V Postgraduate Meeting on Theoretical Physics

Constructing general scalar-tensor theories of gravity: Are they viable?

Jose María EZQUIAGA

Based on:

Phys. Rev. **D94**, 024005 (2016) by **JME**, J. GARCÍA-BELLIDO and M. ZUMALACÁRREGUI

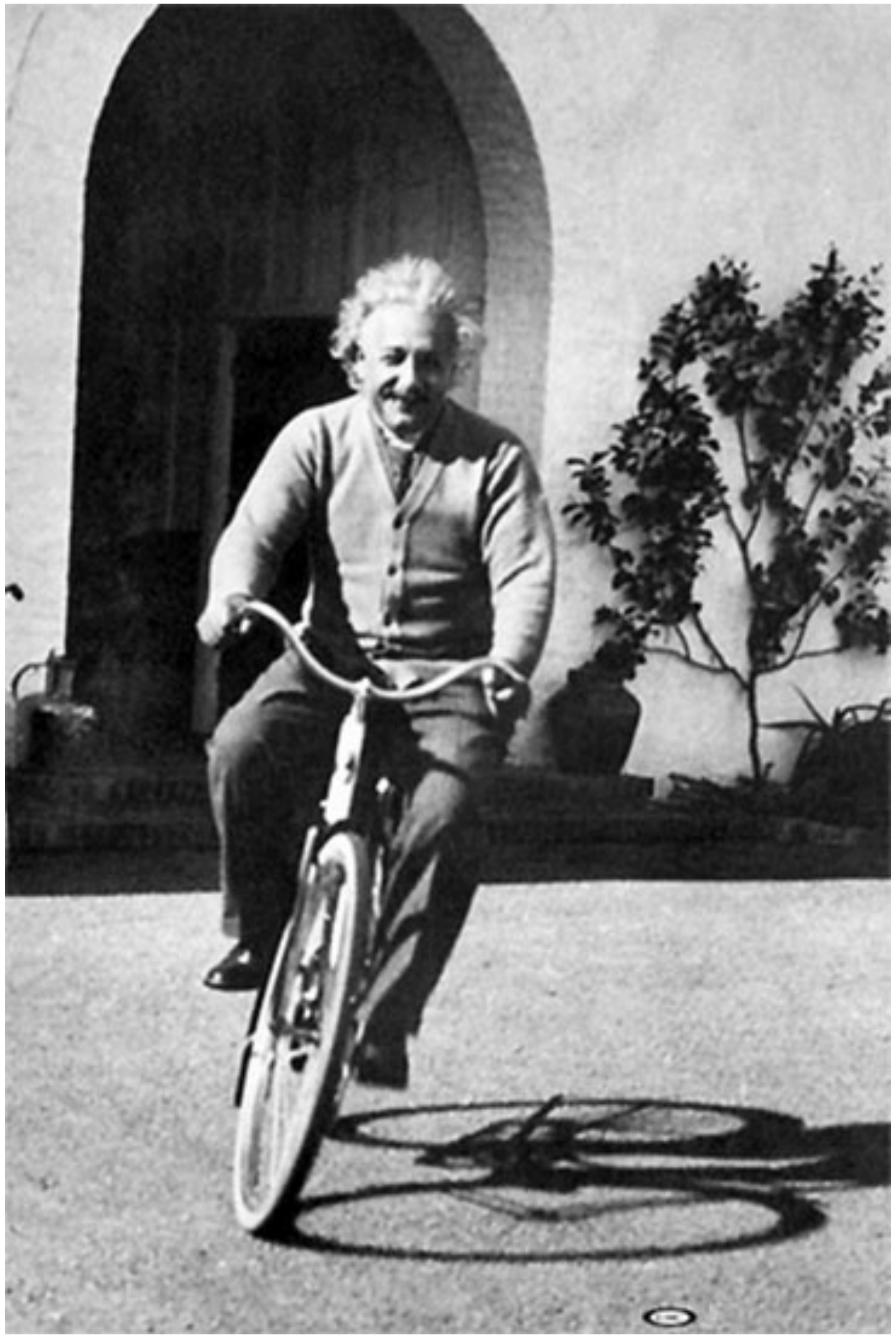
arXiv 1608.01982 by D. BETTONI, **JME**, K. HINTERBICHLER and M. ZUMALACÁRREGUI

UNIVERSIDAD AUTONOMA DE MADRID

Instituto de Física Teórica UAM-CSIC

GOBIERNO DE ESPAÑA MINISTERIO DE EDUCACIÓN, CULTURA Y DEPORTE

GR is in very good shape...

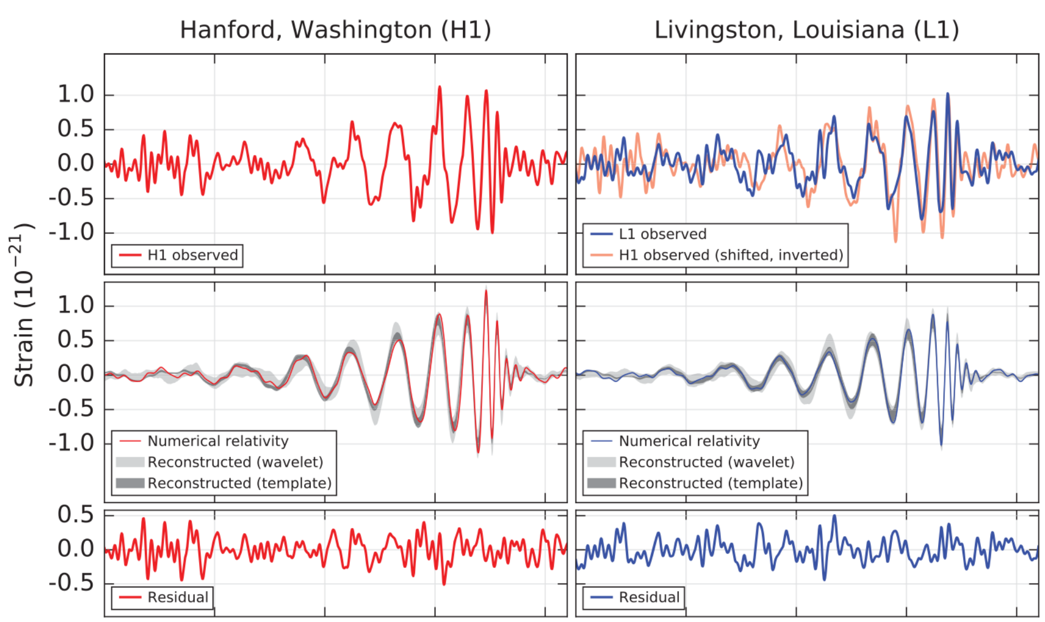


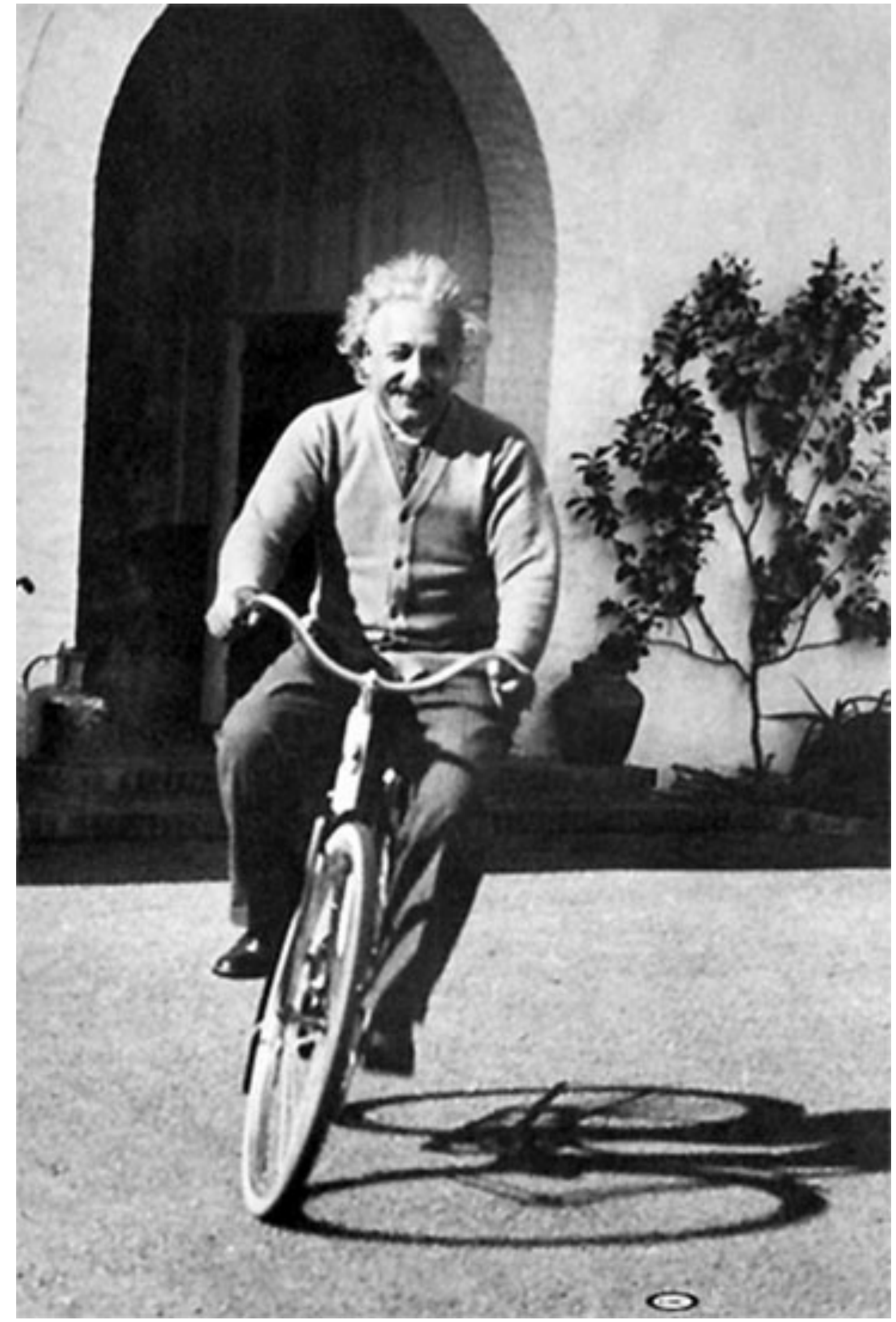
Planck Scale — 10^{-35}
Laboratory — 10^{-4}
Solar System — 10^{14}
Cosmos — 10^{26}
m

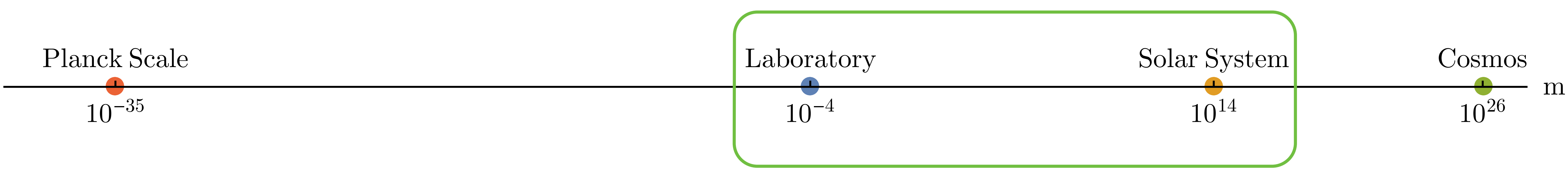
GR is in very good shape...

- 100 years old and still in great agreement at very different scales

- Direct detection of GWs







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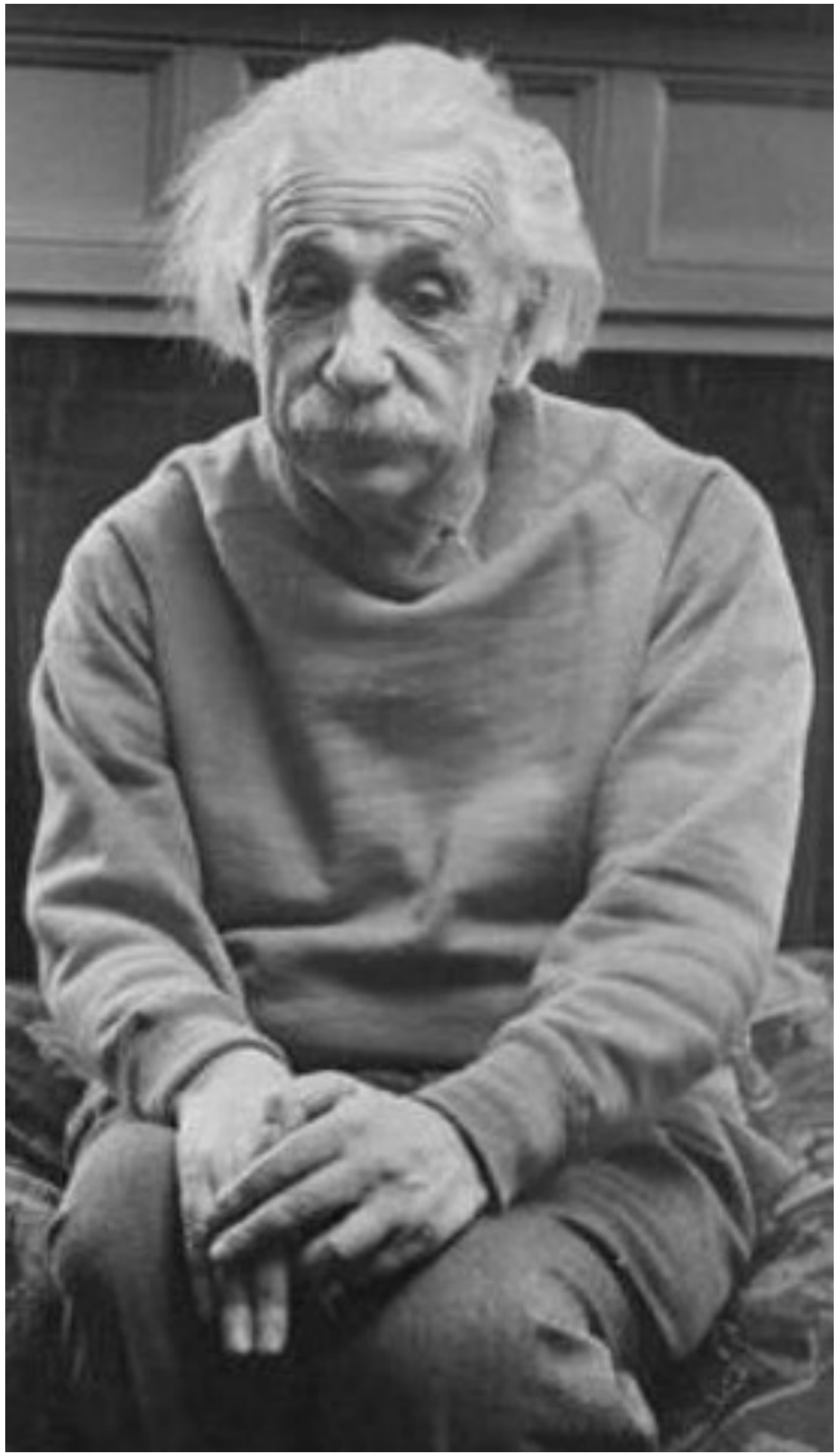
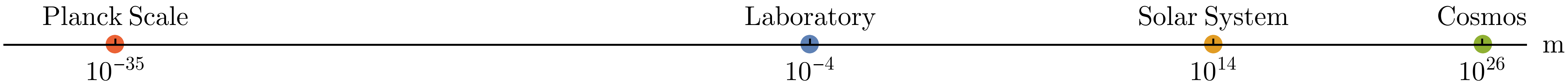
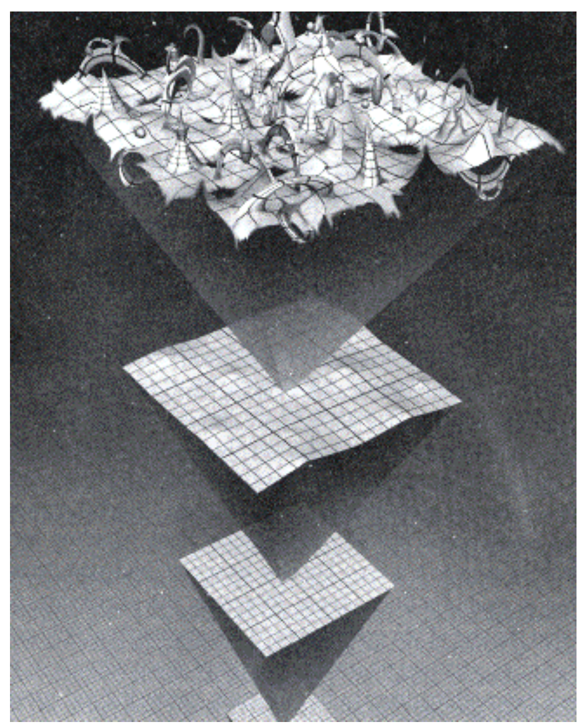


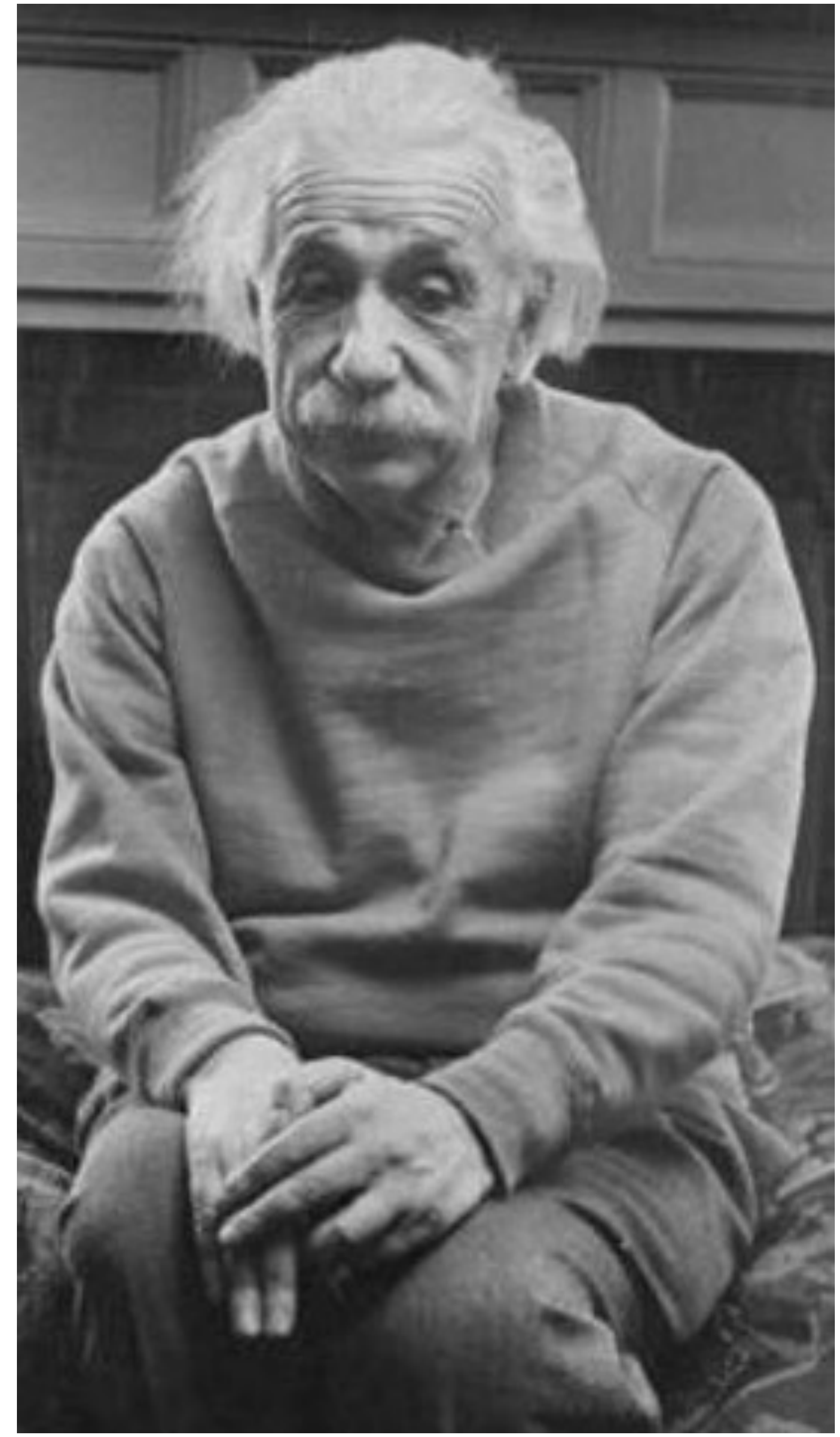


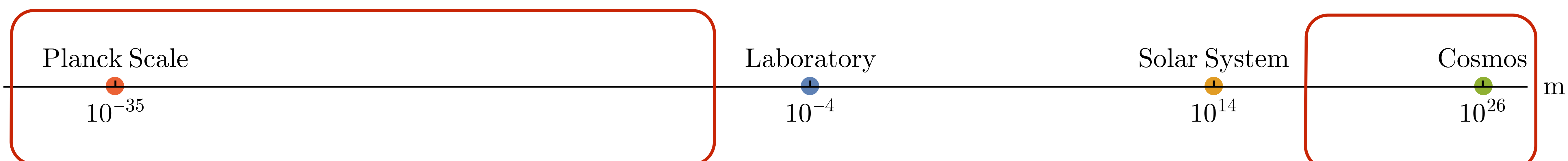
... but it is not enough.

- Singularities and Quantization

- Dark Sector of the Universe







Contents

Contents

1 Why scalar-tensor theories?

- Simplest modification of GR: add only 1 degree of freedom

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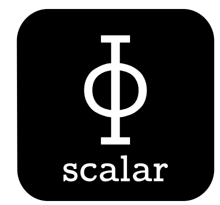
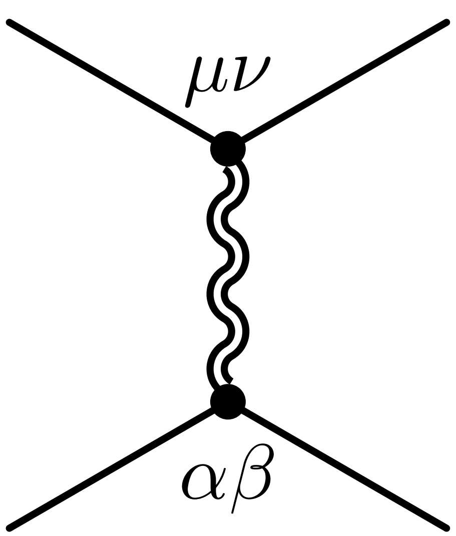


1 Why scalar-tensor theories?

- Simplest modification of GR: add only 1 degree of freedom

...in fact theories with only massless spin-2 particles are fixed to follow linearized GR in the IR (Diff) [Lectures by Arkani-Hamed at IAS]

*Also Unimodular Gravity (TDiff) [Van der Bij *et al.* 1982]



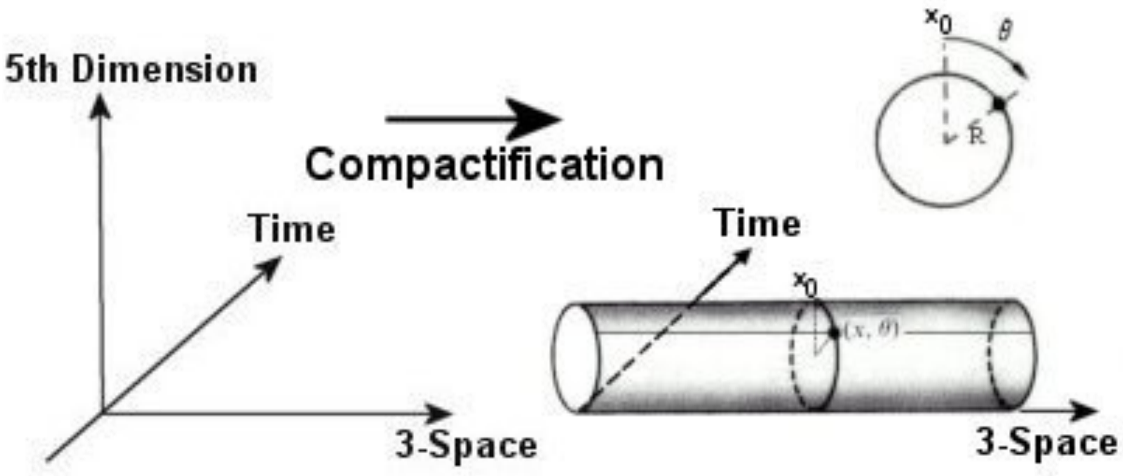
$$\frac{i}{p^2 + i\epsilon} T^{\mu\nu} N_{\mu\nu,\alpha\beta} T^{\alpha\beta}$$

$$N_{\mu\nu,\alpha\beta} \sim \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}$$

[deWitt 1967]

- At the end, any fancy modification of GR is just adding extra DoF (typically scalar fields)

  Example: Kaluza-Klein tower of states



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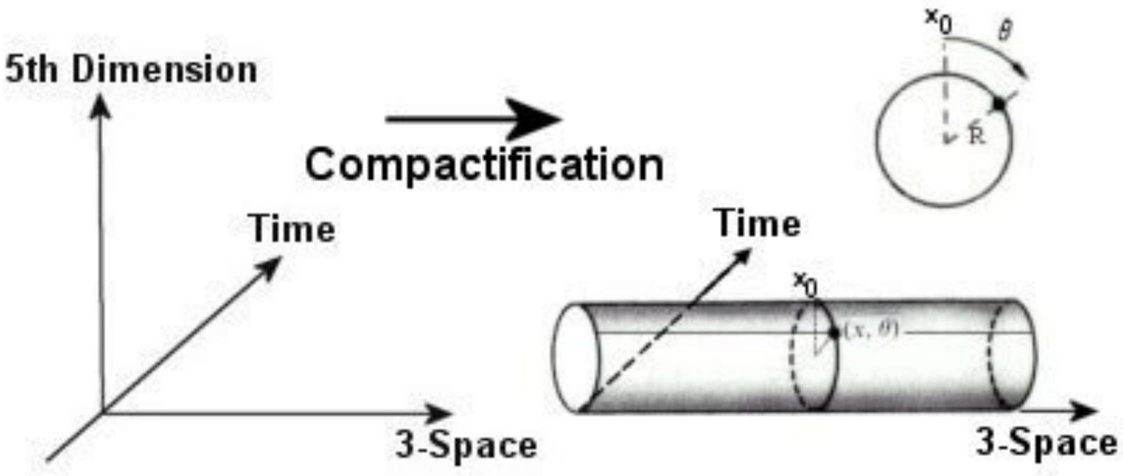
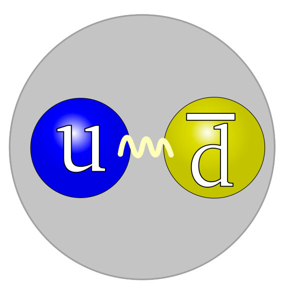
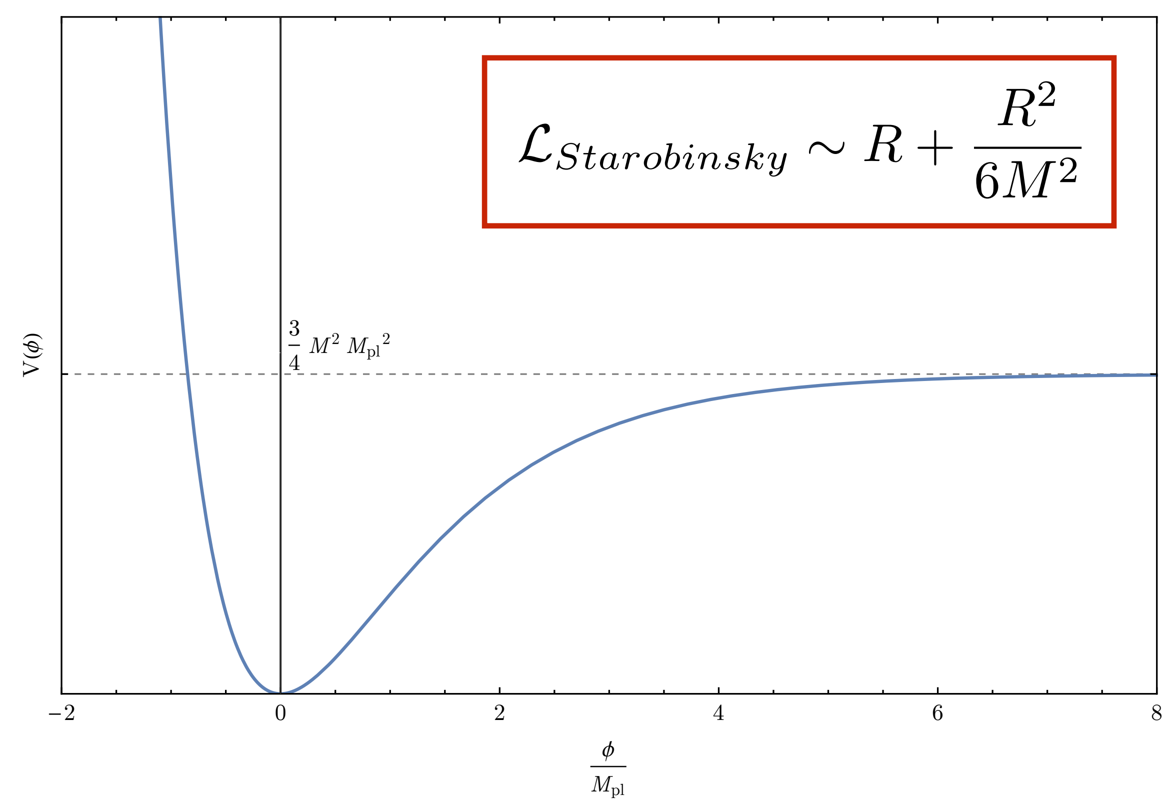


- Moreover, scalar field can be seen as an EFT

Example: Pion in Particle Physics



Example: Scalaron in Gravity



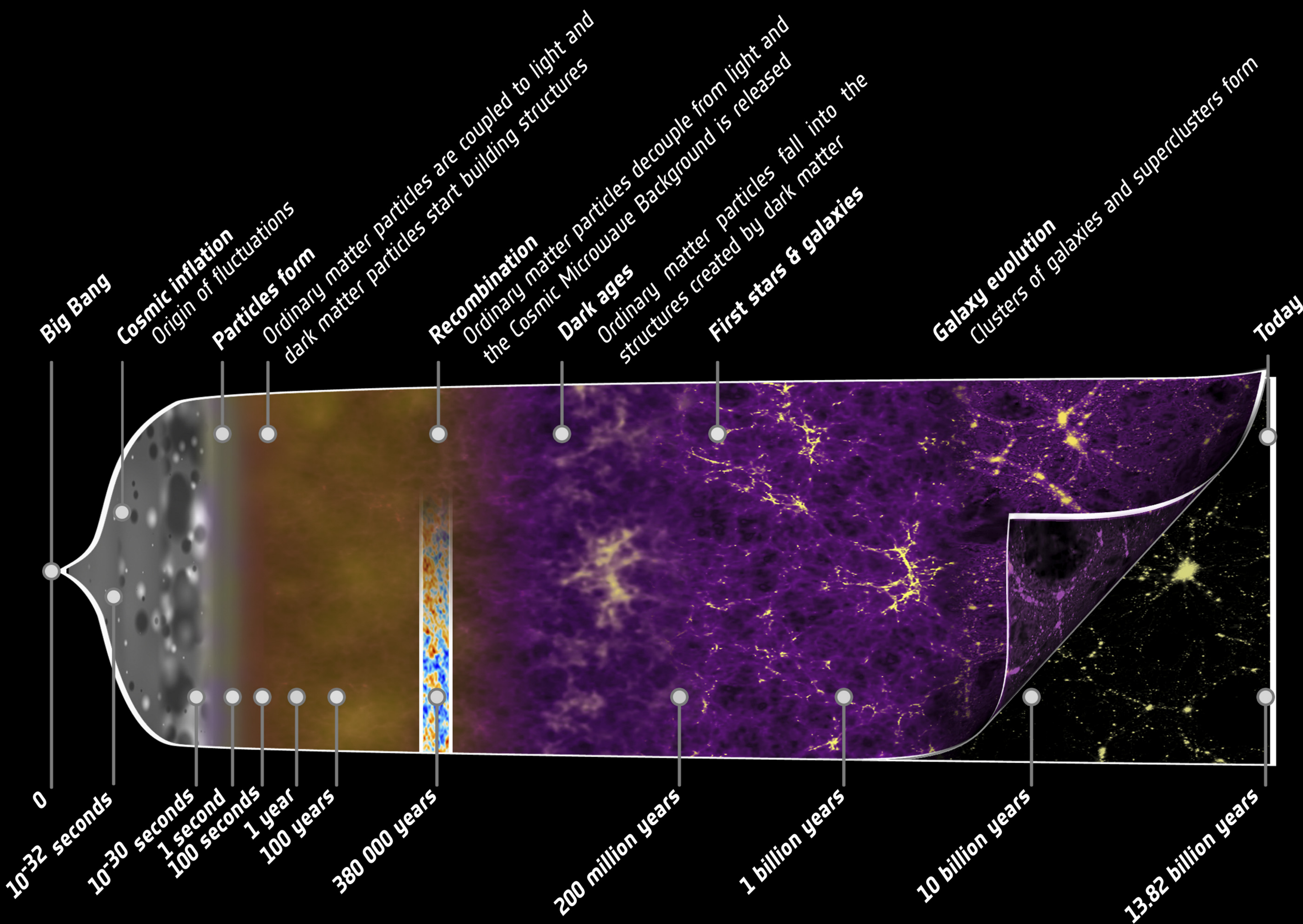
- Today, we will focus on adding a scalar field, but...

Vector-Tensor Theories	[Will and Nordtvedt 1972]
Multi-scalar-tensor	[Damour and Esposito-Farese 1992]
Scalar-Vector-Tensor Theories	[Bekenstein 2004]
Massive Gravity	[de Rham, Gabadadze and Tolley 2011]
Bi-gravity	[Hassan and Rosen 2011]
Multi-gravity	[Hinterbichler and Rosen 2012]

- Alternatively, one could break first principles (locality, Lorentz inv.)

Horava-Lifshitz Gravity	[Horava 2009]

- Scalars can describe periods of accelerated expansion



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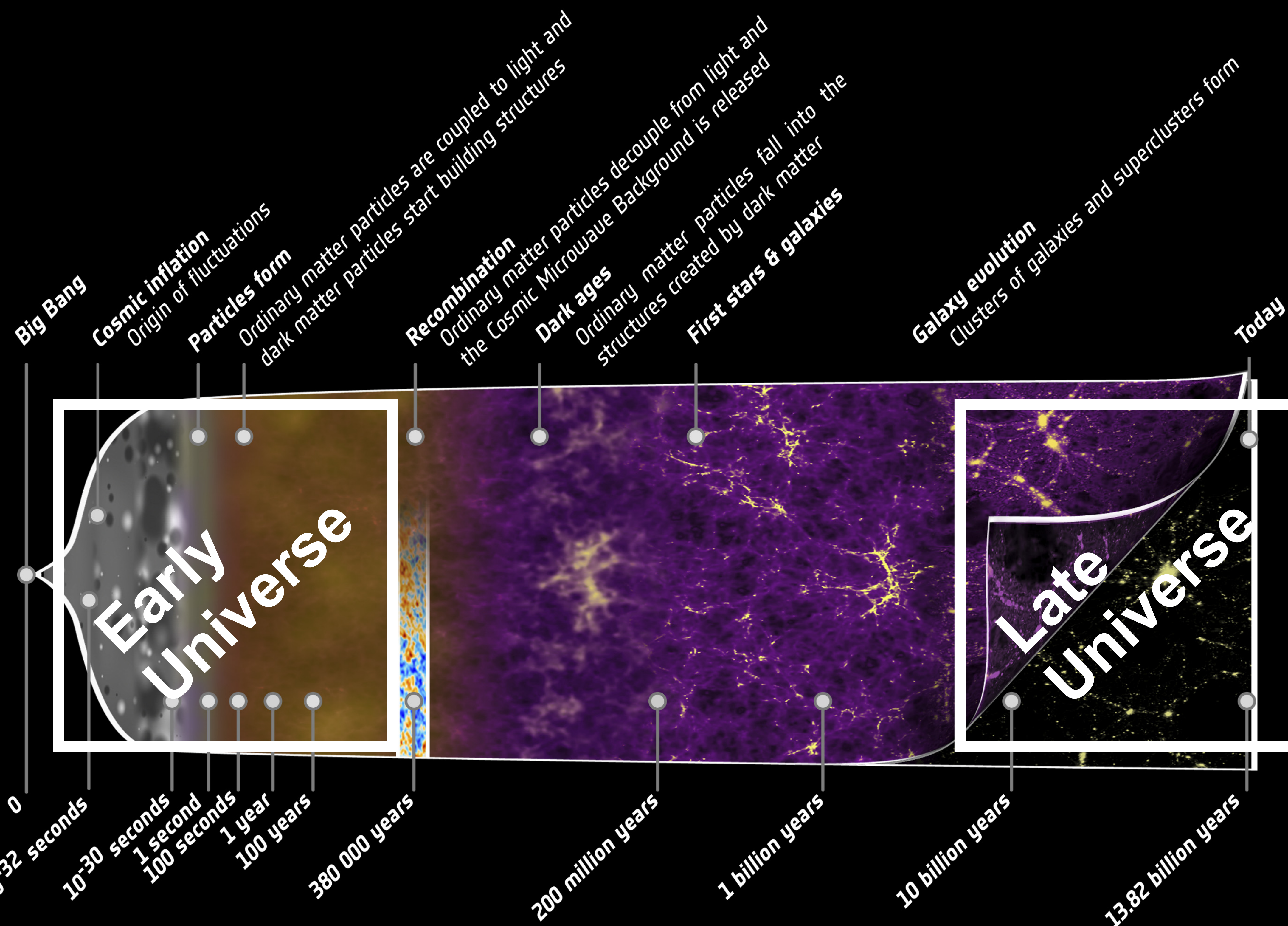
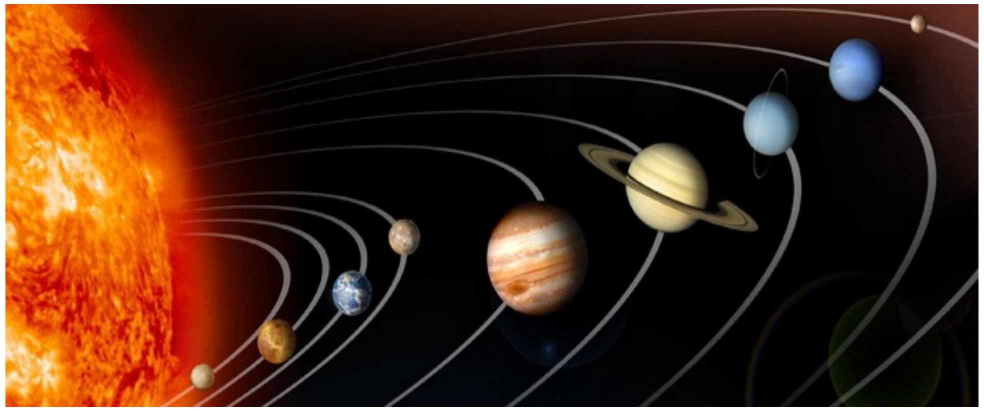
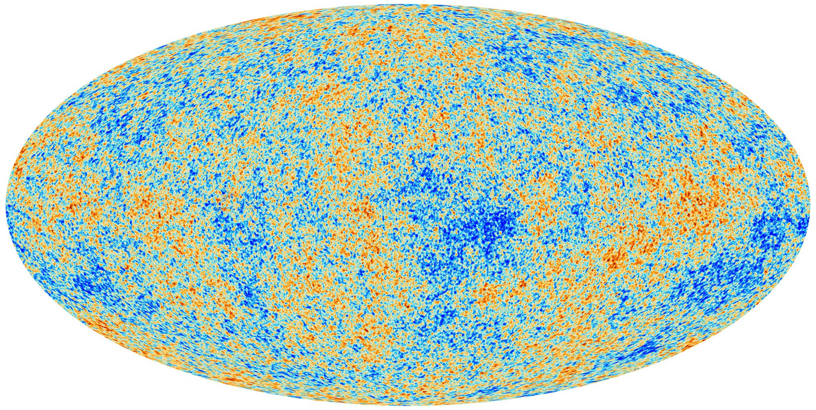


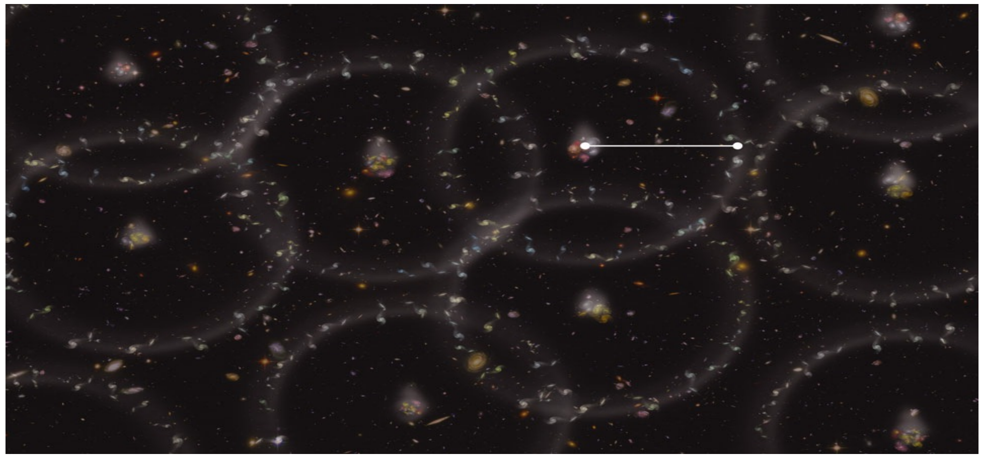
- Useful for testing our current models (GR and ΛCDM)



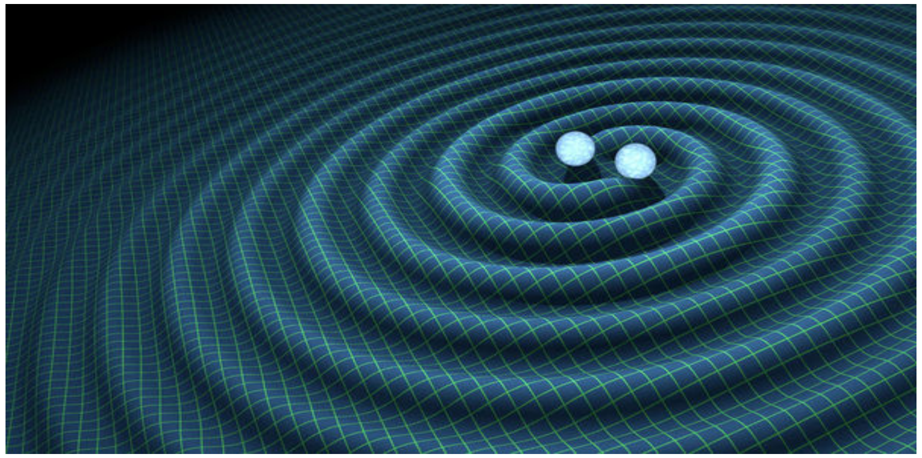
[Solar System Tests]



[CMB data]

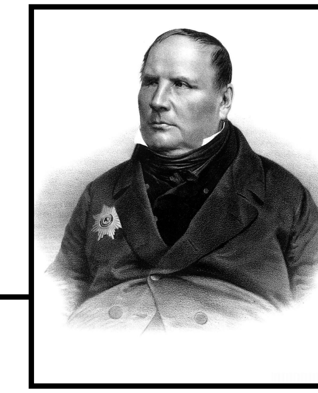


[LSS observations]



[Gravitational Waves]

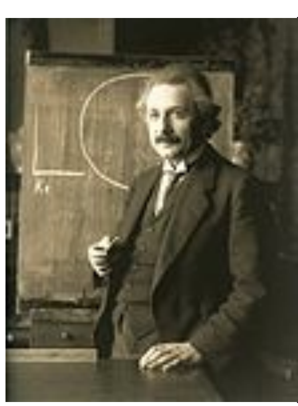
2 Ostrogradski's Theorem

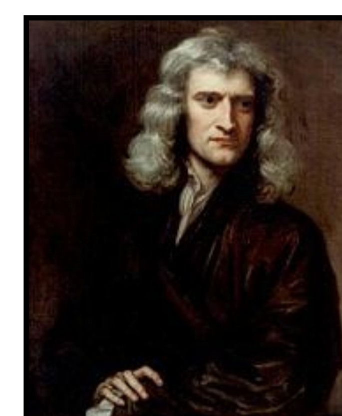


[Ostrogradski 1850]

- Generalization of Hamilton's analysis for arbitrary # of time derivatives

$$L = L(q, \dot{q}, \ddot{q}, \cdots)$$





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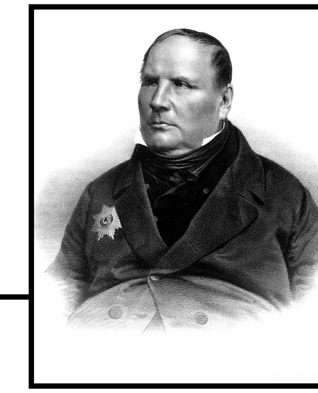

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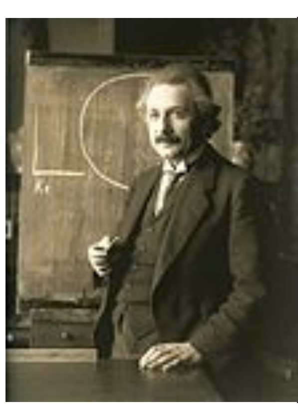
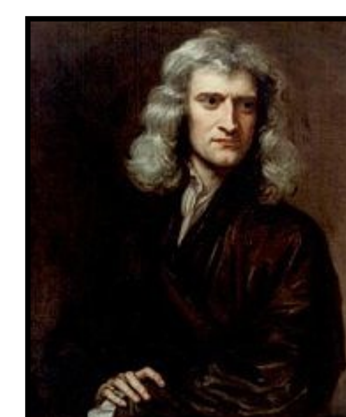
- Non-degenerate theories with two time derivatives or higher are unstable. There are terms linear to the momentum in the Hamiltonian





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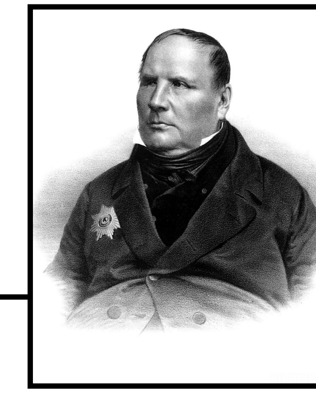

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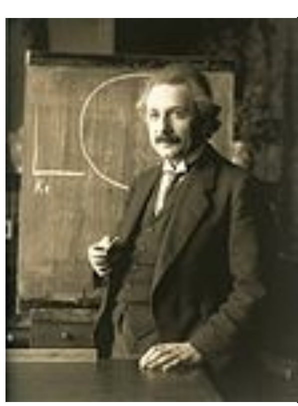

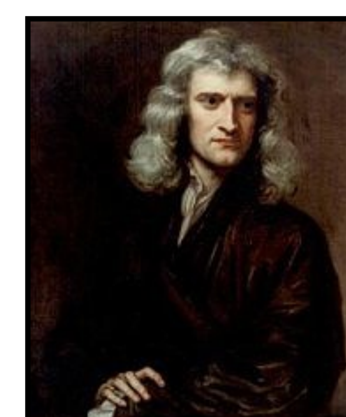

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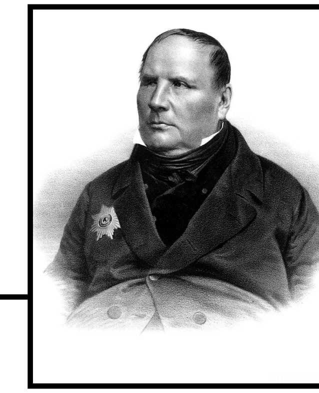

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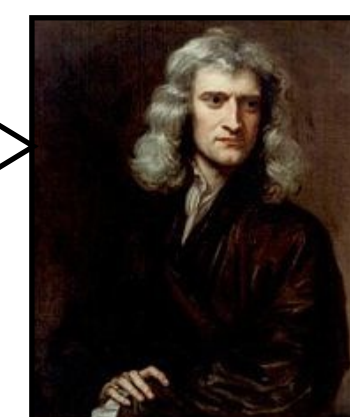
2nd order EoM
rules!



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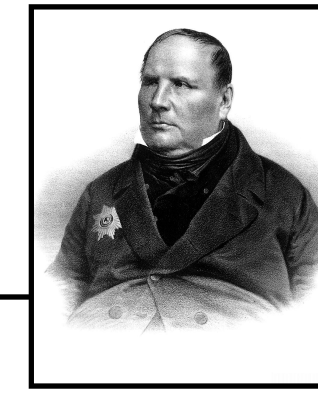

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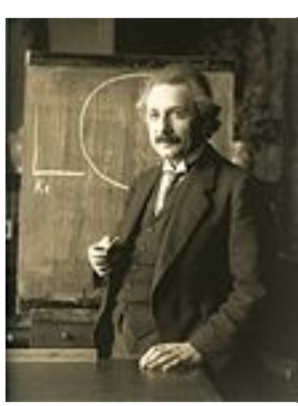
So...
what about us?



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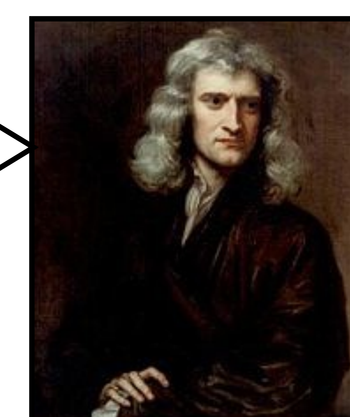

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E.g. GR or Gauge theories

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3rd derivatives: $\nabla_\mu\nabla_\nu\nabla^\alpha\phi$

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$$R^\alpha{}_{\beta[\mu\nu;\rho]} = R^\alpha{}_{[\beta\mu\nu]} = 0$$

2 Horndeski's theory

- Horndeski's work: [Horndeski 1974]

"The Valley Erupts in Flames"
Gregory Horndeski

2 Horndeski's theory

- Horndeski's work: (local+Diff. inv. theories) [Horndeski 1974]

  1st: find most general scalar-tensor 2nd order EoM in 4D

  2nd: find a Lagrangian that reproduces them

"The Valley Erupts in Flames"
Gregory Horndeski

$$
\begin{aligned}
\mathscr{L} = {} & \sqrt{(g)}\,\mathscr{K}_1 \delta^{cde}_{hjk}\phi_{|c}{}^{|h} R_{de}{}^{jk} - \tfrac{4}{3}\sqrt{(g)}\,\dot{\mathscr{K}}_1 \delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
& + \sqrt{(g)}\,\mathscr{K}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - 4\sqrt{(g)}\,\dot{\mathscr{K}}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
& + \sqrt{(g)}(\mathscr{F}+2\mathscr{W})\delta^{cd}_{fh}R_{cd}{}^{fh} + 2\sqrt{(g)}(2\mathscr{K}_3 - 2\mathscr{K}'_1 + 4\rho\dot{\mathscr{K}}_3)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\
& - 3\sqrt{(g)}(2\mathscr{F}' + 4\mathscr{W}' + \rho\mathscr{K}_8)\phi_{|c}{}^{|c} + 2\sqrt{(g)}\,\mathscr{K}_8\delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\
& + \sqrt{(g)}\{4\mathscr{K}_9 - \rho(2\mathscr{F}'' + 4\mathscr{W}'' + \rho\mathscr{K}'_8 + 2\dot{\mathscr{K}}_9)\} \qquad (4.21)
\end{aligned}
$$

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& + \sqrt{(g)}(\mathscr{F} + 2\mathscr{W})\delta^{cd}_{fh} R_{cd}{}^{fh} + 2\sqrt{(g)}(2\mathscr{K}_3 - 2\mathscr{K}'_1 + 4\rho\dot{\mathscr{K}}_3)\delta^{cd}_{fh} \phi_{|c}{}^{|f} \phi_{|d}{}^{|h} \\
& - 3\sqrt{(g)}(2\mathscr{F}' + 4\mathscr{W}' + \rho\mathscr{K}_8)\phi_{|c}{}^{|c} + 2\sqrt{(g)}\,\mathscr{K}_8 \delta^{cd}_{fh} \phi_{|c} \phi^{|f} \phi_{|d}{}^{|h} \\
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"The Valley Erupts in Flames"
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→ There are 4 free functions of ϕ and $X \equiv -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$

→ There are non-minimal couplings (with derivatives)

- Full Horndeski's theory (modern notation): $\mathcal{L}^H = \sum_{i=2}^{5} \mathcal{L}_i^H$

$$\boxed{\begin{aligned}
\mathcal{L}_2^H &= G_2(\phi, X) \\
\mathcal{L}_3^H &= G_3(\phi, X)[\Phi] \\
\mathcal{L}_4^H &= G_4(\phi, X)R + G_{4,X}([\Phi]^2 - [\Phi^2]) \\
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\end{aligned}}$$

$$({\Phi^n}_{\mu\nu} \equiv \phi_{\mu\alpha_1}{\phi^{;\alpha_1}}_{;\alpha_2} \cdots {\phi^{;\alpha_{n-1}}}_{;\nu},\ [\Phi^n] \equiv {\Phi^n}_{\mu\nu} g^{\mu\nu})$$

- Full Horndeski's theory (modern notation): $\mathcal{L}^H = \sum_{i=2}^{5} \mathcal{L}_i^H$

$$\begin{aligned}
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- Incorporates most inflationary and dark energy models!

- Simplest subcases:

Einstein-Hilbert+Λ: $G_2 = \frac{-\Lambda}{8\pi G},\ G_4 = \frac{1}{16\pi G},\ G_3 = G_5 = 0$

Jordan-Brans-Dicke: $G_2 = \frac{1}{w(\phi)}X - V(\phi),\ G_4 = \frac{\phi}{16\pi G},\ G_3 = G_5 = 0$

- There are codes to test the cosmology of your favorite model

[Bellini, Lesgourgues, Sawicki and Zumalacárregui]

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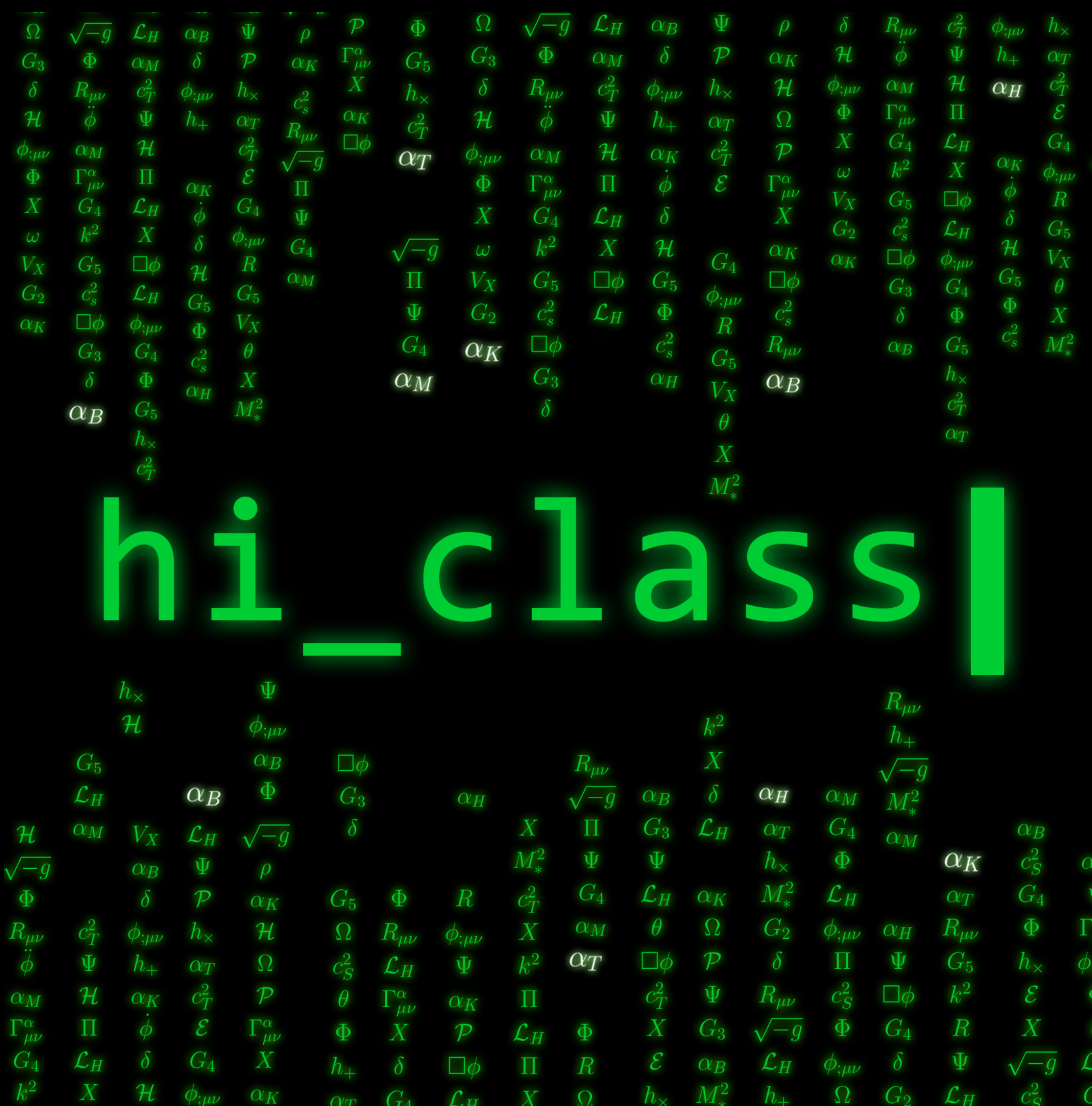

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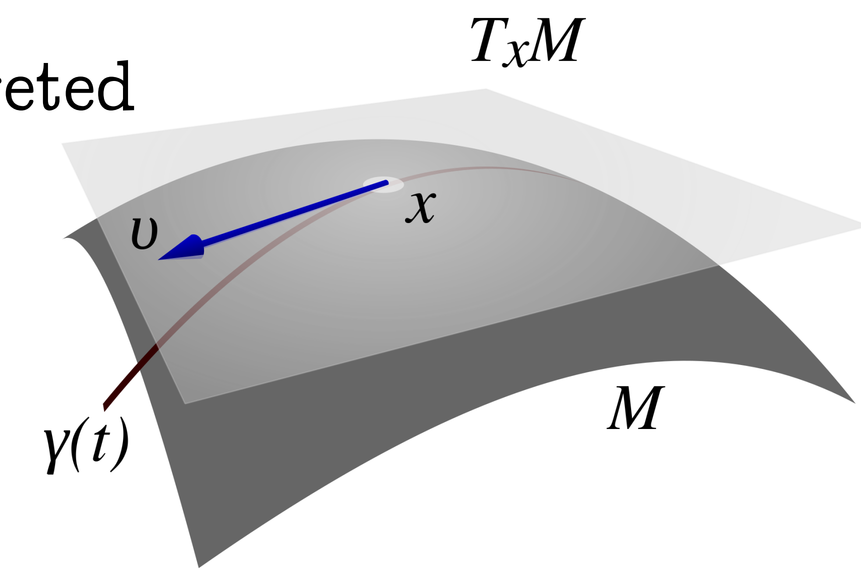
[Reminder] The mathematical objects that describes antisymmetric quantities are the differential forms

3 Differential Forms and Gravity

- General Covariance (Diff Inv.) can be reinterpreted as the invariance under Local Lorentz Transformations (LLT) in the Tangent Space



- Needed to couple fermions to gravity!

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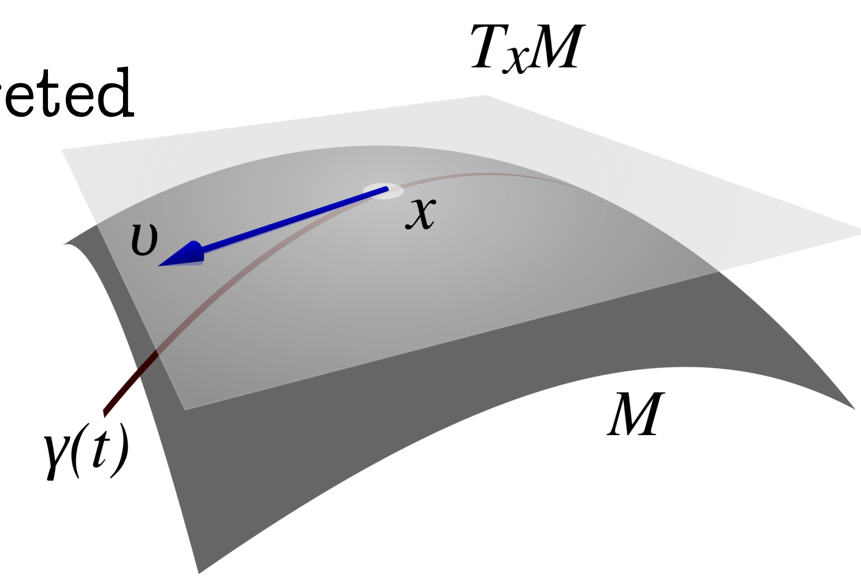

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- Example: Lovelock's Theory

[Lovelock 1971, Zumino 1986]

$$\boxed{\mathcal{L}_{(l)} = \bigwedge_{i=0}^{l} \mathcal{R}^{a_i b_i} \wedge {\theta^\star}_{a_1 b_1 \cdots a_l b_l}} \qquad \text{where } 2l \leq D \qquad \text{and}$$

$$ {\theta^\star}_{a_1 \cdots a_k} = \frac{1}{(D-k)!} \epsilon_{a_1 \cdots a_k a_{k+1} \cdots a_D} \theta^{a_{k+1}} \wedge \cdots \wedge \theta^{a_D}$$

Differential Forms Dictionary

$$\mathfrak{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \theta^a \otimes \theta^b$$

Metric Formalism	**Vielbein Formalism**
$g_{\mu\nu}$	$\theta^a = e^a{}_\mu dx^\mu$
$\Gamma^\lambda{}_{\mu\nu}$	$\omega^a{}_b$
$R^\lambda{}_{\mu\nu\rho}$	$\mathcal{R}^a{}_b$
$\nabla_\mu g_{\alpha\beta} = 0$	$\omega_{ab} = -\omega_{ba}$
$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$	$T^a = \mathcal{D}\theta^a = 0$

- Invariant objects: η_{ab} and $\epsilon_{a_1 a_2 \cdots a_D}$
- Basic operations: wedge product, exterior differential, integration...
- Basic identities: Cartan's structure equations and Bianchi's identities

3 The differential forms formalism

[PRD94.024005 (2016)]

- Define 1-forms with derivatives of the scalar field (at lowest order)

$$\boxed{\Psi^a \equiv \nabla^a\phi\nabla_b\phi\ \theta^b} \qquad \boxed{\Phi^a \equiv \nabla^a\nabla_b\phi\ \theta^b}$$

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...construct a basis of Lagrangians invariant under LLT in a pseudo-Riemannian manifold

$$\boxed{\mathcal{L}_{(lmn)} = \bigwedge_{i=1}^{l} \mathcal{R}^{a_i b_i} \wedge \bigwedge_{j=1}^{m} \Phi^{c_j} \wedge \bigwedge_{k=1}^{n} \Psi^{d_k} \wedge \theta^{\star}{}_{a_1 b_1 \cdots a_l b_l c_1 \cdots c_m d_1 \cdots d_n}}$$

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→ Clear structure in terms of the number of fields: $p \equiv 2l + m + n \leq D$

→ *Finite* basis due to antisymmetry

→ Contains well-known theories, e.g. Horndeski and Beyond Horndeski

- Action of a general scalar-tensor theory: $\boxed{S = \sum_{l,m,n}^{p \leq D} \int_{\mathcal{M}} \alpha_{lmn} \mathcal{L}_{(lmn)}}$

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- Examples: some 4D Lagrangians

$(\eta \equiv \sqrt{-g} d^4 x, \quad \Phi^n{}_{\mu\nu} = \phi_{;\mu\alpha_1} \phi^{;\alpha_1}{}_{;\alpha_2} \cdots \phi^{;\alpha_{n-1}}{}_{;\nu}, \ [\Phi^n] \equiv \Phi^n{}_{\mu\nu}\, g^{\mu\nu})$

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\end{aligned}$$

- The basis is *closed* under exterior derivatives if contractions with the gradient field are included
  - Notation: over bar indicates contractions with $\nabla^a \phi$

    e.g. $\mathcal{L}_{(0\bar{1}0)} = \nabla_a \phi \Phi^a \wedge \theta^\star{}_b \nabla^b \phi$

  - Additional Lagrangians: $\mathcal{L}_{(\bar{l}mn)}$ and $\mathcal{L}_{(l\bar{m}n)}$

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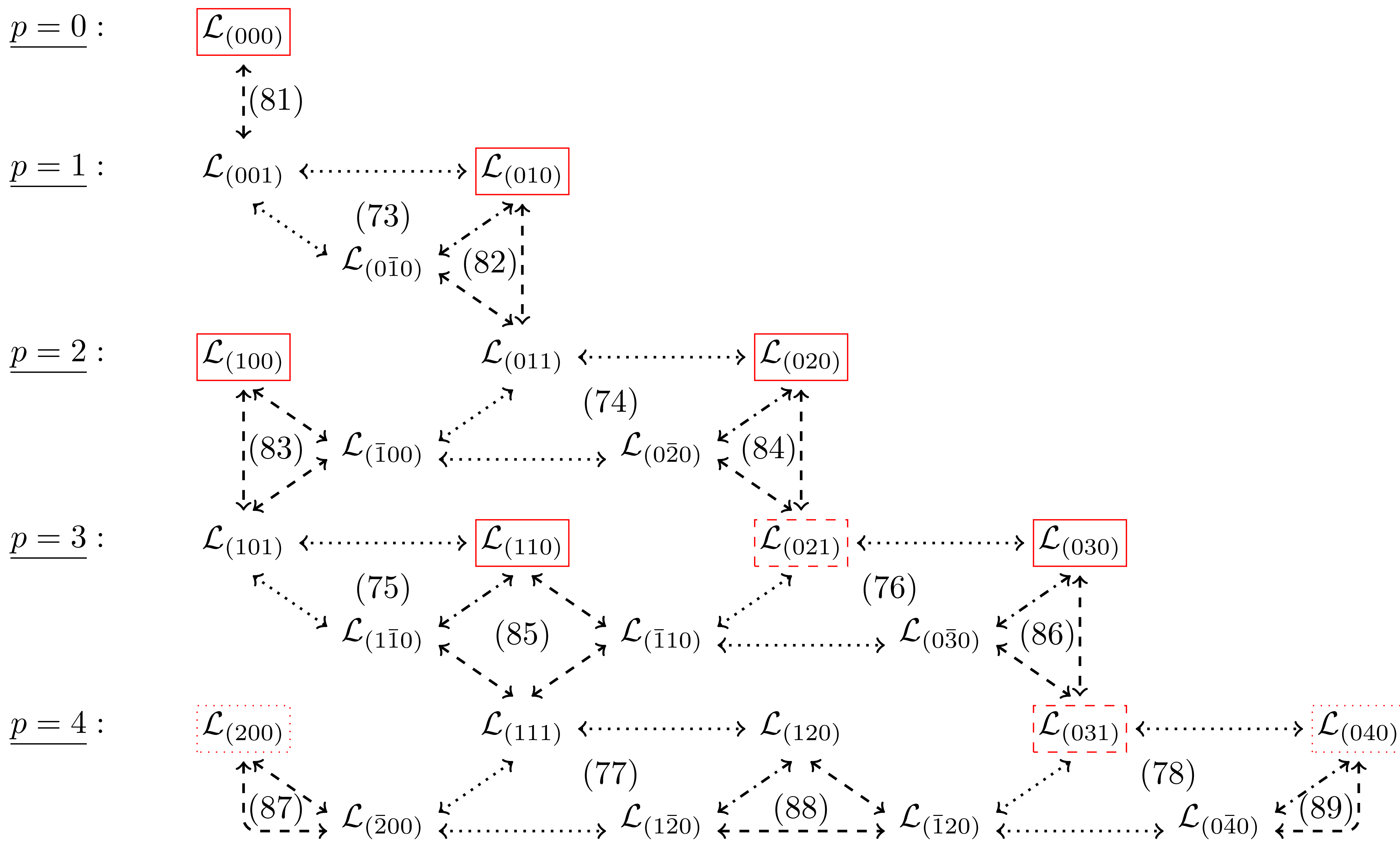
- Results: (4D)

  → There are 10 independent elements in the basis of Lagrangians

  → Only 4 independent linear combinations give rise to 2nd order EoM

  -This set can be associated with Horndeski's theory

- How scalar-tensor theories are related?



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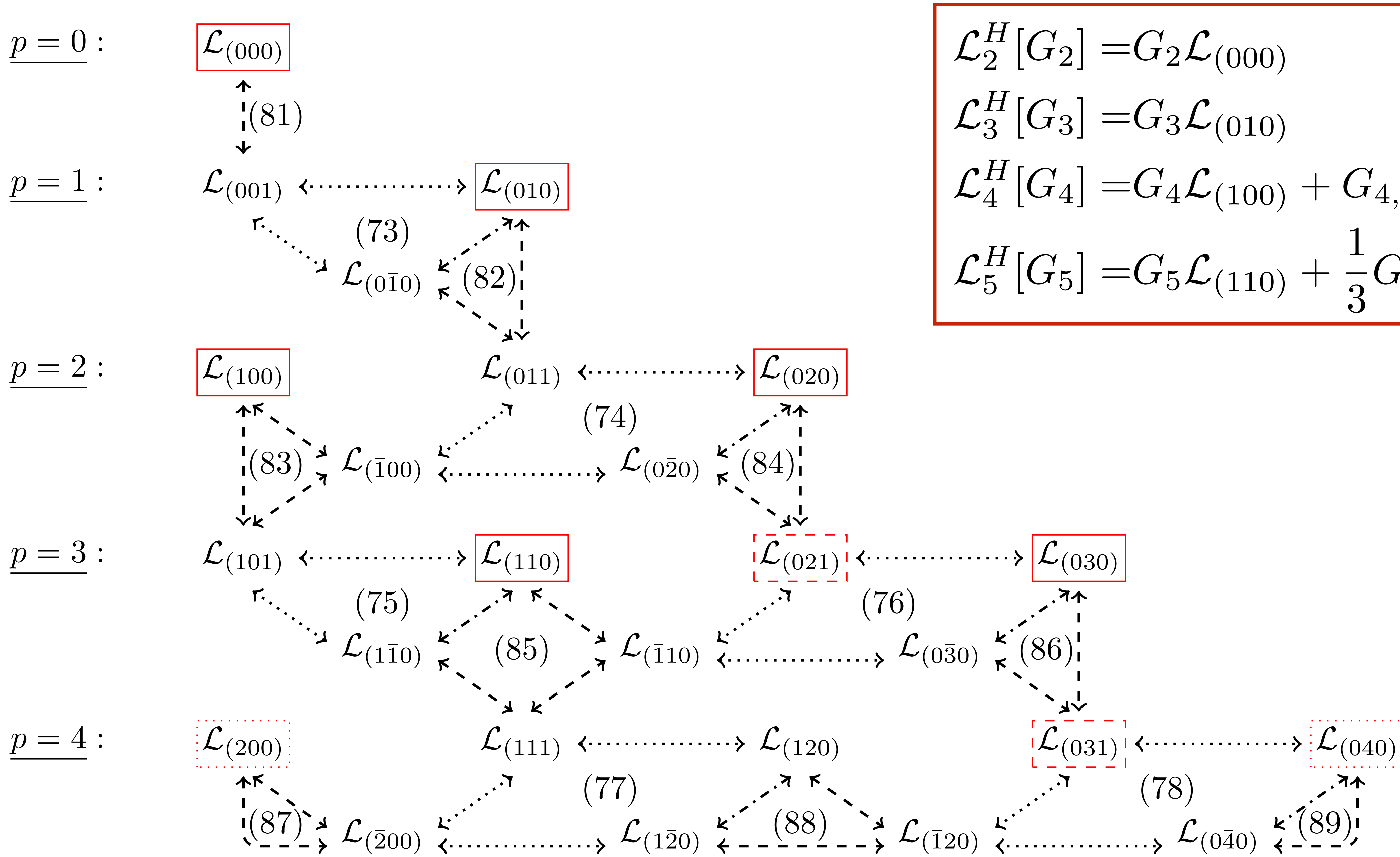

$\longleftrightarrow$ (dotted) $\quad \mathcal{D}\mathcal{L}^{D-1}_{(lmn)}[G_i] = G_{i,\phi}\mathcal{L}_{(lm(n+1))} - G_{i,X}\mathcal{L}_{(l(\overline{m+1})n)} + G_i\left(\mathcal{L}_{(l(m+1)n)} - m\mathcal{L}_{((\overline{l+1})(m-1)n)} - n\mathcal{L}_{(l(m+1)n)}\right)$

$\longleftrightarrow$ (dashed) $\quad \mathcal{L}_{(lm1)} = -2l\mathcal{L}_{(\bar{l}m0)} - m\mathcal{L}_{(l\bar{m}0)} - 2X\mathcal{L}_{(lm0)}$

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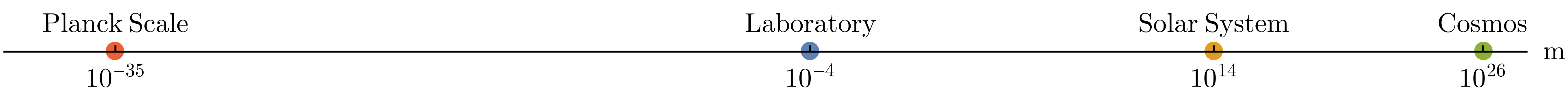
[Question] How can we test these general models?

4 Testing Modified Gravity

- Gravity can be tested at very different scales [Review by C. Will 2014]



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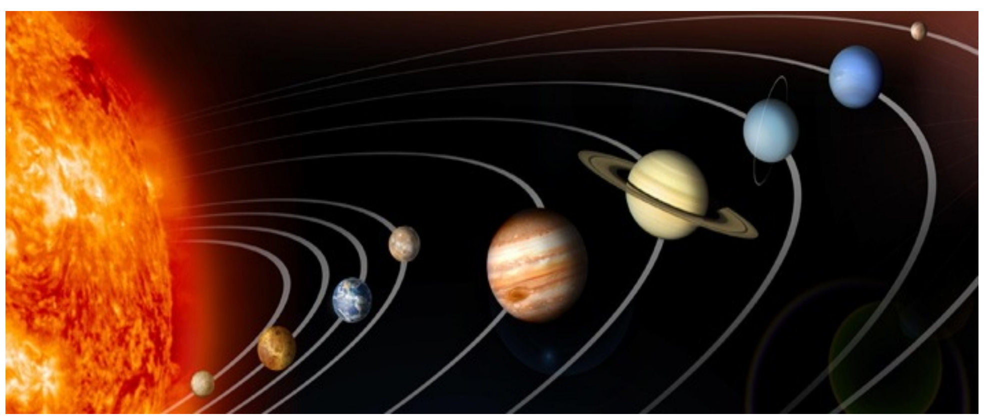

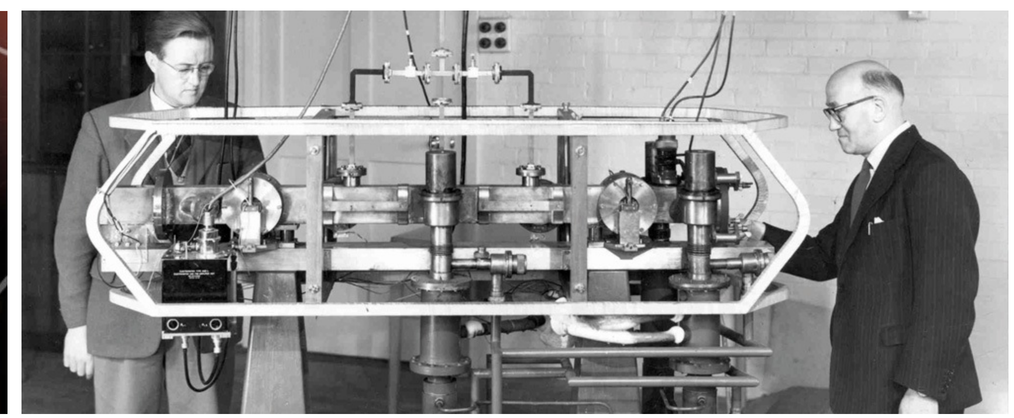

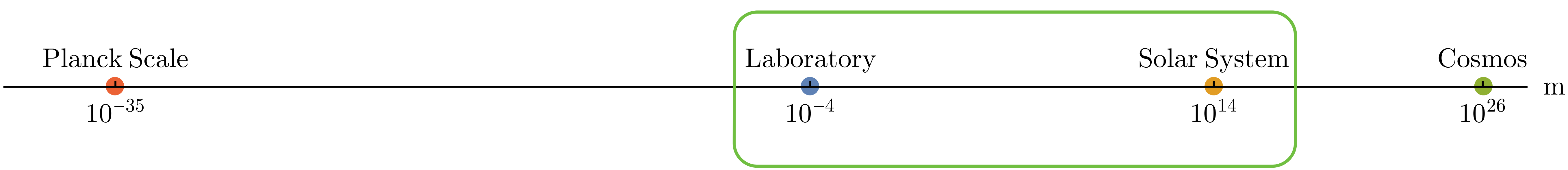

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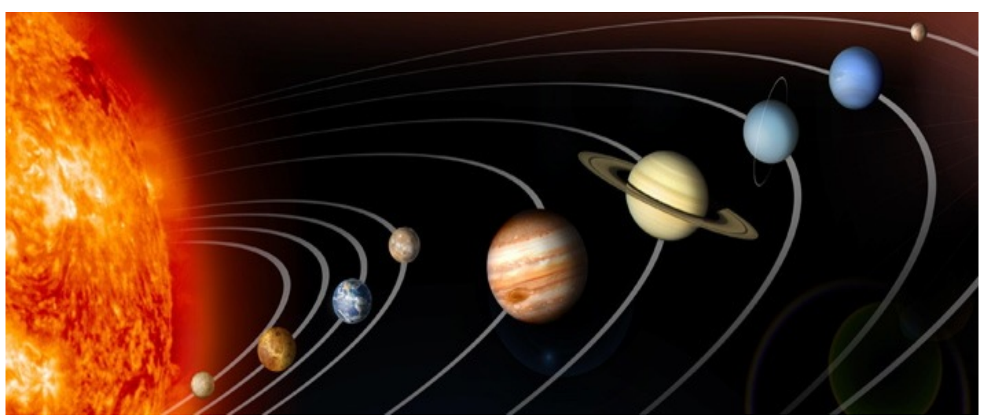
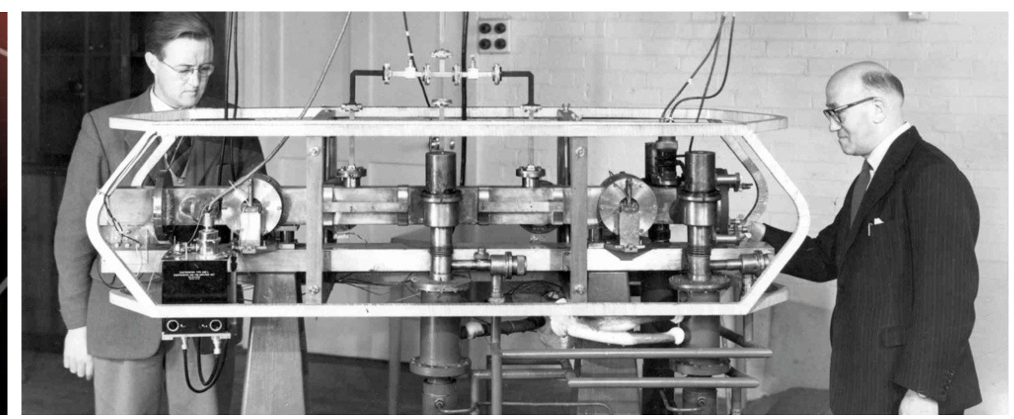
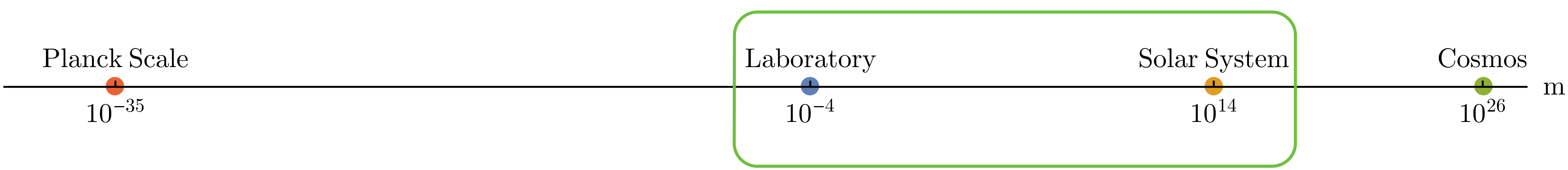



- Modified Gravity: Screening Mechanism [Review by P. Brax 2013]



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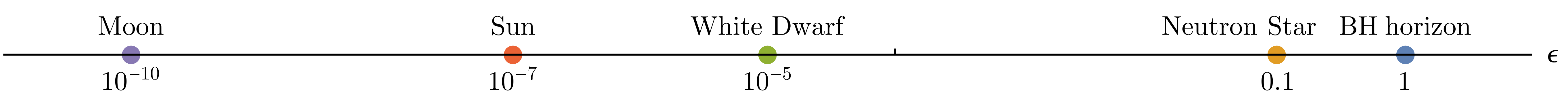

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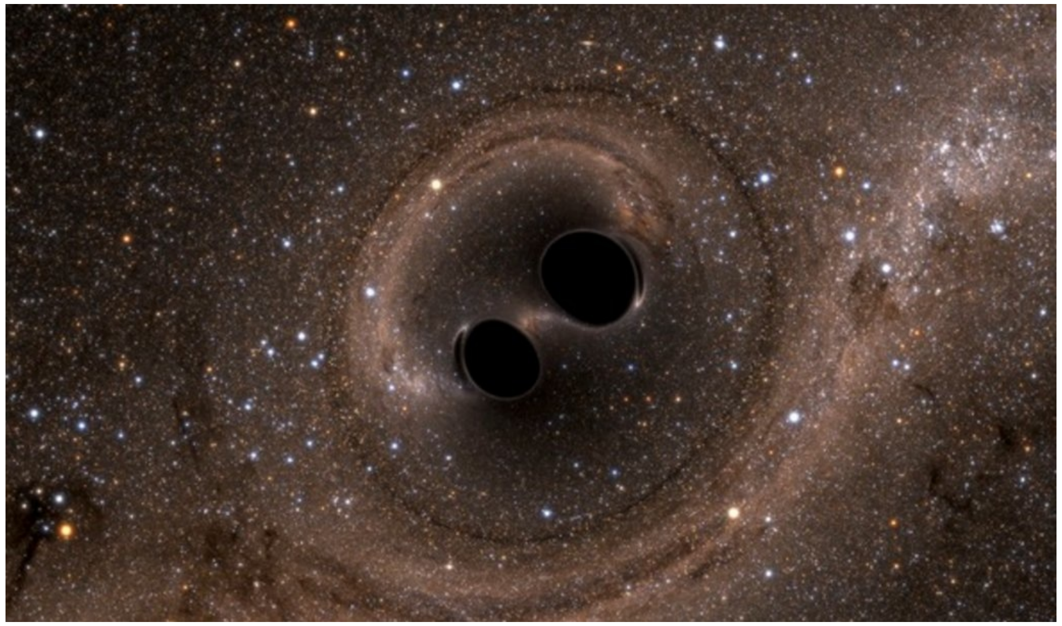
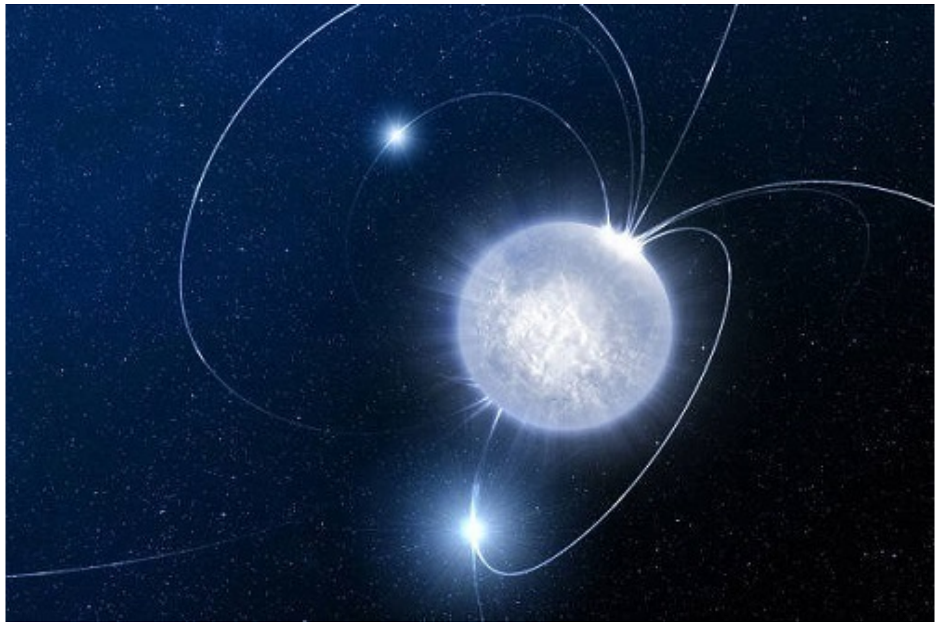
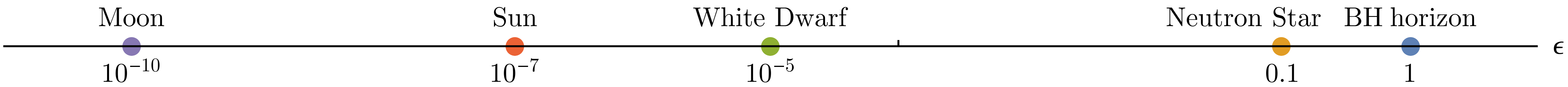
$$\boxed{\epsilon \equiv \frac{GM}{2rc^2}}$$







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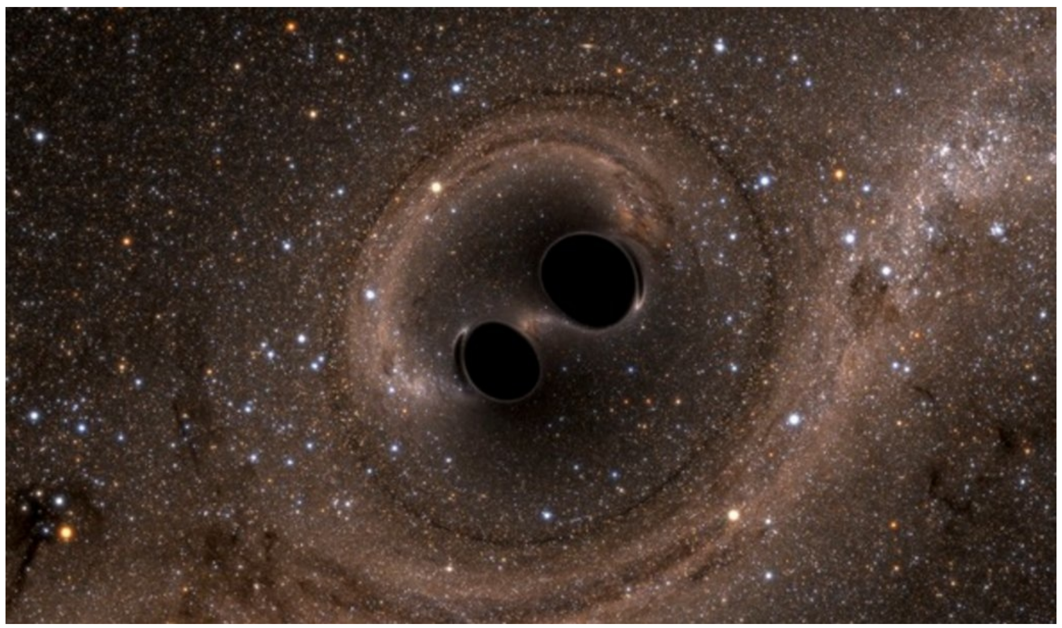
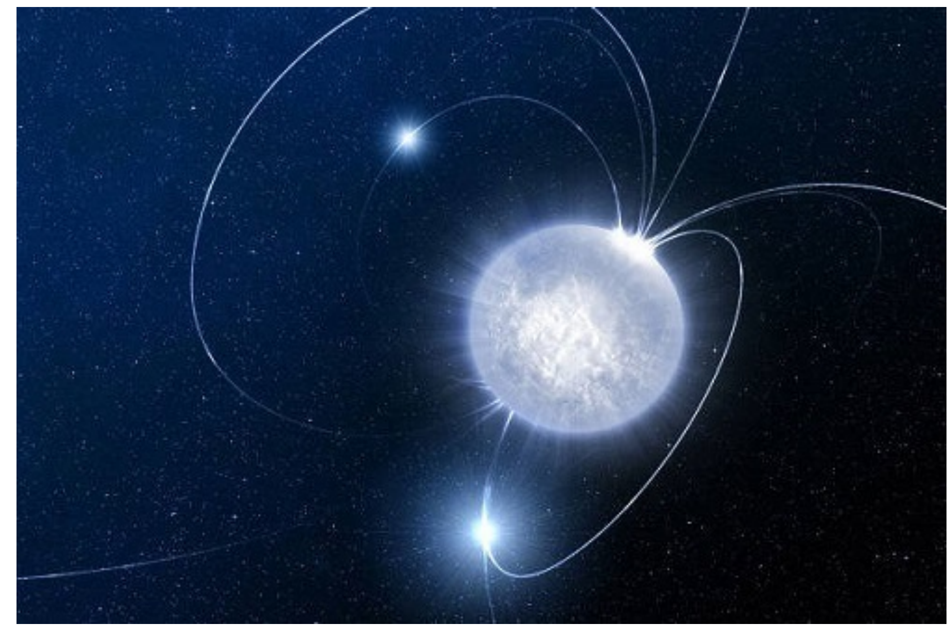
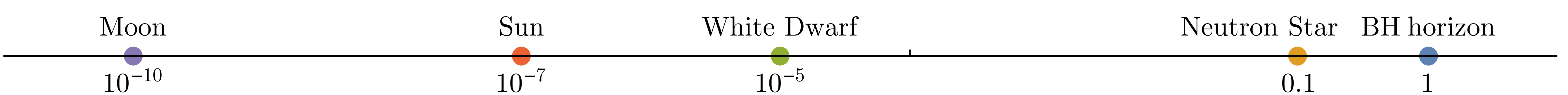




- Specific signatures in alternatives to GR, e.g. scalar radiation [Eardley 1974]



- Cosmological tests: CMB (T, B-modes), LSS (Lensing, Clustering), 21-cm...

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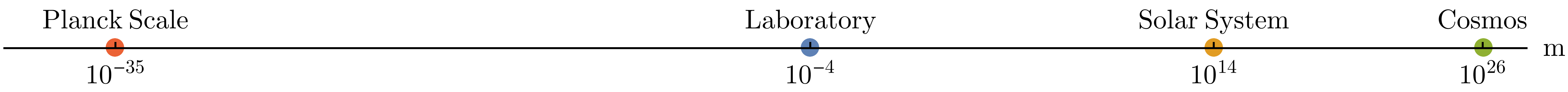

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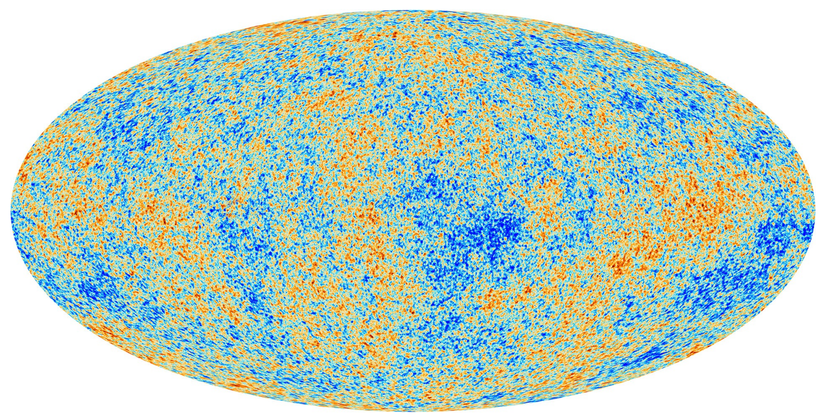

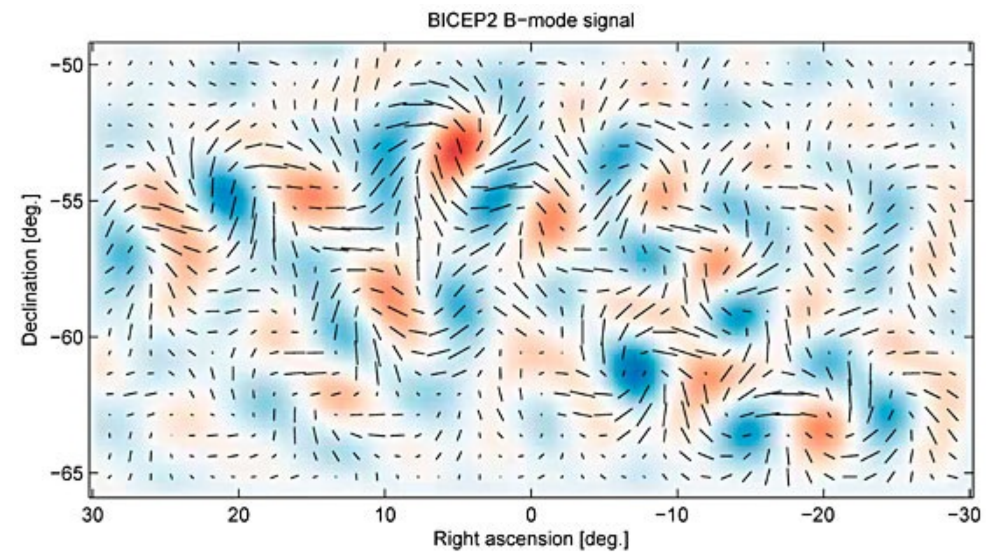

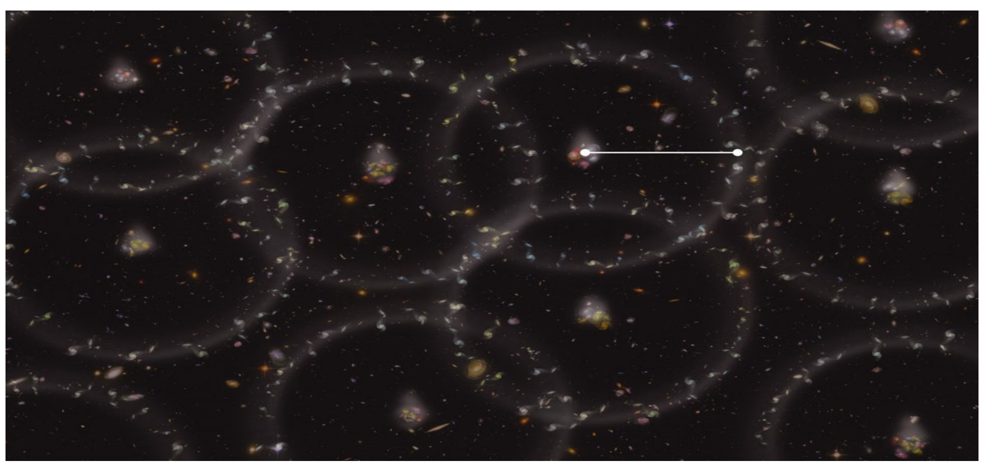

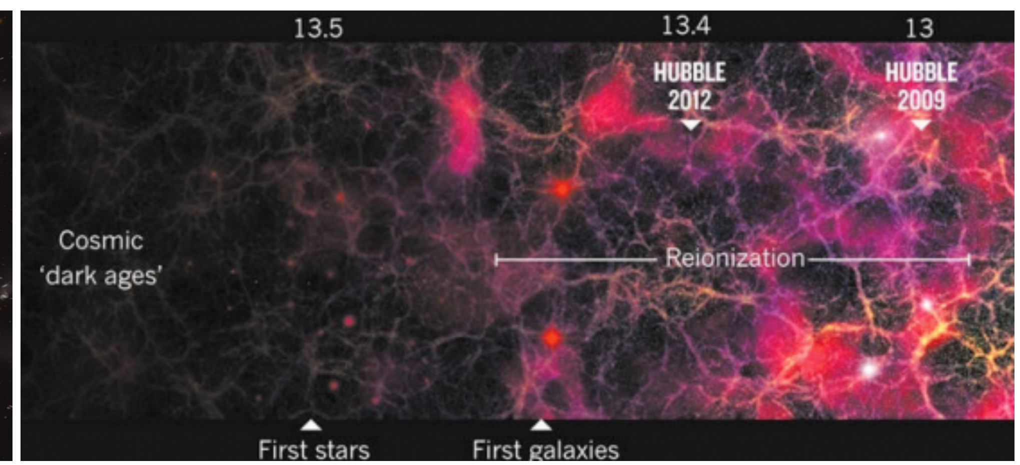

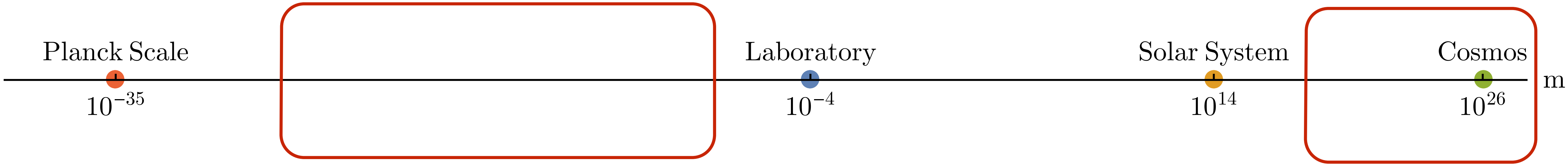

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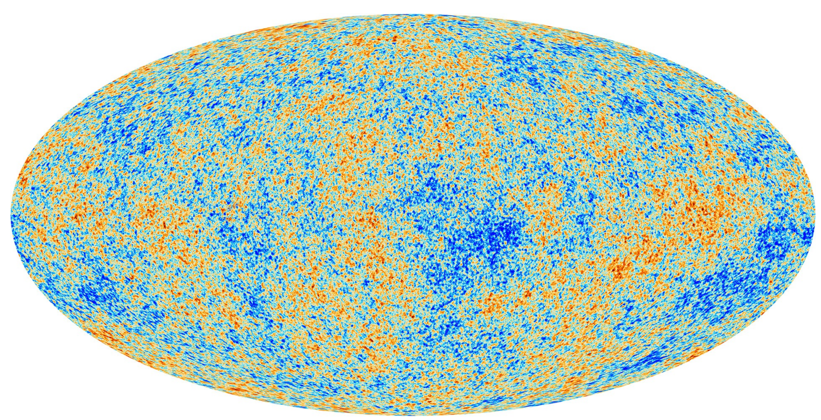
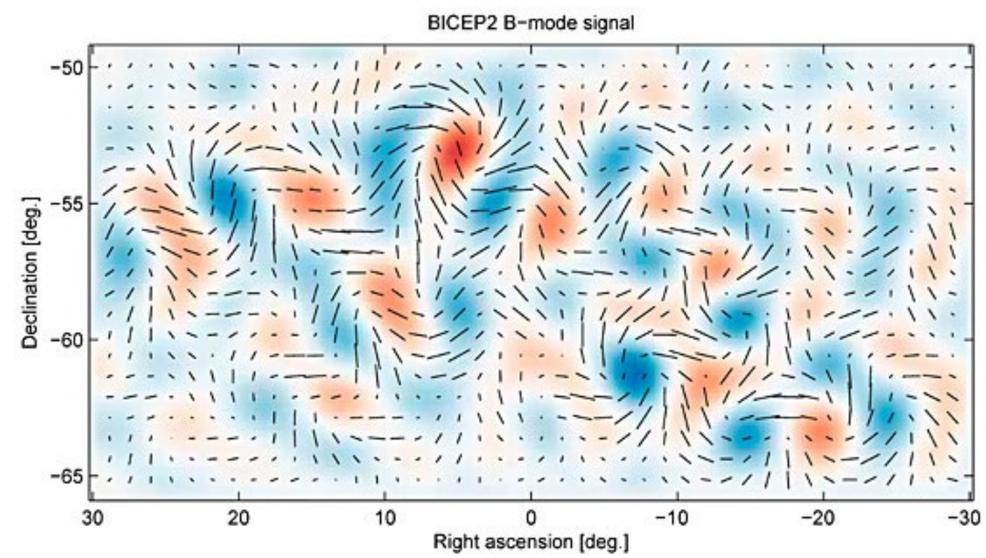
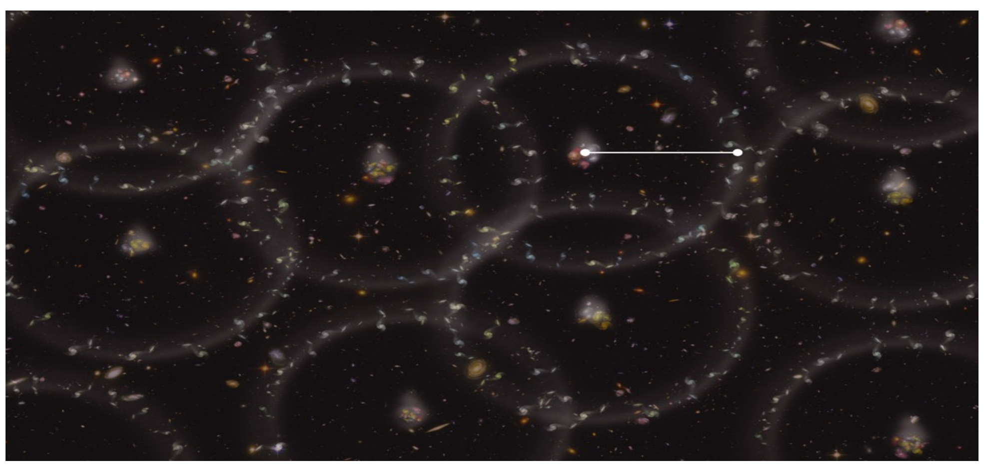
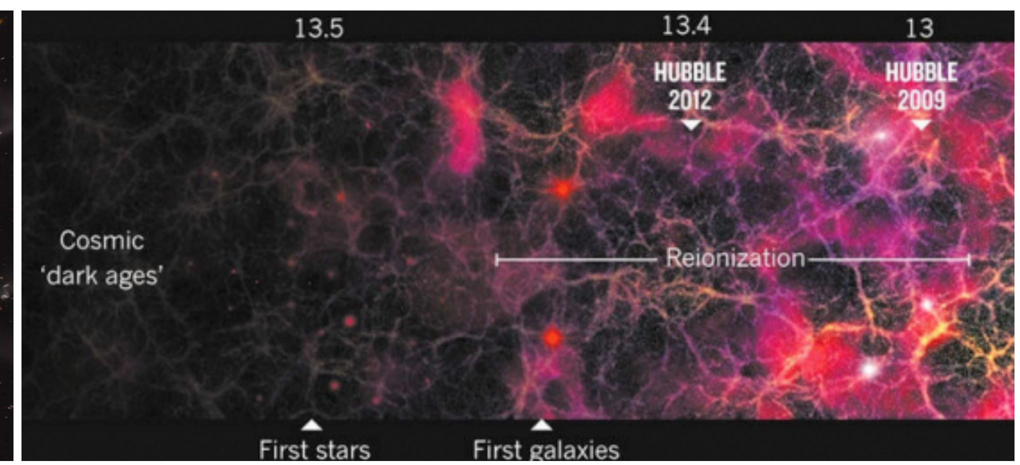
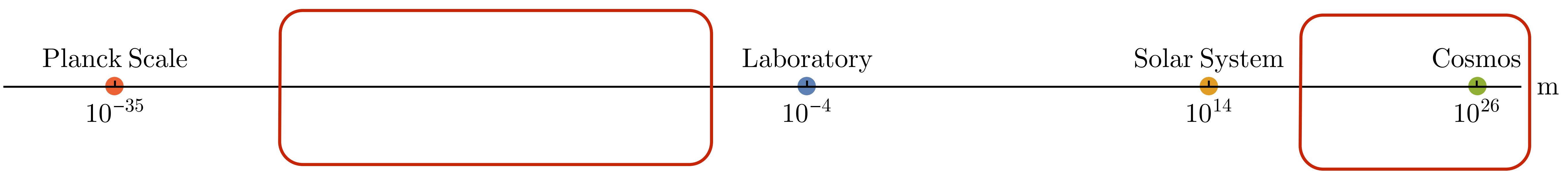




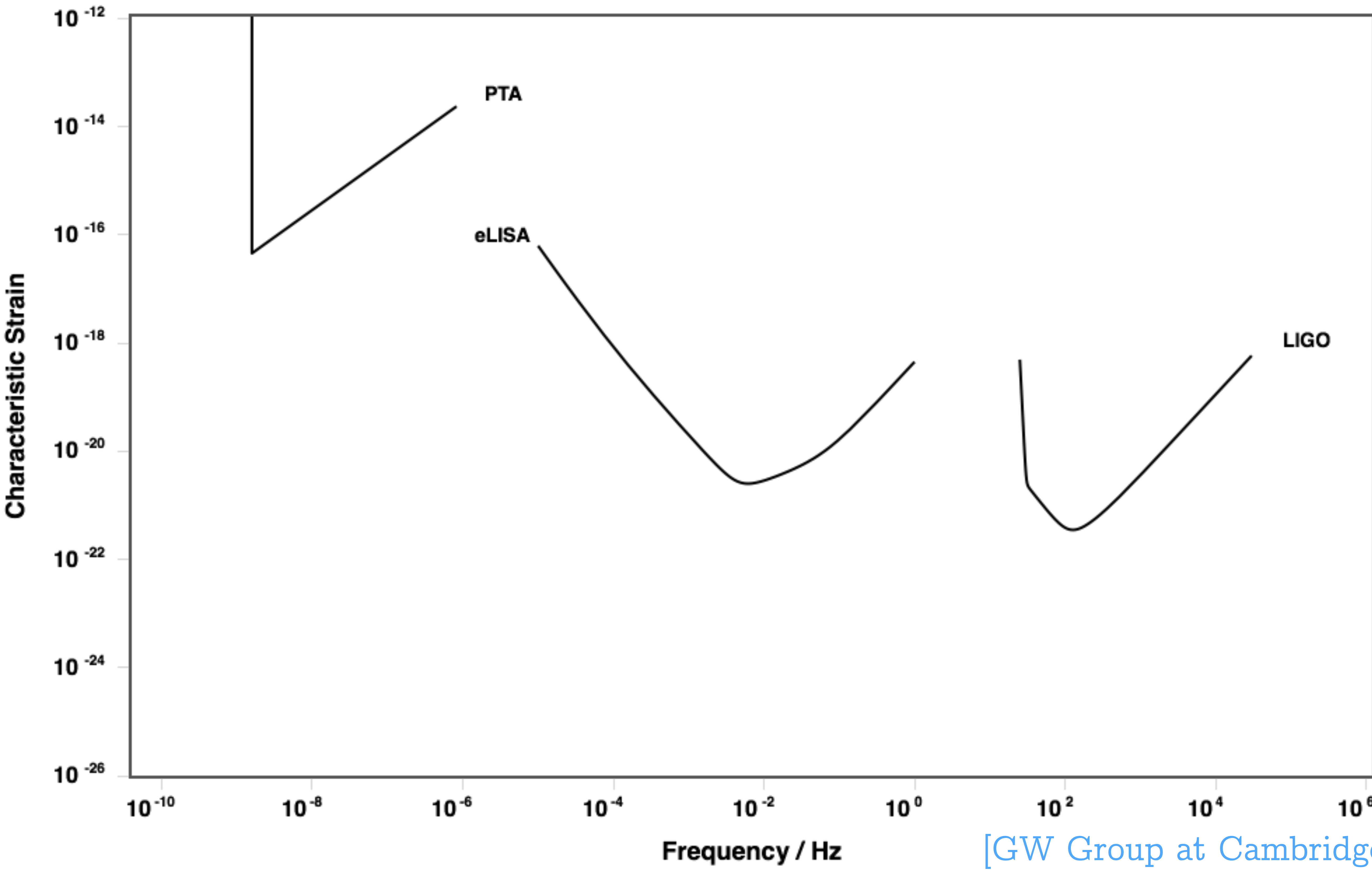




- Constraints on Horndeski: Present $\mathcal{O}(1-0.5)$ and Future $\mathcal{O}(0.1-0.01)$
[Bellini et al. 2016] [Alonso et al. 2016]



- New window to the Universe with Gravitational Wave Astronomy



[GW Group at Cambridge]

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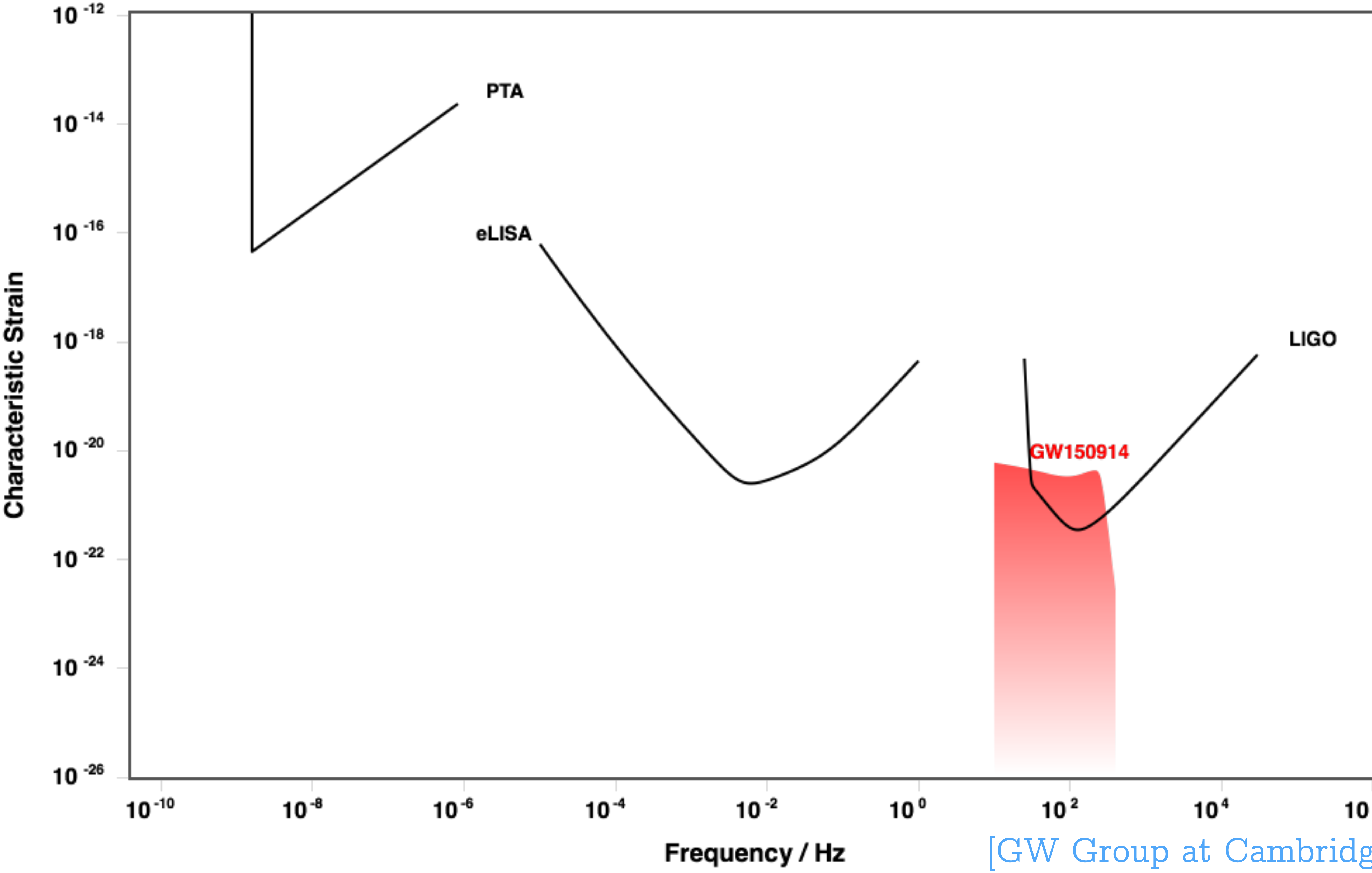

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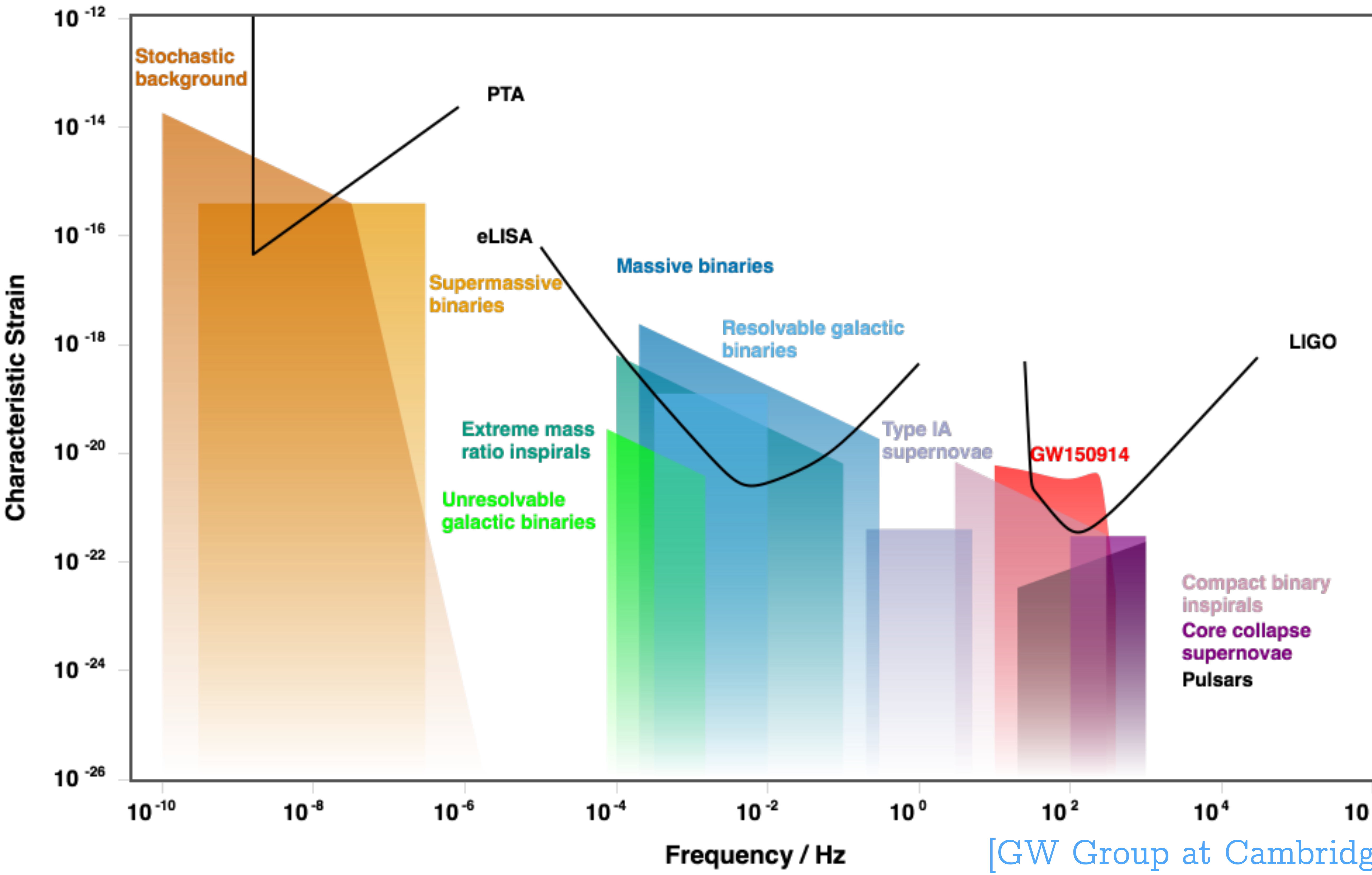

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5 The fate of Scalar-Tensor gravity

[arXiv 1608.01982]

- Fundamental analysis: Test speed of gravity

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- Fundamental analysis: Test speed of gravity
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- At small scales for arbitrary backgrounds

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$$\mathcal{G}_{\mu\nu} = G(X)g_{\mu\nu} + G'(X)\partial_\mu\phi\partial_\nu\phi \longrightarrow c_g^2 = \frac{G}{G - G'\dot{\phi}^2}$$

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B) $c_g \neq c$: No possible counterpart at cosmological scales

→ *Difference in the time of arrival becomes cosmological!*

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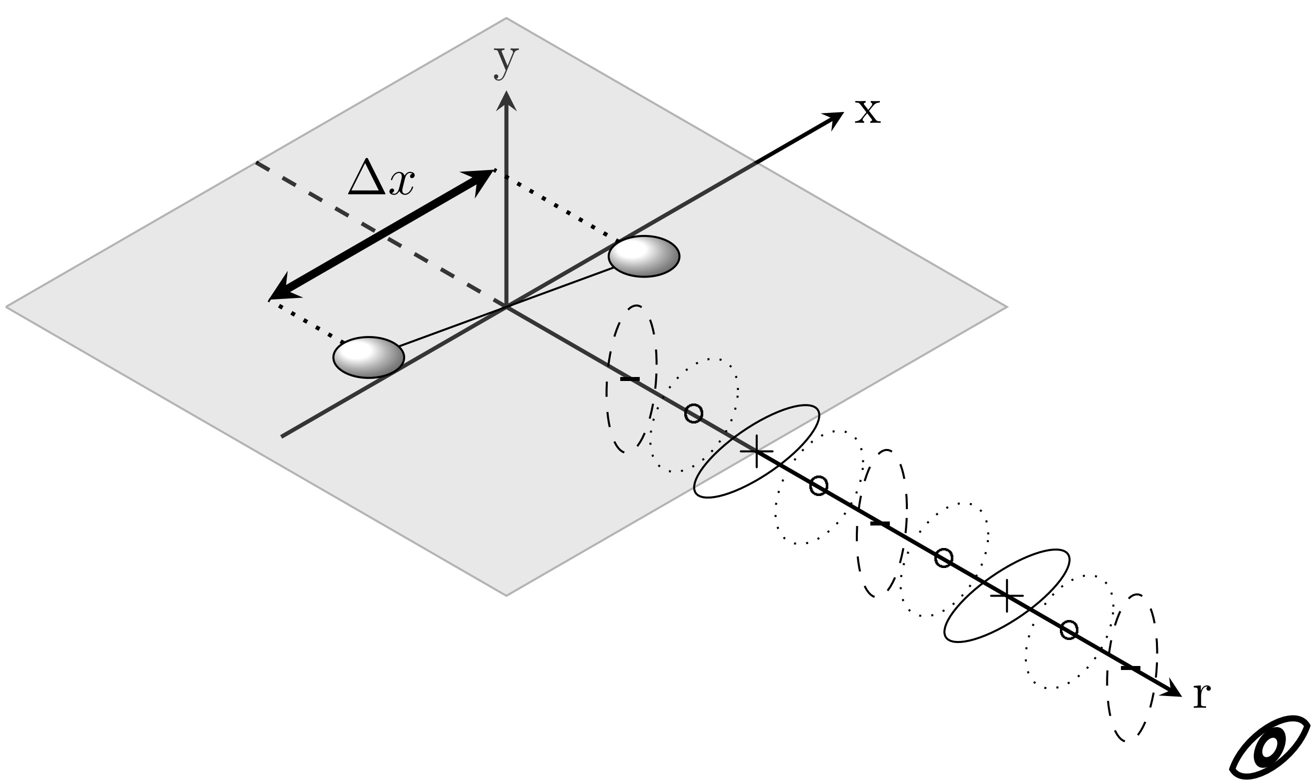

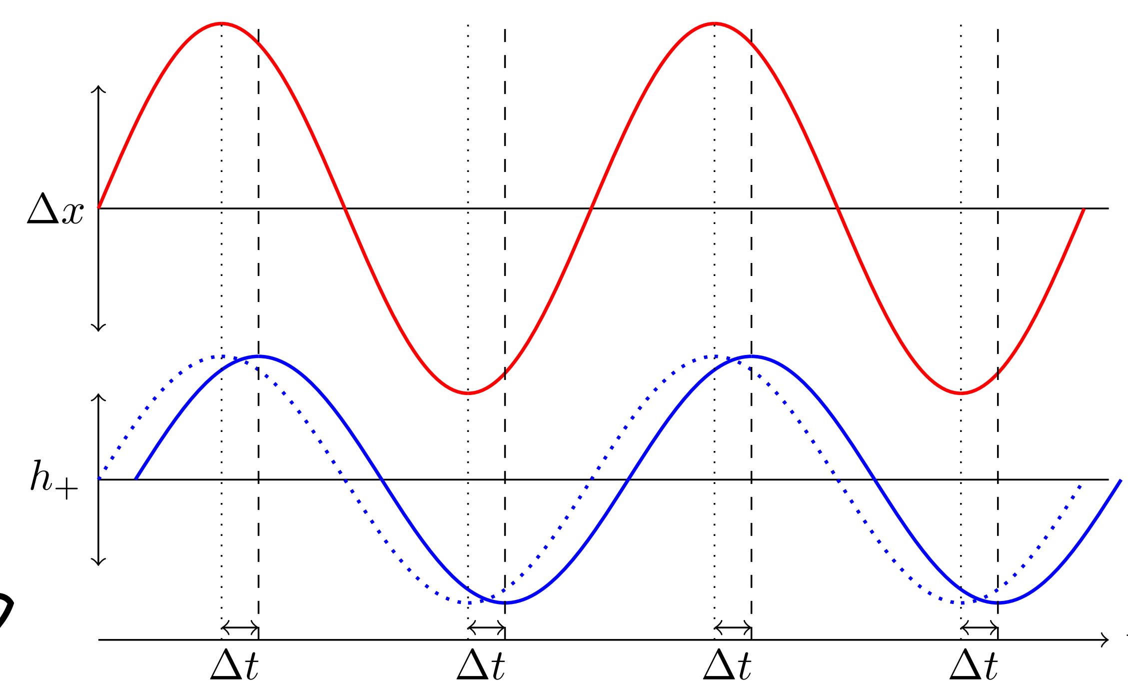

- Test speed of gravity with periodic sources [arXiv 1608.01982]

- Phase Lag Test: measure difference in phase of GWs and EM radiation

- There are sources already identified: eLISA verification binaries

  E.g. WDS J0651+2844 [Brown etal. 2012]

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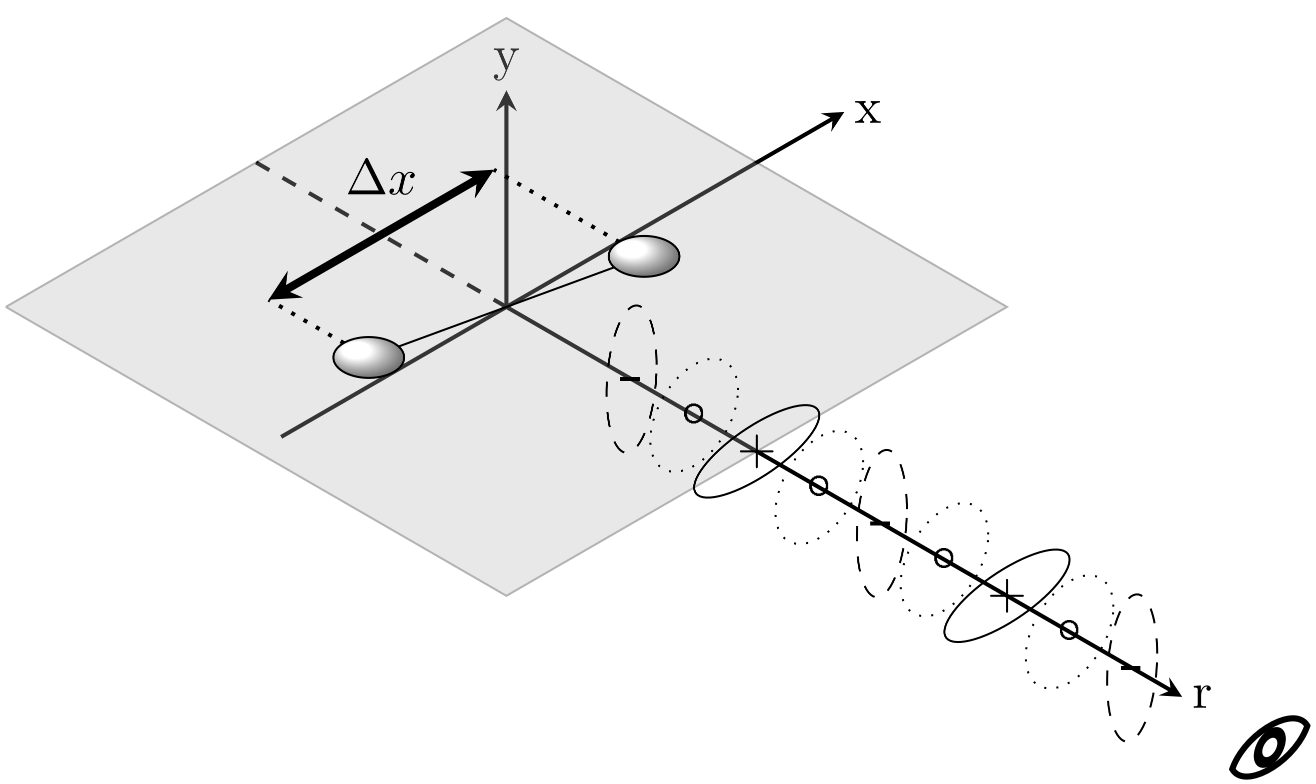

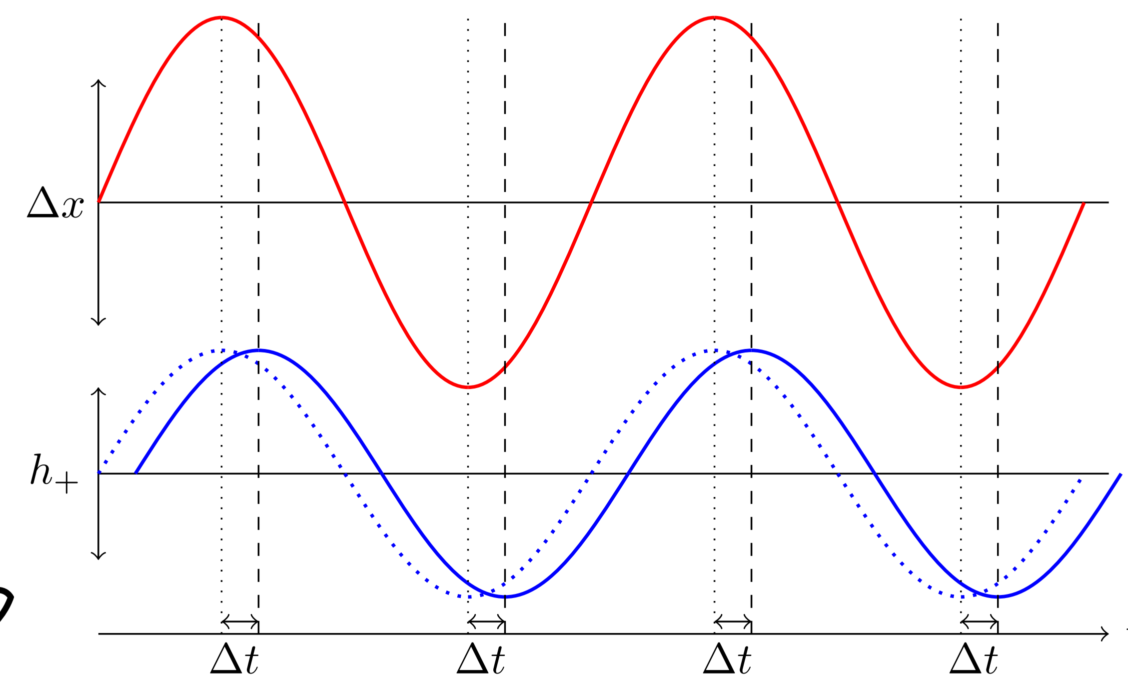

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- There are interesting potential applications of this new formalism both at the practical and conceptual level:
    - E.g. **fermions** in ST theories of gravity, explore general **field redefinitions** or systematically study ST theories with **higher derivative** EoM.

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- There are two possible scenarios:
    - If $c_g = c$: a GW-EM measurement will kill many ST theories
    - If $c_g \neq c$: need periodic sources (phase lag test)

Thank you

Find more details at

Phys. Rev. **D94**, 024005 (2016) by JME, J. GARCÍA-BELLIDO, M. ZUMALACÁRREGUI

arXiv **1608.01982** by D. BETTONI, **JME**, K. HINTERBICHLER and M. ZUMALACÁRREGUI

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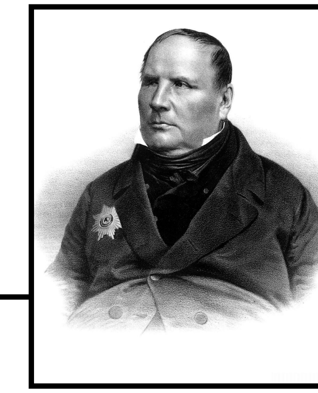
Back slides

2 Ostrogradski's Theorem



[Ostrogradski 1850]

- Hamilton's construction:

$$L = L(q,\dot{q}) \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \Longrightarrow \ddot{q} = F(q,\dot{q})$$

2 initial value data

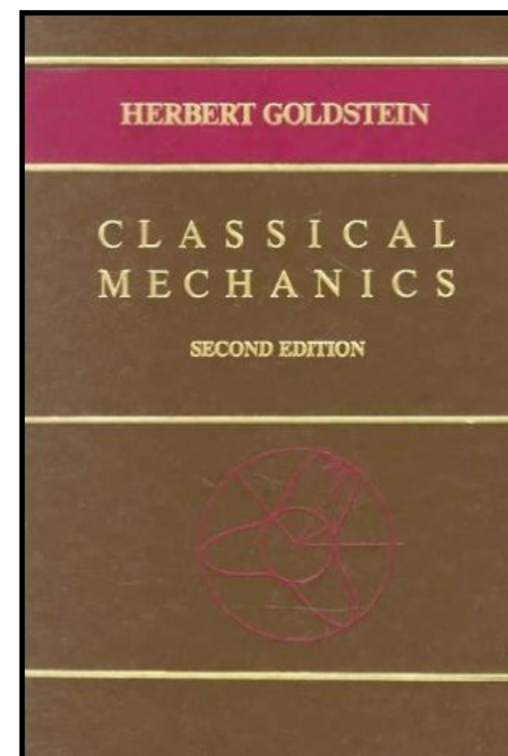
Non-degeneracy: $\dfrac{\partial^2 L}{\partial \dot{q}^2} \neq 0$

- Canonical variables (2D phase space)

$$Q \equiv q \qquad\qquad P \equiv \frac{\partial L}{\partial \dot{q}}$$

- Phase space can be inverted (non-degeneracy)

$$\boxed{H(Q,P) \equiv P\dot{q} - L}$$



- Ostrogradski's construction: [See review by Woodard 2015]

$$L = L(q, \dot{q}, \ddot{q}) \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0 \Longrightarrow \dddot{q} = F(q, \dot{q}, \ddot{q}, \dddot{q})$$

4 initial value data

Non-degeneracy: $\frac{\partial^2 L}{\partial \ddot{q}^2} \neq 0$

- Canonical variables (4D phase space)

$$Q_1 \equiv q \qquad Q_2 \equiv \dot{q} \qquad P_2 \equiv \frac{\partial L}{\partial \ddot{q}} \qquad P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}$$

- Phase space can be inverted (non-degeneracy)

$$\boxed{H(Q_1, Q_2, P_1, P_2) \equiv \sum_{i=1}^{2} P_i q^{(i)} - L = \underbrace{P_1 Q_2}_{\text{Linear in momentum: INSTABILITY!}} + P_2 A(Q_1, Q_2, P_2) - L(Q_1, Q_2, A)}$$

- There can be arbitrarily high positive and negative energy states!

- Healthy theories with higher derivatives EoM (*Beyond Horndeski*)

  Disformal transformations: $\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \nabla_\mu \phi \nabla_\nu \phi$

  Hidden Constraints [Zumalacárregui and García-Bellido 2013]

  ADM and Unitary Gauge: Generalized Generalized Galileons

  Two new Lagrangians [Gleyzes, Langlois, Piazza and Vernizzi 2014]

  Not healthy with full Horndeski [Crisostomi, Hull, Koyama, and Tasinato 2016]

  Degenerate Theories: Extended Scalar-Tensor Theories

  Hamiltonian Analysis [Langlois and Noui 2015]

  Full Classification [Crisostomi, Koyama, and Tasinato 2016]

- **Key Point**: Ostrogradski's Th. only limits time derivatives (Horndeski's theory was derived covariantly)

  $\longrightarrow$ There can be higher order spatial derivatives (inducing Lorentz breaking) [de Rham and Matas 2016]

- More on differential geometry:

Differential Forms Dictionary (extended)

$$\mathfrak{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \theta^a \otimes \theta^b$$

Metric Formalism

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$$

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

$$R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda{}_{\mu\rho} - \partial_\rho \Gamma^\lambda{}_{\mu\nu} + \Gamma^\gamma{}_{\mu\rho} \Gamma^\lambda{}_{\gamma\nu} - \Gamma^\gamma{}_{\mu\nu} \Gamma^\lambda{}_{\gamma\rho}$$

$$\nabla_\mu g_{\alpha\beta} = 0$$

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$$

Vielbein Formalism

$$\theta^a = e^a{}_\mu dx^\mu$$

$$\omega^{ab} = \frac{1}{2} \left(\mathfrak{i}_{e^b}(d\theta^a) - \mathfrak{i}_{e^a}(d\theta^b) + \mathfrak{i}_{e^a}(\mathfrak{i}_{e^b}(d\theta_c))\theta^c \right)$$

$$\mathcal{R}^a{}_b = \mathcal{D}\omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

$$\omega_{ab} = -\omega_{ba}$$

$$T^a = \mathcal{D}\theta^a = 0$$

- How to compute the EoM?

(i) Take perturbations:

$$\delta\Phi^a = \mathcal{D}\nabla^a\delta\phi,\ \delta\mathcal{R}^{ab} = 0,\ \delta G_i(\phi, X) = G_{i,\phi}\delta\phi - G_{i,X}\nabla_z\phi\nabla^z\delta\phi$$

(ii) Use (simple) relations between building blocks:

$$\mathcal{D}\Psi^a = \Phi^a \wedge \mathcal{D}\phi,\ \mathcal{D}\Phi^a = \mathcal{R}^a{}_z\nabla^z\phi,\ \mathcal{D}\mathcal{R}^{ab} = 0$$

(iii) Identify higher order terms:

$$\nabla^z\Phi^a = \nabla^a\Phi^z + \mathfrak{i}_{\nabla\phi}\mathcal{R}^{az}$$

$$\nabla^z\mathcal{R}^{ab} \wedge \theta^\star{}_{ab} = -2\nabla^a\mathcal{R}^{bz} \wedge \theta^\star{}_{ab}$$

(iv) Choose appropriate coefficients:

$$\left.\begin{array}{l}\delta(G_i\mathcal{L}_{(lmn)}) \to \text{ higher order } \sim \nabla^z\mathcal{R}^{ab} \\ \delta(F_i\mathcal{L}_{(l'm'n')}) \to \text{ higher order } \sim \nabla^a\mathcal{R}^{bz}\end{array}\right. \Rightarrow \boxed{F_i(G_i)}$$

- Example: EoM for quartic Horndeski $\begin{cases} G_4(\phi, X)\mathcal{L}_{(100)} \\ F_4(\phi, X)\mathcal{L}_{(020)} \end{cases}$

$$\begin{aligned}\delta(G_4\mathcal{L}_{(100)}) =&\delta G_4 \wedge \mathcal{R}^{ab} \wedge \theta^{\star}{}_{ab} \\ =&\delta\phi \wedge \left(G_{4,\phi} \wedge \mathcal{R}^{ab} + \nabla^z\left(G_{4,X}\nabla_z\phi\right) \wedge \mathcal{R}^{ab} + G_{4,X}\nabla_z\phi \wedge \boxed{\nabla^z\mathcal{R}^{ab}}\right) \wedge \theta^{\star}{}_{ab}\end{aligned}$$

$$\begin{aligned}\delta(F_4\mathcal{L}_{(020)}) =&\delta F_4 \wedge \Phi^a \wedge \Phi^b \wedge \theta^{\star}{}_{ab} + 2F_4 \wedge \delta\Phi^a \wedge \Phi^b \wedge \theta^{\star}{}_{ab} \\ =&\delta\phi \wedge \left(F_{4,\phi} \wedge \Phi^a + \nabla^z\left(F_{4,X}\nabla_z\phi\right) \wedge \Phi^a + 2F_{4,X}\nabla_z\phi \wedge \boxed{\nabla^z\Phi^a}\right) \wedge \Phi^b \wedge \theta^{\star}{}_{ab} \\ &+2\delta\phi \wedge \left(\nabla^a\left(F_{4,\phi}\mathcal{D}\phi\right) - \nabla^a\left(F_{4,X}\nabla_z\phi\right) \wedge \Phi^z\right) \wedge \Phi^b \wedge \theta^{\star}{}_{ab} \\ &+2\delta\phi \wedge \left(-F_{4,X}\nabla_z\phi \wedge \boxed{\nabla^a\Phi^z} \wedge \Phi^b \wedge \theta^{\star}{}_{ab} + \mathcal{D}\left(F_4\right) \wedge \nabla^a\Phi^b \wedge \theta^{\star}{}_{ab}\right) \\ &+2\delta\phi \wedge \left(\nabla^a\left(F_4\nabla_z\phi\right) \wedge \mathcal{R}^{bz} \wedge \theta^{\star}{}_{ab} + F_4\nabla_z\phi \wedge \boxed{\nabla^a\mathcal{R}^{bz}} \wedge \theta^{\star}{}_{ab}\right)\end{aligned}$$

$$\boxed{\begin{aligned}\mathcal{L}_4^H =&G_4 \wedge \mathcal{R}^{ab} \wedge \theta^{\star}{}_{ab} + G_{4,X} \wedge \Phi^a \wedge \Phi^b \wedge \theta^{\star}{}_{ab} \\ =&\left(G_4R + G_{4,X}\left([\Phi]^2 - [\Phi^2]\right)\right)\eta\end{aligned}}$$

- Example: EoM for arbitrary $\alpha_{lmn}(\phi, X)\mathcal{L}_{(lmn)}$

$$
\begin{aligned}
\delta(\alpha_{lmn}\mathcal{L}_{(lmn)}) =& \delta\alpha_{lmn}\wedge\mathcal{L}_{(lmn)} + \alpha_{lmn}\wedge\delta\mathcal{L}_{(lmn)} \\
=& \delta\alpha_{lmn}\wedge\mathcal{L}_{(lmn)} + m\alpha_{lmn}\wedge\delta\Phi^a\wedge[\mathcal{L}_{(l(m-1)n)}]_a + n\alpha_{lmn}\wedge\delta\Psi^a\wedge[\mathcal{L}_{(lm(n-1))}]_a \\
=& \delta\phi\wedge\Big(\big(\alpha_{lmn,\phi} + \nabla^z(\alpha_{lmn,X}\nabla_z\phi)\big)\wedge\mathcal{L}_{(lmn)} + \alpha_{lmn,X}\nabla_z\phi\big(l\nabla^z\mathcal{R}^{ab}\wedge[\mathcal{L}_{((l-1)mn)}]_{ab} \\
& + m\nabla^z\Phi^a\wedge[\mathcal{L}_{(l(m-1)n)}]_a + n\nabla^z\Psi^a\wedge[\mathcal{L}_{(lm(n-1))}]_a\big) + m\big(\nabla^a(\alpha_{lmn,\phi}\mathcal{D}\phi\wedge[\mathcal{L}_{(l(m-1)n)}]_a) \\
& - \alpha_{lmn,X}\nabla_z\phi\nabla^a\Phi^z\wedge[\mathcal{L}_{(l(m-1)n)}]_a - \Phi^z\wedge\nabla^a(\alpha_{lmn,X}\nabla_z\phi[\mathcal{L}_{(l(m-1)n)}]_a)\big) \\
& + m(m-1)\big(\alpha_{lmn}\nabla_z\phi\nabla^a\mathcal{R}^{bz}\wedge[\mathcal{L}_{(l(m-2)n)}]_{ab} + \mathcal{R}^{bz}\wedge\nabla^a(\alpha_{lmn}\nabla_z\phi[\mathcal{L}_{(l(m-2)n)}]_{ab})\big) \\
& + mn\nabla^a(\alpha_{lmn}\mathcal{D}\Psi^b\wedge[\mathcal{L}_{(l(m-1)(n-1))}]_{ab}) + n\alpha_{lmn}\wedge\delta\Psi^a\wedge[\mathcal{L}_{(lm(n-1))}]_a\Big)
\end{aligned}
$$

$$
\boxed{\mathcal{L}^{2^{nd}}(\alpha_{lmn}) = \alpha_{lmn}\mathcal{L}_{(lmn)} + \sum_{j=1}^{l}\alpha_{(l-j)(m+2j)n}\mathcal{L}_{((l-j)(m+2j)n)} + \sum_{k=1}^{m/2}\alpha_{(l+k)(m-2k)n}\mathcal{L}_{((l+k)(m-2k)n)}}
$$

$$
\begin{aligned}
\alpha_{(l-j)(m+2j)n} &= \frac{2(l-(j+1))}{(m+2j)(m+2j-1)}\frac{\partial(\alpha_{(l-(j-1))(m+2(j-1))n})}{\partial X}, \\
\alpha_{(l+k)(m-2k)n} &= \frac{(m-2(k-1))(m-1-2(k-1))}{2(l+k)}\int\alpha_{(l+(k-1))(m-2(k-1))n}dX
\end{aligned}
$$