KU LEUVEN

On the many uses of Squashed S^3

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Based on *N. Bobev, T. Hertog, YV: Arxiv:1610.01497*
and *N. Bobev, P. Bueno, YV: Arxiv:1612.xxxxx*

What is Squashed S^3?

Study of field theories on:

$$ds^2_{sq\ S^3} = \frac{r_0}{4}\left((\sigma_1)^2 + \frac{1}{1+\beta}(\sigma_2)^2 + \frac{1}{1+\alpha}(\sigma_3)^2\right)$$

with

$$\begin{aligned}
\sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi\ , \\
\sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi\ , \\
\sigma_3 &= d\psi + \cos\theta d\phi\ .
\end{aligned}$$

Different special cases

$$\begin{aligned}
\alpha = \beta = 0 \quad &\Rightarrow \quad S^3 \\
\alpha = \beta \neq 0 \text{ or } \alpha = 0, \beta \neq 0 \quad &\Rightarrow \quad \text{single squashed } S^3 \\
\alpha \neq \beta \neq 0 \quad &\Rightarrow \quad \text{double squashed } S^3
\end{aligned}$$

What is Squashed S^3?

Gravitational homogeneous theories (in AdS) with metric

$$ds^2 = dr^2 + l_1(r)(\sigma_1)^2 + l_2(r)(\sigma_2)^2 + l_3(r)(\sigma_3)^2 \; .$$

$l_1(r) = l_2(r) = l_3(r) \Rightarrow$ Homogeneous, isotropic spaces
$l_1(r) \neq l_2(r) = l_3(r) \Rightarrow$ AdS Taub-NUT/Bolt
$l_1(r) \neq l_2(r) \neq l_3(r) \Rightarrow$ Mixmaster universe

At large r the boundary will be squashed S^3 (holography!)

Why care about Squashed S^3?

CFTs on S^3 that we deform by squashing the background useful for: [1]

- Universal properties of CFTs
- Entanglement entropy

[1] With N. Bobev and P. Bueno
[2] With N. Bobev and T. Hertog

Why care about Squashed S^3?

CFTs on S^3 that we deform by squashing the background useful for: [1]

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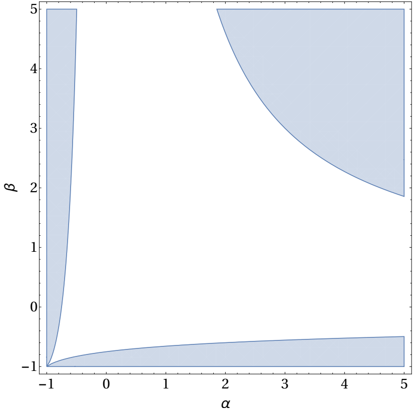
Gravity solutions that asymptote to squashed S^3: [2]

- New GR solutions
- AdS/CFT
- dS/CFT

[1] With N. Bobev and P. Bueno
[2] With N. Bobev and T. Hertog

$$R = \frac{6 + 8\alpha + 8\beta + 2\alpha\beta(6 - \alpha\beta)}{(1+\alpha)(1+\beta)}$$



Overview

1. Fields on squashed S^3
   - Universality?
2. Taub-NUT/Bolt solutions
   - Single Squashed AdS Taub-NUT/Bolt solutions
   - Double squashed Taub-NUT/Bolt solutions
   - Thermodynamics: From NUT to Bolt
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Free fields on squashed S^3

N Scalars (Free O(N) vector model):

$$Z_{\mathrm{sc}}[\alpha,\beta] = \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^3x\sqrt{g}\left[(\partial\phi)^2 + \frac{R\phi^2}{8}\right]\right)$$

Dirac fermion:

$$Z_{\mathrm{f}}[\alpha,\beta] = \int \mathcal{D}\psi \exp\left(-\int d^3x\sqrt{g}\left[\psi^\dagger(i\not{D}\psi)\right]\right)$$

Find free energy $\mathcal{F} = -\log Z = (-1)^f \frac{1}{2} \log\det\left[\mathfrak{D}/\Lambda^\sigma\right]$

Free fields on squashed S^3

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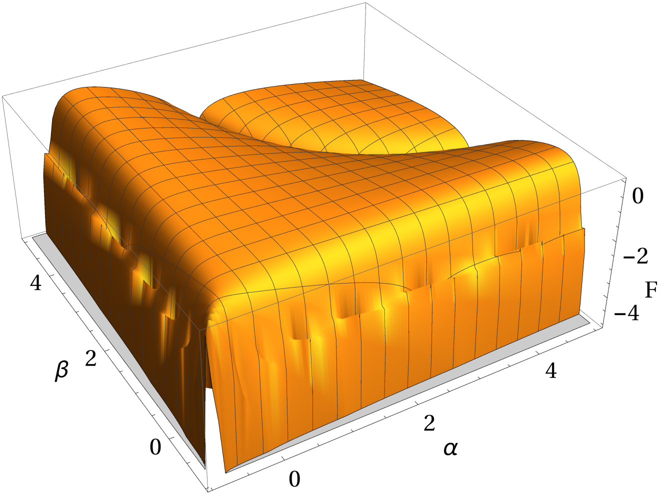
Find free energy $\mathcal{F} = -\log Z = (-1)^f \frac{1}{2}\log\det\left[\mathfrak{D}/\Lambda^\sigma\right]$
How to calculate this?

- Find eigenvalues of $\mathfrak{D}$ (1 squashing vs 2 squashings)
- Numerical renormalization

Double Squashed Free O(N) model



Universal behaviour of CFTs?

Consider CFTs on backgrounds with general metric perturbations (d=odd)

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$$

In general (for general backgrounds with deformations) then

$$\mathcal{F}(\epsilon) = -\log Z = \mathcal{F}(0) + \frac{\epsilon^2}{2}\mathcal{F}''(0) + \mathcal{O}(\epsilon^3)$$

where $\mathcal{F}''(0) \propto C_T$
$C_T =$ central charge controlling two-point stress-tensor(theory dependent)
$\Rightarrow$ Similar to entanglement entropy on deformed backgrounds!

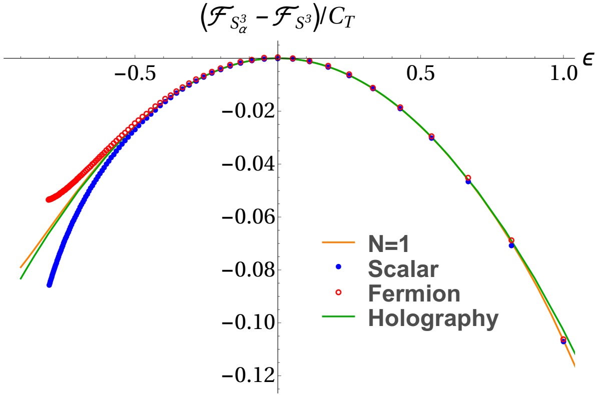
$$s(\epsilon) = s(0) - \frac{\epsilon^2}{2}s''(0) + (O)(\epsilon^3) \ ,$$

with $s''(0) \propto C_T$

For single squashed S^3 (β=0): $\epsilon = -\frac{\alpha}{1+\alpha}$
Do some ugly calculations...

$$\mathcal{F}_{S^3_\alpha} = -\log Z = \mathcal{F}_{S^3} - \frac{\pi^2 C_T}{96}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

For single squashed S^3 (β=0): $\epsilon = -\frac{\alpha}{1+\alpha}$



Holography: $$\mathcal{F}_{\mathrm{E},\,S^3_\alpha} = \mathcal{F}_{\mathrm{E},\,S^3} - \frac{\pi^2 C_T}{96}\epsilon^2$$

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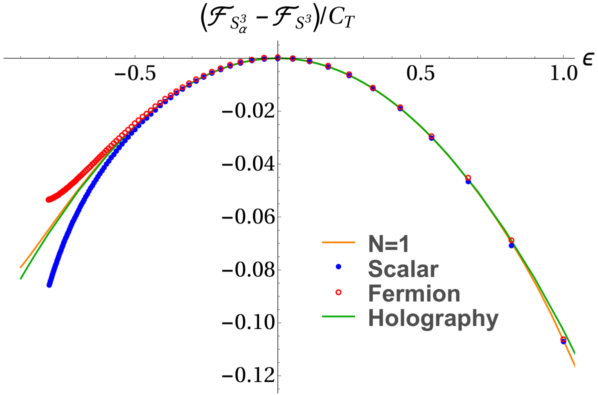

$$\text{Holography:} \qquad \mathcal{F}_{\mathrm{E},\, S^3_\alpha} = \mathcal{F}_{\mathrm{E},\, S^3} - \frac{\pi^2 C_T}{96}\epsilon^2$$

Unreasonable effectiveness!

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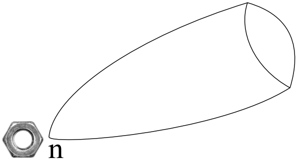
Taub-NUT/Bolt solutions

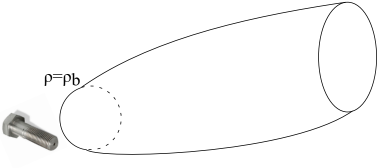
$$ds^2 = 4n^2 V(\rho)(\sigma_3)^2 + \frac{d\rho^2}{V(\rho)} + (\rho^2 - n^2)(\sigma_1^2 + \sigma_2^2) \; ,$$

$$V \equiv \frac{(\rho^2 + n^2) - 2m\rho + l^{-2}(\rho^4 - 6n^2\rho^2 - 3n^4)}{\rho^2 - n^2}$$

NUT: Around $\rho = n \Rightarrow \mathbb{R}^4$



Bolt: Around $\rho = \rho_b \Rightarrow \mathbb{R}^2 \times S^2$



Bolt solutions exist only for large enough squashing ($\alpha \geq \alpha_{\text{crit}} = 5 + 3\sqrt{3}$)

There are two different types of bolt solutions

Double squashed Taub-NUT solutions

Ansatz: $dr^2 + l_1(r)(\sigma_1)^2 + l_2(r)(\sigma_2)^2 + l_3(r)(\sigma_3)^2$

$$
\begin{aligned}
l_1(r) &= \frac{1}{4}(r-r^*)^2 + \beta_4(r-r^*)^4 + \mathcal{O}\left((r-r^*)^6\right) \ , \\
l_2(r) &= \frac{1}{4}(r-r^*)^2 + \gamma_4(r-r^*)^4 + \mathcal{O}\left((r-r^*)^6\right) \ , \\
l_3(r) &= \frac{1}{4}(r-r^*)^2 + \delta_4(r-r^*)^4 + \mathcal{O}\left((r-r^*)^6\right)
\end{aligned}
$$

Plug this into the Einstein equations and numerically integrate to the UV:

$$
l_1(r) = \frac{1}{4}e^{2r} + A_k e^{(2-k)r} \ , \qquad l_2(r) = \frac{1}{4(1+\beta)}e^{2r} + B_k e^{(2-k)r} \ ,
$$
$$
l_3(r) = \frac{1}{4(1+\alpha)}e^{2r} + C_k e^{(2-k)r}
$$

Double squashed Taub-NUT solutions

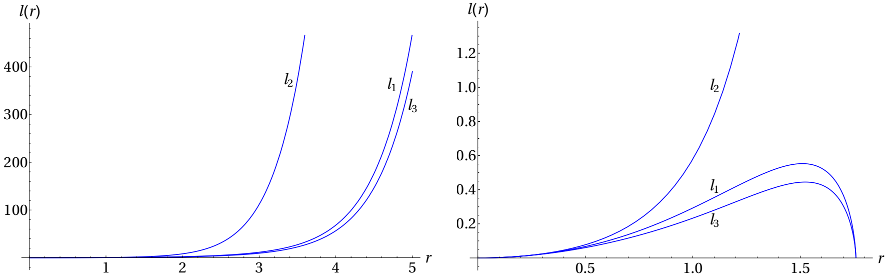
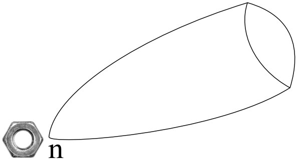


Figure: Left: a solution with $\beta_4 = 1/12$ and $\gamma_4 = 1/6$ Right: a solution with $\beta_4 = 1/12$ and $\gamma_4 = 3/14$.

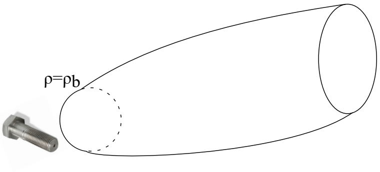
Taub-NUT/Bolt solutions

$$ds^2 = 4n^2 V(\rho)(\sigma_3)^2 + \frac{d\rho^2}{V(\rho)} + (\rho^2 - n^2)(\sigma_1^2 + \sigma_2^2) \ ,$$
$$V \equiv \frac{(\rho^2 + n^2) - 2m\rho + l^{-2}(\rho^4 - 6n^2\rho^2 - 3n^4)}{\rho^2 - n^2}$$

NUT: Around $\rho = n \Rightarrow \mathbb{R}^4$



Bolt: Around $\rho = \rho_b \Rightarrow \mathbb{R}^2 \times S^2$



Bolt solutions exist only for large enough squashing ($\alpha \geq \alpha_{\rm crit} = 5 + 3\sqrt{3}$)
There are two different types of bolt solutions

Double squashed Taub-Bolt solutions

$$l_1(r) = \beta_0 + \beta_2(r - r^*)^2 + \mathcal{O}\left((r - r^*)^4\right) \ ,$$
$$l_2(r) = \gamma_0 + \gamma_2(r - r^*)^2 + \mathcal{O}\left((r - r^*)^4\right) \ ,$$
$$l_3(r) = \frac{1}{4}(r - r^*)^2 + \delta_4(r - r^*)^4 + \mathcal{O}\left((r - r^*)^6\right) \ ,$$

Again numerical integrate Einstein equations to the UV

$$l_1(r) = \frac{1}{4}e^{2r} + A_k e^{(2-k)r} \ , \qquad l_2(r) = \frac{1}{4(1+\beta)}e^{2r} + B_k e^{(2-k)r} \ ,$$
$$l_3(r) = \frac{1}{4(1+\alpha)}e^{2r} + C_k e^{(2-k)r}$$

Double squashed Taub-Bolt solutions

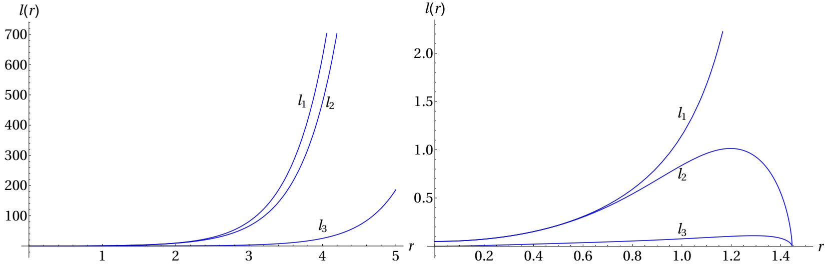
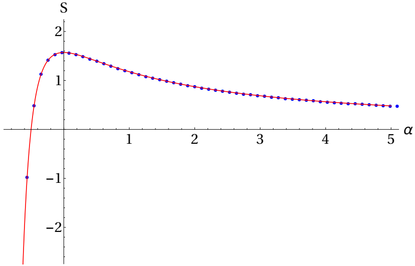
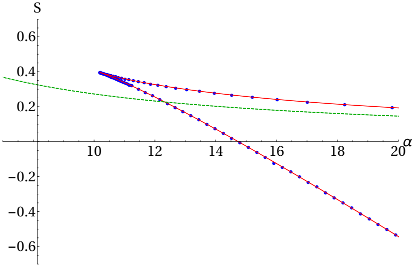


Figure: Left: a solution with $\gamma_0 = 1/20$ and $\gamma_4 = 0.382281$. Right: a solution with $\gamma_0 = 1/20$ and $\gamma_4 = 0.380208$.

Single Squashed Taub-NUT/Bolt Thermodynamics

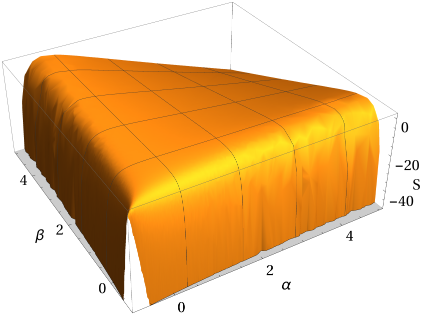
$$S = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3\sqrt{h}K + S_{ct} \ ,$$



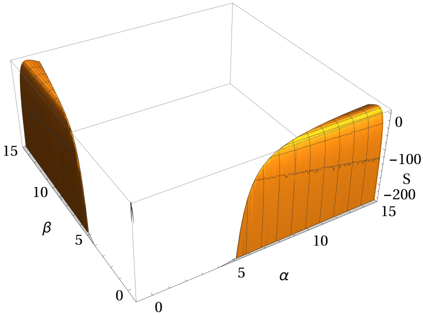


Notice: Phase transition from NUT to Bolt

Double Squashed Taub-NUT/Bolt Thermodynamics



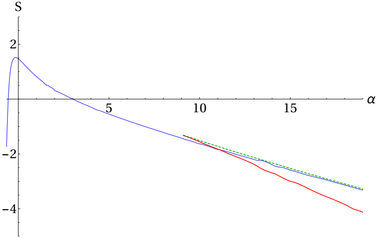
(a) NUT solution



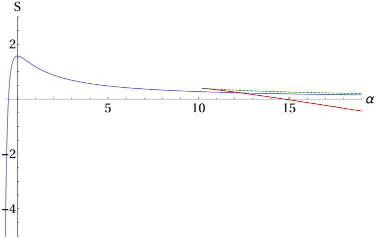
(b) Bolt solution

NUT solutions exist everywhere while Bolt solutions only exist in a restriced area

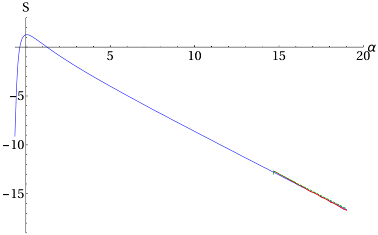
Double Squashed Taub-NUT/Bolt Thermodynamics



(a) $\beta = -0.2$



(b) $\beta = 0$



(c) $\beta = 0.8$

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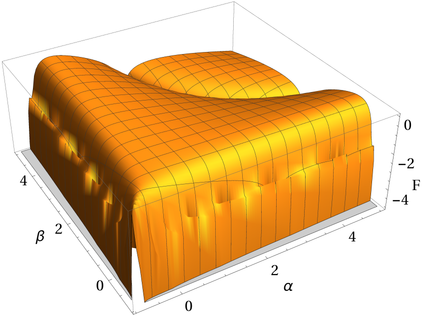
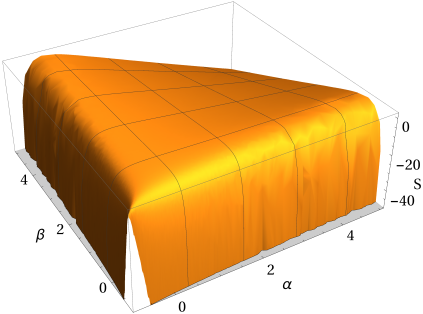
Compare gravity with free O(N) model





Compare gravity with free O(N) model

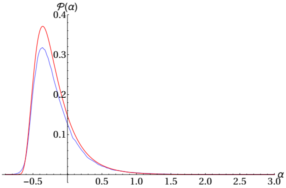
- Same global maxima
- Same maxima for slices of β or α
- No NUT to Bolt transition in (free) CFT
- F for CFT diverges when $R = 0$

dS/CFT

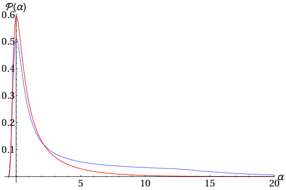
$$\Psi_{HH}[h_{ij},\phi] = \int \mathcal{D}g_{ij}\mathcal{D}\phi' e^{-S_E(g_{ij},\phi')}$$

On a saddle point level, relate dS solutions to Euclidean AdS solutions

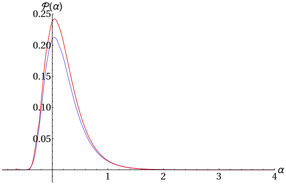
$$\Psi_{HH}[h_{ij},\phi] = Z_{QFT}^{-1}[\tilde{h}_{ij},\zeta]\exp(iS_{ct}[h_{ij},\phi]/\hbar)$$



(a) $\beta = -0.4$



(b) $\beta = 0.05$



(c) $\beta = 1.15$

dS/CFT

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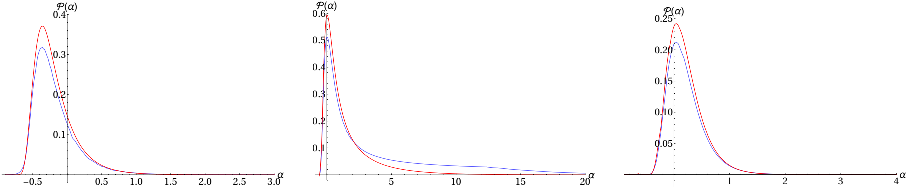

(a) $\beta = -0.4$ (b) $\beta = 0.05$ (c) $\beta = 1.15$

$\Rightarrow$ Anisotropic universes are less likely to occur

Summary

- Studied CFTs for which we deformed the background to have two squashings
- Unreasonable effectiveness when we expand the free energy in terms of the squashing
- Relationship with entanglement entropy?
- New GR solutions which asymptote to a double squashed sphere
- Compared the new GR solutions with CFTs on same geometry
- In cosmological context: anisotropic universes are not favoured