{A}{}^{p}{}_{n} + 2 d_{n\,im\,jq} Z^{jqp} + Z_{jn} d^{j}{}_{im}{}^{p} &= 0\,,\\
\vartheta_{im}{}^{A} T_{A\,jk} + 2 Z_{(j|n} d_{|k)\,im}{}^{n} &= 0\,,\\
d^{i}{}_{jp}{}^{[m|} Z^{jp|n]} + Z^{i}{}_{p} d^{pmn} &= 0\,,\\
\tfrac{1}{2} d^{k}{}_{(ip|}{}^{m} d_{k|jq)}{}^{n} + d^{mnp} d_{p\,ip\,jq} + 3 d^{[m|}{}_{ip\,jq\,lr} Z^{lr|n]} &= 0\,,\\
Z_{im} d^{m}{}_{jn\,kp\,lq} - d_{i(jn|}{}^{m} d_{m|kp\,lq)} &= 0\,,
\end{aligned}$$

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\end{aligned}$$
We have found a set of deformation parameters which are a solution for all contraints
$$\vartheta_{im}{}^{n} = v_i \delta_m{}^{n}\,, \quad Z^{imn} = v^{i} \delta^{mn}\,, \quad Z_{im} = 0$$

- The tensor hierarchy
- From Noether identities

$$
\begin{aligned}
k_A{}^x \frac{\delta S}{\delta \phi^x} &= \mathscr{B}(K_A)\,, \qquad\qquad A = m, a, \alpha\,, \\
\frac{\delta S}{\delta A^{im}} &= \mathscr{B}(\tilde{F}_{im}) + \left(\delta^1{}_i B_m - \tfrac{1}{6}\varepsilon_{mnp} A^{1n} A^{jp}\right) \mathscr{B}(G^2) - \tfrac{1}{2}\varepsilon_{mnp}\varepsilon_{ij} A^{jn} \mathscr{B}(\tilde{H}^p)\,, \\
\frac{\delta S}{\delta B_m} &= \mathscr{B}(\tilde{H}^m)\,, \\
\frac{\delta S}{\delta C^1} &= \mathscr{B}(G^2) \\
&.
\end{aligned}
$$

We find

$$\frac{\delta S}{\delta A^{im}} = -\mathscr{D}(\mathscr{W}_{ij}\mathscr{M}_{mn}\star F^{jn}) - \varepsilon_{mnp}\varepsilon_{ij}F^{jn}\mathscr{M}^{pq}\star H_q - (\delta_i{}^1\tilde{G} - \delta_i{}^2 G)H_m$$

$$-v_i K_m + \left(\delta^1{}_i B_m - \tfrac{1}{6}\varepsilon_{mnp}A^{1n}A^{jp}\right)\frac{\delta S}{\delta C} - \tfrac{1}{2}\varepsilon_{mnp}\varepsilon_{ij}A^{jn}\frac{\delta S}{\delta B_p},$$

$$\frac{\delta S}{\delta B_m} = -\mathscr{D}(\mathscr{M}^{mn}\star H_n) + F^{1m}\tilde{G} - F^{2m}G + \tfrac{1}{2}\varepsilon^{mnp}H_nH_p + v^i\mathscr{W}_{ij}\mathscr{M}_{mn}\star F^{jn},$$

$$\frac{\delta S}{\delta C} = -d\tilde{G} + F^{2m}H_m.$$

the scalar equations of motion are

$$\frac{\delta S}{\delta \phi^y} = -\mathscr{G}_{xy}\mathscr{D}\star\mathscr{D}\phi^y + \tfrac{1}{2}\partial_x\left\{\mathscr{W}_{ij}\mathscr{M}_{mn}F^{im}\wedge\star F^{jn} + \mathscr{M}^{mn}H_m\wedge\star H_n\right.$$

$$\left. + e^{-\varphi}G\wedge\star G - aG\wedge G - V(\phi)\right\}.$$

- The kinetic terms in the action

$$S^{(0)} = \int \Big\{ -\star R + \tfrac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1} \wedge \star \mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right) + \tfrac{1}{4}\mathrm{Tr}\left(d\mathcal{W}\mathcal{W}^{-1} \wedge \star d\mathcal{W}\mathcal{W}^{-1}\right)$$
$$+\tfrac{1}{2}\mathcal{W}_{ij}\mathcal{M}_{mn}F^{im} \wedge \star F^{jn} + \tfrac{1}{2}\mathcal{M}^{mn}H_m \wedge \star H_n + \tfrac{1}{2}e^{-\varphi}G \wedge \star G - \tfrac{1}{2}aG \wedge G - V \Big\}\,.$$

- We add

$$S^{(1)} = \int \Big\{ -dC^1 \Delta G^2 - \tfrac{1}{2}\Delta G^1 \Delta G^2 - \tfrac{1}{12}\varepsilon^{mnp}B_m \mathcal{D}B_n \mathcal{D}B_p + \tfrac{1}{4}\varepsilon^{mnp}B_m H_n H_p$$
$$-\tfrac{1}{24}\varepsilon_{ij}A^{im}A^{jn}\Delta H_m \mathcal{D}B_n \Big\}\,,$$

- Another correction

$$S^{(2)} = \int \Big\{ -\tfrac{1}{12}v_i(F^{im} - v^i B_m)B_m B_n B_n + \tfrac{1}{4}\varepsilon^{mnp}B_m \Delta H_n \Delta H_p - \tfrac{1}{2}\varepsilon_{ij}\Box G^i \Box F^{jm} B_m$$
$$+\tfrac{1}{24}\varepsilon_{ij}A^{im}A^{in}\mathcal{D}B_m \Delta H_n \Big\}\,.$$

- The scalar potential must satisfy:

$$k_A{}^x \frac{\partial V}{\partial \phi^x} = Y_A{}^\sharp \frac{\partial V}{\partial c^\sharp},$$

where the index $\sharp$ labels the deformations $c^\sharp$, which, in this case, are just $\vartheta_{im}{}^A, Z^{imn}$ and Z_{im}.

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- The fermion shifts of SO(3)-gauged $\mathcal{N} = 2, d = 8$ supergravity theory can be written in terms of

$$f_{\mathbf{mn}}{}^{\mathbf{p}} \equiv L_{\mathbf{m}}{}^m L_{\mathbf{n}}{}^n L_p{}^{\mathbf{p}} f_{mn}{}^p \,,$$

where $f_{mn}{}^p = \varepsilon_{mnp}$, the matrix $L_{\mathbf{m}}{}^n$ is the SL$(3,\mathbb{R})/$SO(3) coset representative, and $L_m{}^{\mathbf{n}}$is its inverse.

- The structure of the fermion shifts and of the entire supersymmetry transformations in SS theory does not show the transformation properties of the spinors under the R-symmetry group $SO(2)\times SO(3) \sim U(1)\times SU(2)$, because the fermions obtained in the dimensional reduction from 11 dimensions are not symplectic-Majorana.

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$$\begin{aligned} \delta_\varepsilon \psi_{\mu I} &\sim \cdots + S_{IJ}\varepsilon^J\,, \\ \delta_\varepsilon \lambda^I_{\mathbf{m}} &\sim \cdots + N_{\mathbf{m}}{}^I{}_J \varepsilon^J\,. \end{aligned}$$

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- We propose $\mathscr{V}^M{}_{IJ}$ where the index M labels the vectors available in the theory (electric and magnetic in 4 dimensions) and the indices I, J are R-symmetry indices in the representation carried by the spinors (the fundamental of SU()$\mathscr{N}$) to be written as

$$\mathscr{V}^{im}{}_{IJ} \equiv V^i L_{\mathbf{m}}{}^m \varepsilon_{IK} \sigma^{\mathbf{m}\,K}{}_J \,, \;\; \text{and} \;\; \mathscr{V}^{im}{}_{\mathbf{m}} \equiv V^i L_{\mathbf{m}}{}^m \,,$$

where we have introduced

$$(V_i) \equiv e^{\varphi/2}(\tau \;\; 1) \,,$$

which transforms linearly under SL(2,$\mathbb{R}$) up to a U(1) phase.

- The fermion shifts are

$$
\begin{aligned}
S_{IJ} &= \mathscr{V}^{im}{}_{[I|K}\vartheta_{im}{}^{n}P_{n}{}^{\mathbf{p}}(\sigma^{\mathbf{p}})^{K}{}_{|J]}\,,\\
N_{\mathbf{m}}{}^{I}{}_{J} &= \mathscr{V}^{in}{}_{\mathbf{r}}\vartheta_{in}{}^{p}P_{p}{}^{\mathbf{s}}\left(\delta^{\mathbf{r}}{}_{\mathbf{m}}\delta^{\mathbf{q}}{}_{\mathbf{s}}-\tfrac{1}{2}\delta_{\mathbf{m}}{}^{\mathbf{q}}\delta^{\mathbf{r}}{}_{\mathbf{s}}\right)(\sigma^{\mathbf{q}})^{I}{}_{J}\,,
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where the $(\sigma^{\mathbf{p}})$ are Pauli's sigma matrices.

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- For the class of gaugings that we are considering, with embedding tensor $\vartheta_{im}{}^{n} = v_i\delta_m{}^{n}$

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\end{aligned}
$$

- The dressed structure constants can be expressed in these two different ways:

$$
f_{\mathbf{mn}}{}^{\mathbf{p}}=\begin{cases} L_{\mathbf{m}}{}^{q}\Gamma_{\mathrm{Adj}}(L^{-1})_{q}{}^{A}(T_{A})_{\mathbf{n}}{}^{\mathbf{p}}\,,\\ \varepsilon_{\mathbf{mnq}}T^{\mathbf{qp}}\,,\end{cases}
$$

where we have defined $T^{\mathbf{mn}}\equiv L_{p}{}^{\mathbf{m}}L_{p}{}^{\mathbf{n}}\Gamma$

- We can express the fermion shifts entirely in terms of $T^{\mathbf{mn}}$:

$$
\begin{aligned}
S_{IJ} &= \varepsilon_{IJ} V^i v_i T\,, \\
N_{\mathbf{m}}{}^I{}_J &= V^i v_i \left(T^{\mathbf{mp}} - \tfrac{1}{2}\delta_{\mathbf{mp}} T\right) (\sigma^{\mathbf{p}})^I{}_J\,,
\end{aligned}
$$

The scalar potential

$$
V = -\tfrac{1}{4} S_{IJ} S^{*\,IJ} + \tfrac{1}{8}\delta^{\mathbf{mn}} N_{\mathbf{m}}{}^I{}_J N^*_{\mathbf{n}\,I}{}^J = -\tfrac{1}{2}\mathscr{W}^{ij} v_i v_j \left[\mathrm{Tr}(\mathscr{M})^2 - 2\mathrm{Tr}(\mathscr{M}^2)\right],
$$

where $\mathscr{W}^{ij}$ is the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ symmetric matrix, and where we have used

$$
\mathscr{M}_{mn} \equiv L_m{}^{\mathbf{p}} L_n{}^{\mathbf{p}}\,, \quad \text{so that} \quad T = \mathrm{Tr}(\mathscr{M})\,, \quad \text{and} \quad T^{\mathbf{mn}} T^{\mathbf{mn}} = \mathrm{Tr}(\mathscr{M}^2)\,.
$$

Thank you.