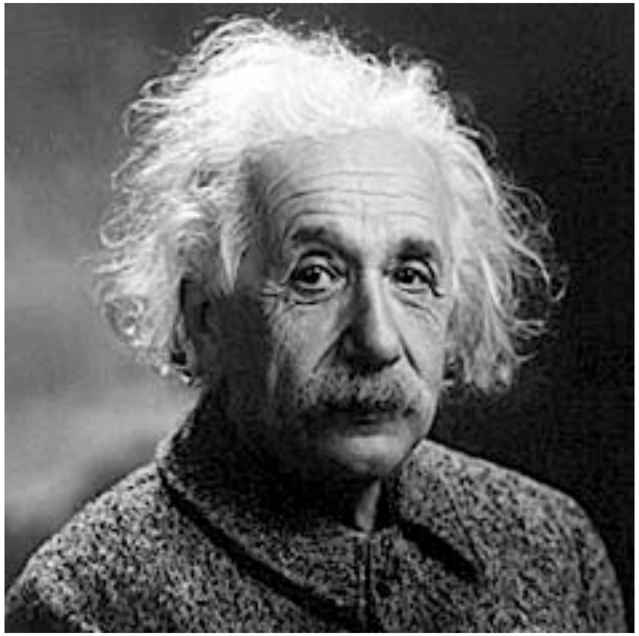
Supersymmetric Lorentzian Einstein-Weyl Spaces



HEP Theory
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Outline of the talk

- Set the stage: Killing spinors according to mathematicians
- Set the stage: Killing spinors according to sugra-ites
- Set the stage: Weyl geometry
- Set the stage: Einstein-Weyl spaces
- Killing spinors in Riemannian Weyl geometry
- A simple weighted Killing spinor equation in Lorentzian Weyl geometry
  - Integrability condition
  - How to extract information of the Killing spinor equations
  - On to the gory details
- Conclusions & outlook (as if I had any!)

Parallel spinors in Riemannian geometry

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multiplying with $R^{\mu\sigma}\gamma_\sigma$ leads to: $R_{\mu\nu} R^{\mu\nu}\, \epsilon = 0$

Whence in Riemannian geometry we see that: $\boxed{R_{\mu\nu} = 0}$

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Whence in Riemannian geometry we see that: $\boxed{R_{\mu\nu} = 0}$

- A Riemannian manifold endowed with a metric that admits a parallel spinor is necessarily Ricci flat.
- In Lorentzian, or pseudo-Riemannian, geometry the same argument leads to the fact that the Ricci tensor has to be null.

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As you surely know the matrix $[R_{\mu\nu}]_a{}^b$ are the generators of the local holonomy algebra $\mathfrak{hol}(\nabla) \subseteq \mathfrak{so}(n)$ in the vector representation.

Likewise, the combination $R_{\mu\nu}{}^{ab}\gamma_{ab}$ is the action of the local holonomy in the spinor representation.

The above condition can therefore be stated as:

Given the symmetry breaking pattern $\mathfrak{so}(n) \longrightarrow \mathfrak{hol}(\nabla)$:

what are the possible holonomy algebras, such that the branching of SO(n)'s spinor representation contains singlets?

Parallel spinors in Riemannian geometry

This question is a group theoretical question, and leads to the famous Berger List.

Taking into account that the manifold has to admit a Ricci flat metric one ends up with:

$\mathfrak{hol}(M)$	$\subset ?$	$\#\epsilon$	Name Geom. Struc.
$\mathfrak{su}(p)$	$\mathfrak{so}(2p)$	2	Kähler
$\mathfrak{sp}(r)$	$\mathfrak{so}(4r)$	$r+1$	Hyper Kähler
$\mathfrak{g}_2$	$\mathfrak{so}(7)$	1	G_2 structure
$\mathfrak{spin}(7)$	$\mathfrak{so}(8)$	1	Spin(7) structure

Killing spinors in Sugra

The Killing Spinor equations studied in physics are not as simple as the one before, and here are some simple examples:

N=2 d=4 minimal sugra: $$\nabla_\mu \epsilon_I = -\epsilon_{IJ} F^{+}{}_{\mu\nu} \gamma^\nu \epsilon^J$$

Fake d=4 De Sitter sugra: $$\nabla_a \epsilon_I = -\tfrac{iH}{2}\gamma_a \, \varepsilon_{IJ}\epsilon^J + H \, A_a \epsilon_I + i F^{+}_{ab}\gamma^b \varepsilon_{IJ}\epsilon^J$$

N=1 d=5 minimal sugra: $$\nabla_\mu \epsilon^i = \frac{1}{8\sqrt{3}} F^{ab}\left(\gamma_{\mu ab} - 4 g_{\mu a}\gamma_b\right)\epsilon^i$$

$$\vdots$$

M-theory: $$\nabla_a \epsilon = -\tfrac{i}{4!}\left[3 \not{G}_{(4)}\Gamma_a - \Gamma_a \not{G}_{(4)}\right]\epsilon$$

Coupling to matter multiplets, when possible, complicate the above rules. And even though the analysis seems daunting, powerful techniques have been devised over the last decades in order to be able to do just that.

What is Weyl geometry?

In 1924 Weyl was trying to unify General Relativity with Electro-magnetism, and decided that nature has no natural length scale but has a well-defined null-cone structure. In order to advance he introduced

• Weyl transformations, which you all know, is a local rescaling of the metric

$$g \longrightarrow \Omega^2(x)\; g$$

• the principle of Eichinvarianz, gauge invariance, meaning that physics should be invariant under Weyl transformations.

• In fact, he said that nature is not only covariant under diffeomorphism, but rather under conforma-diffeomorphisms.

What is Weyl geometry?

As the metric is charged under Weyl transformations, Weyl introduced the concept of a gauge connection A, and a gauge-covariant derivative such that

$$\mathtt{D}\, g = 2A \otimes g \qquad \text{or in components: } \mathtt{D}_\sigma\, g_{\mu\nu} = 2A_\sigma\, g_{\mu\nu}$$

The covariance of the above rule under Weyl rescalings is

$$g = e^{2\Omega}\, \tilde{g} \quad , \quad A = \tilde{A} + d\Omega$$

Observe: in ordinary differential geometry one deals with a manifold endowed with a metric. In Weyl geometry one deals with a manifold endowed with an equivalence class of metrics [g]. The calculations are done w.r.t. a chosen reference metric, i.e. $g \in [g]$.

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Weyl's covariant derivative can written in terms of the Levi-Cività connection and the gauge field as

$$\mathtt{D}_a Y_b \;=\; \nabla_a^{(g)} Y_b \;+\; \gamma_{ab}{}^c\, Y_c \quad \text{where} \quad \gamma_{ab}{}^c \;=\; g_a{}^c\, A_b \;+\; g_b{}^c\, A_a \;-\; g_{ab}\, A^c$$

In fact, the Weyl connection is torsion-free!

What is Weyl geometry?

Being a polymath, Weyl then defined the Riemann tensor, i.e. the field strength, of his connection by

$$[\mathtt{D}_a, \mathtt{D}_b]\, Y_c = -\mathtt{W}_{abc}{}^d Y_d \qquad \text{and the Ricci tensor by:} \quad \mathrm{W}_{ab} \;\equiv\; -\mathtt{W}_{acb}{}^c$$

Even though the connection is torsion-free, it is not contorsion-free, meaning that the Ricci tensor is not a symmetric tensor. A straightforward calculation shows

$$\mathtt{W}_{[ab]} \quad = \quad -\tfrac{n}{2}\, F_{ab} \qquad\qquad \text{where: } F \equiv dA$$

$$\mathtt{W}_{(ab)} \quad = \quad \mathtt{R}(g)_{ab} \;-\; (n-2)\nabla_{(a}A_{b)} \;-\; (n-2)\; A_a A_b \;-\; g_{ab}\; [\nabla_a A^a \;-\; (n-2)\; A_c A^c]$$

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$$\mathtt{W}_{[ab]} = -\tfrac{n}{2}\, F_{ab} \qquad \text{where: } F \equiv dA$$

$$\mathtt{W}_{(ab)} = \mathtt{R}(g)_{ab} - (n-2)\nabla_{(a}A_{b)} - (n-2)\, A_a A_b - g_{ab}\left[\nabla_a A^a - (n-2)\, A_c A^c\right]$$

By construction, the Riemann tensor and the Ricci tensor for Weyl's connection are invariant under Weyl transformations, i.e. conformal transformation.

The remaining question then is: how to do physics with this?

What is an Einstein-Weyl space?

Remember that the equations of motion in General relativity can be obtained by considering the coupling of a combination of the Ricci tensor and its contractions to a stress energy tensor T. The fact that the stress-energy tensor should be covariantly conserved, then immediately leads to the Einstein equations of motion

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In Weyl geometry, the basic ingredient is conformal invariance and any coupling by means of the Weyl-Ricci tensor and its contractions to an energy-momentum tensor must respect conformal invariance. If we couple this to the fact that the energy-momentum tensor of conformal matter is traceless we see that the only way to couple is

$$\boxed{\mathtt{W}_{(ab)} - \frac{1}{n} g_{ab} \, \mathtt{W} = T_{ab}}$$

What is an Einstein-Weyl space?

Now then we can give the definition of an Einstein-Weyl space: an EW space is a manifold endowed with an equivalence class of metrics such that

$$\mathtt{W}_{(ab)} \; - \; \frac{1}{n} \, g_{ab} \, \mathtt{W} \; = \; 0$$

As this equation is conformally invariant, it suffices to show this for a reference metric in order to be sure that it holds for all metrics in the equivalence class.

Why study EW spaces? Well, they have a richer structure.

Consider for example the 3-dimensional case: if the Weyl structure is closed/trivial, the EW equations imply that we are actually dealing with an ordinary Einstein space, R ~ g, which in 3 dimensions are maximally symmetric spaces and have no real freedom.

For non-closed EW spaces Tod in the eighties showed that the general metric can be given in terms of 4 real functions

Parallel spinors in Riemannian Weyl geometry

Given the Weyl connection, we can take the Killing spinor to satisfy

$$\boxed{\mathrm{D}_X \epsilon = 0}$$

In the Riemannian (Euclidean) case, this case was analysed by Moroianu (1996), who found that

✦ n <> 4: every Weyl geometry admitting Parallel spinors is exact, i.e. the metric is conformally related to a space admitting parallel spinors of the Levi-Cività connection, which are the spaces of exceptional holonomy in Berger's list [SU(N), Sp (N), G2, Spin(7)].

✦ n=4: the 4-dimensional manifold admits a quaternionic structure, which is integrable w.r.t. a combined Levi-Cività and R-connection. Even though Moroianu doesn't say it, this structure can be identified with a HyperKähler-Torsion structure (more later!).

- If the 4-dimensional space is compact, then the Weyl geometry is once again exact and conformally related to either a 4-torus, a K3 or the Hopf surface S1xS3 with the standard conformally flat metric.

Some more information on the literature

Buchholtz (1999/2000) considered the Killing Spinor Equation $\mathtt{D}\epsilon = wA\ \epsilon$ showing that depending on the weight w, the resulting Riemannian space could be a general Weyl space, an EW-space or in 3-dimensions a subclass of EW-spaces, dubbed Gauduchon-Tod.

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The little literature on Lorentzian EW spaces that I've been able to find deals with 3-dimensional EW spaces.

In 2001, Calderbank and Dunajski considered the case of scalar-flat EW spaces (W=0) and were able to show that there are only 2 distinct such spaces, to wit

Proposition 2 *Let (h, ω) be an Einstein–Weyl structure with vanishing scalar curvature. Then either (h, ω) is flat, or the signature is $(++-)$ and there exist local coordinates $x^i = (y, x, t)$ such that $\omega = y\mathrm{d}t$, and h is given by one of two solutions:*

$$h_1 = \mathrm{d}y^2 + 2\mathrm{d}x\mathrm{d}t + \Big(x[R(t) - \frac{y}{2}] + \frac{1}{48}y^4 + \frac{1}{12}R(t)y^3 + S(t)y\Big)\mathrm{d}t^2, \tag{4}$$

$$h_2 = \mathrm{d}y^2 + 2\mathrm{d}x\mathrm{d}t - \frac{4x}{y}\mathrm{d}y\mathrm{d}t + \Big(\frac{x^2}{y^2} + \frac{xy}{2} + \frac{1}{8}y^4 + R(t)y^2 + S(t)y\Big)\mathrm{d}t^2, \tag{5}$$

where $R(t)$ and $S(t)$ are arbitrary functions with continuous second derivatives.

Running ahead of myself, only case 1 will satisfy the spinorial rule that I'll consider.

A simple Killing spinor equation

Consider the following Killing Spinor Equation (KSE):

$$\nabla_a \epsilon = \frac{4-n}{4} A_a \epsilon + \frac{1}{2}\gamma_{ab} A^b \epsilon$$

This is nothing more than the criterion that the spinor is a weighted Killing spinor in Weyl geometry. The weight of the spinor is chosen for convenience!

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Its integrability condition leads to

$$\frac{1}{2}\gamma_a \not{F}\ \epsilon = \frac{1}{2}\ \mathrm{W}_{(ab)}\gamma^b \epsilon \xrightarrow{\text{contraction with } \gamma^a} n\ \not{F}\epsilon = \mathrm{W}\ \epsilon$$

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which allows us to rewrite the integrability condition as:

$$\tfrac{1}{2}\left[\; \mathrm{W}_{(ab)} \;-\; \tfrac{1}{n}\mathrm{W}\eta_{ab} \;\right]\; \gamma^b \epsilon \;=\; 0$$

Hadn't it been for the fact that we are interested in Lorentzian spaces, we would have deduced that we are dealing with EW spaces. As it stands, we will just analyse the KSE and impose the EW condition afterwards!

Observe that if we had chosen the weight differently, we would have obtained a more restrictive integrability condition, meaning that if we are interested in EW spaces, the above simple KSE has to most chance of giving generic results.

Extracting information out off the KSE

i) Assume that the spinor ϵ solves the KSE; by definition it is a classical spinor.
ii) Construct all possible non-vanishing spinor bilinears: these are not completely independent as the Fierz identities give quadratic relations between the various bilinears.

one real bilinear that always exists is: $L_\mu \equiv \bar{\epsilon}\gamma_\mu\epsilon$

and it either timelike, $g(L,L) > 0$, or null, $g(L,L) = 0$.

This bilinear is of the utmost importance and the explicit techniques used depend heavily on its character: the analysis is therefore usually split into a timelike case and a null case, and are considered independently.

iii) Use the KSE to derive differential constraints on the spinorial bilinears.
iv) The Fierz and diff. constraints give enough information as to write down an Ansatz for the metric and the gauge field, that automatically solves the KSE.
v) Of course you need to check that this really is the case!
vi) Impose the Einstein-Weyl equation in order to really find what you were looking for...

Timelike case: immediate consequences

Given the KSE we can calculate for arbitrary dimension

$$\nabla_a L_b \;=\; \frac{4-n}{2}\; A_a L_b \;-\; L_a A_b \;+\; \imath_L A\; g_{ab}$$

and suppose that we are in the timelike case, meaning that we can define the real and positive-definite function

$$f \equiv g(L,L) > 0$$

We can then calculate

$$df \;=\; (4-n)\; A\; f$$

So that for spacetime dimensions different from 4, the Weyl structure is closed/trivial.

Timelike case: 4 dimensions

Take the spinor to be Dirac or equivalently 2 complex Weyl spinors, then the set-up is equivalent to the spinorial structure used in N=2 d=4 supergravity and then we know what the possible bilinears are

- ➡ a complex scalar X ,
- ➡ four real vector bilinears V^a (a=0,1,2,3),
- ➡ a triplet of 2-forms $\Phi^x_{(2)}$ (x=1,2,3), that will play no rôle whatsoever.

The Fierz identities imply that the vector bilinears are locally linearly independent, which allows us to write the metric as

$$4|X|^2\, g \;=\; \eta_{ab}\, V^a \otimes V^b$$

Timelike case: 4 dimensions

A straightforward calculation of the differential constraint then leads to

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$$\Big\downarrow \text{ Linear independency}$$

$$F = 0$$

$$\Big\downarrow \text{ Locally}$$

$$A = dP$$

Timelike case: 4 dimensions

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$$
\begin{tikzcd}
dX = 0 \arrow[rr] & & \text{X is just some complex constant!} \\
dV^a = A \wedge V^a \arrow[rr, "\text{use } d^2=0"] \arrow[dr, "\text{Weyl transf.}"] & & F \wedge V^a = 0 \arrow[d, "\text{Linear independency}"] \\
& dV^a = 0 & F = 0 \arrow[d, "\text{Locally}"] \\
& & A = dP \arrow[ul, "\text{Weyl transf.}"]
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& & F = 0 \arrow[dd, "\text{Locally}"] \\
& dV^a = 0 \arrow[dl, "\text{coordinatise}"'] & \\
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 & & F = 0 \arrow[d, "\text{Locally}"] \\
 & dV^a = 0 \arrow[dl, "\text{coordinatise}"'] & A = dP \arrow[l, "\text{Weyl transf.}"] \\
V^a = dy^a & &
\end{tikzcd}$$

Conclusion: A 4 dimensional solution to the KSE in the timelike case is locally conformal to Minkowski space.

Null case: 4 dimensions

We could now go on and study the null case for the N=2 d=4 spinorial structure. Experience, however, shows that this case is equivalent to the case of the spinorial structure used N=1 d=4 sugra, which is by far simpler as that spinorial structure does not admit a timelike case. Whence….

But first remember that we had the general result:

$$\nabla_a L_b \;=\; \frac{4-n}{2}\, A_a L_b \;-\; L_a A_b \;+\; \imath_L A\, g_{ab}$$

The totally anti-symmetric version of this equation reads

$$d\hat{L} \;=\; \frac{6-n}{2}\, A \wedge \hat{L}$$

So that when n=6, we have that dL=0, so that we can introduce a coordinate u such that

$$\hat{L} \;=\; du$$

We will treat the case n=6 in more detail later on.

Null case: 4 dimensions

If n != 6, we can use $d\hat{L} \;=\; \frac{6-n}{2}\; A\wedge\hat{L}$ to deduce: $\hat{L}\wedge d\hat{L} = 0$

This means that the vector L is hyper-surface orthogonal, for which we can use Frobenius' theorem which ensures the existence of 2 real functions u and P such that

$$\hat{L} \;=\; e^{P}\; du$$

But, according to the above differential equation L has a non-trivial gauge weight, whence we can do a gauge/Weyl transformation in order to get rid of the function P.

After this Weyl transformation we see that

$$\hat{L} \;=\; du \;\xrightarrow{\text{which implies}}\; A \;=\; \Upsilon\;\hat{L} \quad \text{for some real function } \Upsilon$$

And while we are coordinatising, we can introduce another coordinate v, by aligning it with the flow generated by the vector field, i.e.

$$\boxed{L^{\mu}\partial_{\mu} \;=\; \partial_{v}}$$

Null case: 4 dimensions

Anyway, we were going to use the N=1 d=4 spinorial structure to discuss the null case. In that case we are dealing with one complex Weyl spinor, out of which one can build

- ➡ the null vector L, which we dealt with before, and
- ➡ an imaginary self-dual 2-form $\Phi_{(2)}\ \ (\star\Phi_{(2)} = i\Phi_{(2)})$.

Sparing you the details of the analysis, let it suffice to say that the solution to the KSE, w.r.t. the coordinates u, v, and a complex coordinate z, is completely determined by a function H depending on all coordinates, as long as

$$\boxed{\begin{aligned} ds^2_{(4)} \;&=\; 2du\,(dv \;+\; H du) \;-\; 2dzd\bar{z} \\ A \;&=\; -\partial_v H\ du \end{aligned}}$$

Null case: 4 dimensions

Having found the solution to the KSE, we must then impose the EW-condition: this leads, as predicted by the general theory of Killing Spinor Identities, to only one condition, namely:

$$\partial_u \partial_v H \ - \ H \partial_v^2 H \ = \ \partial \bar{\partial} H$$

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Follows by factorisation
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Follows by factorisation is never trivial

II) $H \; = \; v\partial F + v\bar{\partial}\bar{F} \; + \; \bar{z}\partial_u F + z\partial_u \bar{F}$

where $F = F(u, z)$.

This solution is Weyl-scalar flat and in fact is the general of such solutions.

It gives rise to a non-trivial EW space as long as $\partial^2 F \neq 0.$

Null case: 6 dimensions

In this case we can take the spinor to be Weyl, which corresponds to the spinorial structure used in chiral N=(1,0) d=6 sugra. The bilinear structure is

- ➡ the null vector L, which as we know is closed: L=du
- ➡ a triplet of self-dual 3-forms: $\Phi^x_{(3)} \quad (\star\Phi^x = \Phi^x)$.

The Fierz identities then imply that:

$$
\begin{aligned}
L_a L^a &= 0 \\
\imath_L \Phi^r_{(3)} &= 0 \qquad \longrightarrow \qquad \hat{L}\wedge\Phi^r_{(3)} = 0 \\
\Phi^{r\ fab}\Phi^s_{fcd} &= 4\delta^{rs}\ L^{[a}L_{[c}\ \eta^{b]}_{d]} - \varepsilon^{rst}L^{[a|}\Phi^{t\ |b]}{}_{cd} + \varepsilon^{rst}L_{[c}\Phi^{t\ ab}{}_{d]}
\end{aligned}
$$

Yes, I am only flashing these formulas in order to fill this slide! The full result of the analysis is

Null case: 6 dimensions

The result is that in order to solve the KSE we must have

$$ds^2_{(6)} = 2du\,(dv + H du - 2v\,\mathsf{A} + \varpi) - \mathsf{h}_{mn}\,dy^m dy^n$$

$$A = -\tfrac{1}{2}\partial_v H\; du + \mathsf{A}$$

Observe that the form of the metric is wave-like: this is due to the nullity of L.

Null case: 6 dimensions

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$$A \;=\; -\tfrac{1}{2}\partial_v H\; du \;+\; \mathsf{A}$$

Observe that the form of the metric is wave-like: this is due to the nullity of L.

The only non-explicit v-dependence
resides in H.

Null case: 6 dimensions

The result is that in order to solve the KSE we must have

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This is a 4-dimensional Riemannian metric, called generically the base-space metric. For supersymmetry it must have special properties.

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The 4-dimensional metric h, allows for a quaternionic structure J^r (r=1,2,3) and therefore satisfies $\mathsf{J}^r\mathsf{J}^s = -\delta^{rs} + \varepsilon^{rst}\,\mathsf{J}^t$. This structure is compatible with the metric, which means that

$$\mathsf{h}(\mathsf{J}^r X, Y) \equiv \mathsf{K}^r(X, Y)$$

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The condition of the integrability of the above structure is best written by introducing the totally anti-symmetric torsion $\mathsf{T}_{(3)} = -\star_{(4)}\mathsf{A}$, and then

$$(\nabla + \mathsf{T})_m\; \mathsf{K}^r_{pq} = 0$$

The resulting structure is called Hyper-Kähler-Torsion, and if we couple it to Moroianu's results we see that the pair (h, A), forms a 4-dimensional Riemannian EW space.

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Due to the HKT property A must be a selfdual connection

$$\star_{(4)}\mathsf{F} = \mathsf{F}$$

where $\mathsf{F} = d\mathsf{A}$, *i.e.* it is an instanton.

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Define the covariant derivative of ϖ as $\mathsf{D}\varpi \equiv \mathsf{d}\varpi - 2\mathsf{A}\wedge\varpi$, then (for u-independent h)

$$\star_{(4)}\mathsf{D}\varpi = \mathsf{D}\varpi$$

Null case: 6 dimensions

Imposing then the EW condition leads to, restricting ourselves to base-space metrics that are u-independent, one imposing differential equation for H:

$$2\partial_u\partial_v H - 2H\partial_v^2 H \; + \; (\partial_v H)^2 \; = \; \left(\nabla_m^{(\mathsf{h})} - S_m\partial_v - 4\mathsf{A}_m \right) \left(\partial_m - S_m\partial_v - 2\mathsf{A}_m \right) \; H \; .$$

where S is short-hand for $S \; = \; -2v \; \mathsf{A} \; + \; \varpi$

We are looking for interesting solutions.

Null case: n !=4, 6 dimensions

In general dimensions applying the bilinear approach becomes a daunting task, but by knowing that we have a null vector L that must satisfy

$$\nabla_a L_b \;=\; \frac{4-n}{2}\, A_a L_b \;-\; L_a A_b \;+\; \imath_L A\; g_{ab}$$

- ★ Make a wave-like Ansatz for the metric (follows from nullity
- ★ Use the fact that we can do a Weyl transformation such that L=du
- ★ choose the base-space metric to be u-independent (not really needed but simplifies the resulting equations)
- ★and analyse the KSE by hand

The result is that

Null case: n !=4, 6 dimensions

$$ds^2_{(n)} = 2du\,(dv + Hdu + \varpi_m dy^m) - h_{mp}dy^m dy^n$$

$$A = -\frac{2}{n-2}\,\partial_v H\,du$$

only v-dependence resides in H

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The EW-condition then implies that H must satisfy

$$2\partial_u\partial_v H - 2H\partial_v^2 H + 2\tfrac{n-4}{n-2}\ (\partial_v H)^2 = -\left(\nabla^{(h)} - \varpi\right)^m (\partial_m - \varpi_m\partial_v)H$$

Conclusions & outlook

➡ We introduced the concept of Einstein-Weyl spaces

➡ put forward a simple Killing Spinor Equation which if it allows for a solution then it is almost an EW space

➡ We analysed the KSEs and found the conditions for it to be solved and give rise to an EW space.

➡ What to do with it? Well, it might be interesting to mathematicians (about 6 of them..)

➡ It may find applications in Conformal supergravities, but this needs looking into..