Background Material for the course

"Solitons and Instantons for senior graduates"

— Consider the oscillator $\mathcal{L} = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$

$\longrightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \rightsquigarrow m\ddot{x} = -kx$

$$\boxed{\ddot{x} + \omega^2 x = 0}$$ linearity $\omega^2 = \frac{k}{m}$

$$\boxed{x(t) = \underset{Im}{Re}\{A e^{i\omega t}\}} \rightsquigarrow \boxed{\ddot{x} = -\omega^2 x(t)}$$ ✓

Consider now a "field" $\phi(x,t) = \phi(x^\mu)$

whose ~~Lagrangian is~~ eq of motion is

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) - \frac{\partial^2}{\partial x^2}\phi(x,t) = 0} \longrightarrow \text{wave eq } c=1$$

linear eq

$\eta_{\mu\nu} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix}$

Solution $\phi(x,t) = Re\, A e^{i(\omega t - kx)}$

$\partial_t^2 \phi = -\omega^2 \phi$

$\partial_x^2 = -k^2 \phi$

$(\omega^2 - k^2)\phi(x,t) = 0 \rightsquigarrow \boxed{\omega^2 = k^2}$

if the field has a mass $\boxed{\partial_t^2\phi - \partial_x^2\phi = -m^2\phi}$ linear eq

$\phi(x,t) = Re\, A e^{i(\omega t - kx)}$ where $\boxed{-\omega^2 + k^2 = -m^2} \rightsquigarrow \boxed{\omega^2 = k^2 + m^2}$

the Lagrangian from where this is derived

$$\boxed{\mathcal{L} = \frac{1}{2}\underbrace{\partial_\mu\phi\,\partial^\mu\phi}_{\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi} - \frac{m^2}{2}\phi^2}$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu(\eta^{\mu\nu}\partial_\nu\phi) = \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\boxed{\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = -m^2\phi} \longrightarrow \boxed{\partial_t^2\phi - \partial_x^2\phi = -m^2\phi}$$

OK?

Now come back to the oscillator

$$\mathcal{L} = m\frac{\dot{x}^2}{2} - \frac{kx^2}{2} - \frac{\lambda x^4}{4!} \longrightarrow V = \frac{\lambda}{4!}x^4 + \frac{kx^2}{2}$$

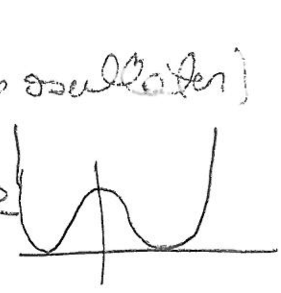
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} \longrightarrow \boxed{m\ddot{x} = -kx - \frac{\lambda x^3}{3!}}$$ non linear

if $\lambda \sim 0$ perturbation theory in λ

if $\lambda \sim 1$? non perturbative problem $\boxed{\text{what can we do?}}$

one can choose a problem (not a perturbation of the oscillator)

$$\mathcal{L} = m\frac{\dot{x}^2}{2} - V(x) \qquad V(x) = \frac{\lambda}{4!}(x^2 - v^2)^2$$



and study things close to $\underline{x = +v}$

$$x = v + \varepsilon z(t) \longrightarrow \dot{x} = \varepsilon \dot{z}$$

$$(x^2 - v^2)^2 = \left(2\varepsilon v z + \varepsilon^2 z^2\right)^2 = 4\varepsilon^2 v^2 z^2(t) + \mathcal{O}(\varepsilon^3)$$

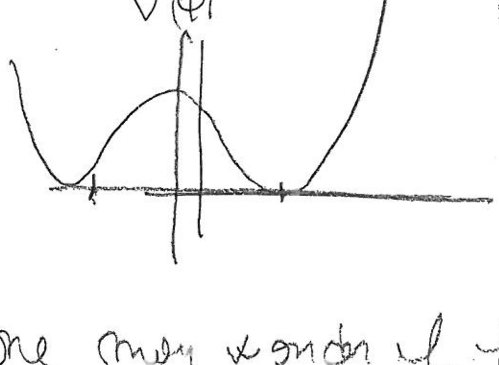
$$\mathcal{L} = m\frac{\dot{z}^2}{2} - \frac{\lambda}{\cancel{16}}4\varepsilon^2 v^2 z^2 \quad \rightsquigarrow \text{ an oscillator close to } \underline{\underline{x = v}}$$

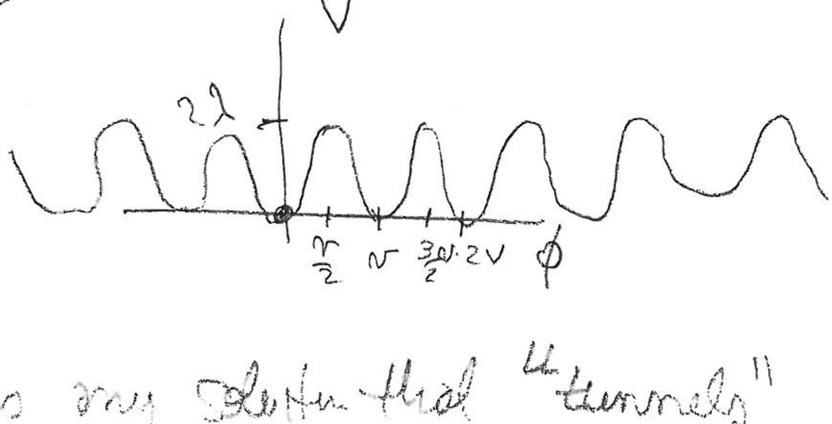
Let us do the same for the field $\underline{\phi(x,t)}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\,\eta^{\mu\nu} - V(\phi) \qquad V(\phi) = \frac{\lambda}{16}(\phi^2 - v^2)^2$$

$$V(\phi) = \lambda\left(1 - \cos\frac{2\pi\phi}{v}\right)$$

$$\boxed{\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = -\frac{\partial V}{\partial\phi}} \longrightarrow$$





One may wonder if there is any solution that $\underline{\text{"tunnels"}}$

Expansion $\phi = v + \eta$

$$V(\phi) = \left(\frac{v^2\lambda}{4}\right)\eta^2 + \left(\frac{\lambda v}{\cancel{6}}\right)\eta^3 + \left(\frac{\lambda}{16}\right)\eta^4 + \dots$$

$$\frac{m^2}{2} = \frac{\lambda v^2}{\dots} \qquad g = \frac{\lambda v}{6}$$

expand close to $\phi \approx 0$

$$\phi = \eta$$

$$V(\phi) = \left(\frac{2\pi^2\lambda}{v^2}\right)\eta^2 - \left(\frac{2\pi^4\lambda}{3v^4}\right)\eta^4$$

Solitons

- Domain wall / kink
- Vortices
- monopoles

relation with SUSY

→ ideas that are precursors of all non-perturbative techniques [dualities]

Quite useful in many non perturbative context

(technicolor, ~~SUSY~~, condensed matter, effective theory)

— Informal → not rigorous in numerical factors

— without topology

— No instantons in this meeting (great loss!)

— No application → great loss!

books/ papers to read :

Rajaraman "Solitons and Instantons"
Yung + Shifman "SUSY Solitons"

books

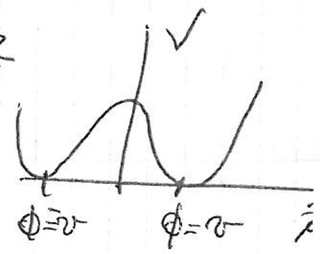
Preskill "Vortices and Monopoles" + Many more | David Tong TASI lectures hep-th/0509216

Solitons and Instantons

- Consider a very simple model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

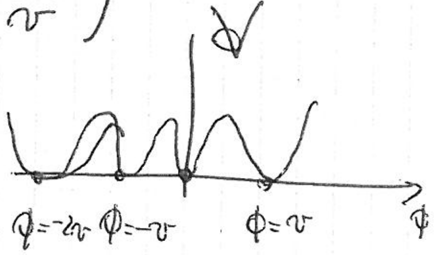
where $$V = \frac{\lambda}{4!}(\phi^2 - v^2)^2$$



vacua in $\phi = \pm v$

or

$$V = \lambda\left(1 - \cos\frac{2\pi\phi}{v}\right)$$



vacua in $\phi = n v$

~o~

Standard perturbation theory: we sit around a vacuum say $\phi = +v$ and expand

$$\phi = v + \eta \qquad \eta = \text{small fluctuation}$$

$$V \approx \overset{m^2/2}{\frac{\lambda}{4} v^2}\, \eta^2 + \overset{g/3!}{\frac{\lambda}{4} v}\, \eta^3 + \dots = \frac{m^2}{2}\eta^2 + \frac{g}{3!}\eta^3$$

$$\rightsquigarrow \quad m^2 = \frac{\lambda v^2}{2}$$

$$g = 3\lambda v$$

For the cosine potential

$$V = \lambda\left(1 - \cos\frac{2\pi\phi}{v}\right)$$

we love around $\phi = \eta$

$$V \simeq \overset{m^2/2}{\left(\frac{2\pi^2\lambda}{v^2}\right)}\eta^2 - \overset{g/4!}{\left(\frac{2\pi^4\lambda}{3v^4}\right)}\eta^4 = \frac{m^2}{2}\eta^2 - \frac{g}{4!}\eta^4$$

$$\boxed{\begin{aligned} m^2 &= \frac{4\pi^2\lambda}{v^2} \\ g &= \frac{32\pi^4\lambda}{3v^4} \end{aligned}}$$

Expanding around $\phi = 0$ for the first potential or $\phi = v\,\frac{(2n+1)}{4}$ generates tachyons $\rightsquigarrow$ $m^2 < 0$

Notice: the mass of these "particles" η and their interactions are small if λ is small (perturbative regime of the original model)

Now, let us consider the equation of motion coming from $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$.

$$\boxed{\partial_\mu(\partial^\mu\phi) = -\frac{\partial V}{\partial\phi}} \rightsquigarrow \boxed{\Box\phi = -\frac{\partial V}{\partial\phi}}$$

Convention here

$$\boxed{\Box\phi = +\partial_t^2\phi - \partial_x^2\phi}$$

Assumptions

Suppose that we search for static solutions $\phi(x,t) \rightsquigarrow \phi(x)$

The equation reads

$$\frac{1}{2}\cdot 2\,\frac{\partial}{\partial x}(\partial\phi)\cdot\frac{\partial\phi}{\partial x} - \frac{\partial V}{\partial\phi}\cdot\frac{\partial\phi}{\partial x} = 0$$

$$\boxed{\partial_x^2\phi - \frac{\partial V}{\partial\phi} = 0} \leftrightsquigarrow \overbrace{\frac{\partial}{\partial x}\left[\frac{1}{2}(\partial_x\phi)^2 - V(\phi)\right] = 0}^{(\partial\phi)\cdot\partial_x^2\phi - V'(\phi)\frac{\partial\phi}{\partial x} = 0 \;\checkmark}$$

Ok?

$$\rightsquigarrow \frac{1}{2}(\partial_x\phi)^2 - V(\phi) = \text{Constant} = C$$

$$(\partial_x\phi)^2 = 2[C + V(\phi)] \rightsquigarrow \frac{d\phi}{dx} = \sqrt{2(C + V(\phi))}$$

$$\frac{d\phi}{\sqrt{2(C + V(\phi))}} = dx \rightsquigarrow \int^{\phi}\frac{d\phi}{\sqrt{2(C + V(\phi))}} = (x - x_0)$$

If we take $\underline{c=0}$

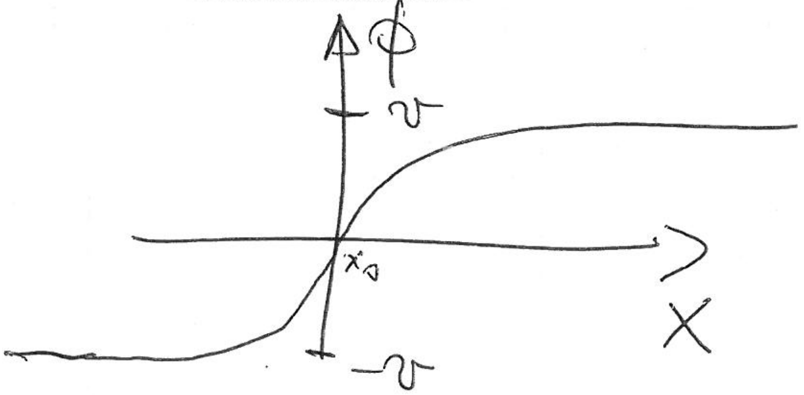
$$\int \frac{d\phi}{\sqrt{2V(\phi)}} \quad \text{gives}$$

$$V = \frac{\lambda}{16}(\phi^2 - v^2)^2$$

$$(x - x_0) = \frac{2\sqrt{2}}{\sqrt{\lambda}\, v}\, \text{arctanh}\left(\frac{\phi}{v}\right)$$

or

$$\boxed{\phi = v \cdot \tanh\left(\frac{\sqrt{\lambda}\, v}{2\sqrt{2}}(x - x_0)\right)}$$



Interpolates between the two vacua!

Notice the units $[\phi] = m^0$ $\quad (\partial_\mu \phi)^2 \sim m^2$

in $\underline{d=2}$ $\quad [v] = m^0$

$[\lambda] = m^2$

the solution has a "characteristic ~~mass~~ length"

~~that is the particle mass~~

$$\boxed{\cancel{\Delta x = \sqrt{\frac{\lambda}{8} v^2}}}$$

this is the distance on which "it decays"

the object is localised for $\lambda \to \infty$

$$\boxed{\Delta x = \sqrt{\frac{8}{\lambda v^2}}}$$

notice that this object is VERY Heavy

Δx

$\sqrt{\frac{8}{\lambda v^2}}$

$\lambda \to 0$

$\Delta x \to \infty$

$\lambda \to \infty$

$\Delta x \to 0$

(pointlike)

in perturbation theory. But it becomes light when the theory becomes strongly coupled

(at the same time the "perturbative excitations" become heavy)

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for the case of the $V = \lambda\left(1 - \cos\frac{2\pi\phi}{v}\right)$

$$x - x_0 = \int^{\phi} \frac{d\phi}{\sqrt{2V(\phi)}}$$

$$x - x_0 = \frac{v}{2\pi\sqrt{\lambda}} \log\left(tg\left(\frac{\pi\phi}{2v}\right)\right)$$

or

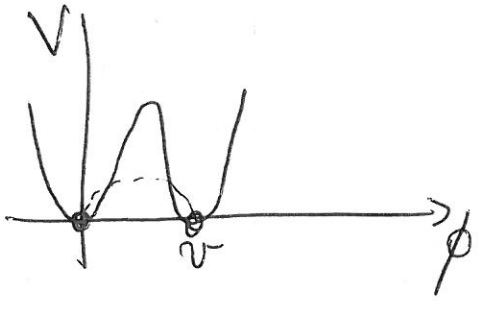
$$\phi = \frac{2v}{\pi} \cdot \arctan\left[e^{m(x-x_0)}\right]$$

$$\boxed{m = \frac{v}{2\pi\sqrt{\lambda}}}$$

So, when $x \to +\infty$ $\qquad \phi \to v$

$\arctan(\infty) = \pi/2$

$x \to -\infty$ $\qquad \phi \to 0.$



again, notice that the mass is very high in perturbation theory, becoming light as $\lambda \to \infty$

Suppose that we calculate the Energy of these solutions.

$$H = p\dot{q} - \mathcal{L} \qquad \text{classical Mechanics}$$

we can define in classical field theory

$T_{\mu\nu}$ such that T_{00} is the ~~[illegible]~~ Energy $\equiv$ Hamiltonian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi).$$

$$\boxed{T_{\mu\nu} = \frac{1}{\cancel{2}}\left(\partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha \phi)^2\right) + \eta_{\mu\nu} V(\phi)}$$

Auxiliary calculation of $T_{\mu\nu}$

$$\mathcal{L} = \sqrt{g}\left[\frac{1}{2} g^{\mu\nu} \partial_\mu\phi \partial_\nu\phi - V(\phi)\right]$$

$$T_{\mu\nu} = +\frac{2}{\sqrt{g}} \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \qquad \left(\text{use } \frac{\partial\sqrt{g}}{\partial g^{\mu\nu}} = -\sqrt{g}\,\frac{g_{\mu\nu}}{2}\right)$$

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \equiv \frac{2}{\sqrt{g}}\left\{-\frac{\sqrt{g}}{2} g_{\mu\nu}\left(\frac{1}{2}(\partial\phi)^2 - V(\phi)\right) + \frac{\sqrt{g}}{2}\partial_\mu\phi\partial_\nu\phi\right\}$$

$$\boxed{T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{g_{\mu\nu}}{2}(\partial\phi)^2 + g_{\mu\nu} V(\phi)}$$

$$T_{00} = \partial_t\phi\,\partial_t\phi - \frac{g_{tt}}{2}\left[\partial_t\phi\partial_t\phi - \partial_x\phi\,\partial_x\phi\right] + g_{tt} V(\phi)$$

$g_{tt} = +1$

$$\boxed{T_{00} = \frac{\partial_t\phi\,\partial_t\phi}{2} + \frac{\partial_x\phi\,\partial_x\phi}{2} + V(\phi)}$$

for static configuration $\boxed{T_{00} = \frac{(\partial_x\phi)^2}{2} + V(\phi)}$

Se calcula

$$T_{00} = (\partial_x \phi)^2 + V(\phi)$$

Sobre las soluciones

$$T_{00} = (\partial_x \phi)^2 + \lambda(\phi^2 - v^2)^2$$

$$T_{00} = \frac{9}{8}\,\frac{v^4 \lambda}{\cosh^4\left(x\sqrt{\frac{v^2\lambda}{7}}\right)}$$

$$E = \int_{-\infty}^{\infty} T_{00} = 3\sqrt{2}\, v^{3}\sqrt{\lambda} \sim M$$

ﾄ eq.

$$\partial_x \phi = \sqrt{2V} = \sqrt{\frac{\lambda}{8}}(\phi^2 - v^2) = \frac{\partial W}{\partial \phi}$$

$$\boxed{\partial_x \phi = \frac{\partial W}{\partial \phi}}$$ BPS eq.

$$E = \int_{-\infty}^{\infty} dx \sqrt{2V}\, \partial_x \phi = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} W(\phi)\, dx$$

$$= W(\infty) - W(-\infty)$$

$$W = \left(\frac{\phi^2}{3} - v^2\right)\phi\sqrt{\frac{\lambda}{8}} = \frac{v^3\sqrt{\lambda}}{6\sqrt{2}} \cdot \tanh\left(\sqrt{\frac{\lambda v^2}{8}}x\right)\left(-3 + \tanh^2\left(\sqrt{\frac{\lambda v^2}{8}}x\right)\right)$$

$x \to \infty$ $\quad W(\infty) = \frac{v^3}{6}\sqrt{\frac{\lambda}{2}}\left\{\underbrace{(-3+1) + (-3+1)}_{-4}\right\}$

$x \to -\infty$ $\quad -W(-\infty)$

$$\boxed{\Delta W = \frac{v^3}{3}\sqrt{\frac{\lambda 2}{8}}}$$

$$\boxed{E \geq \sqrt{\frac{2\lambda}{8}}\, v^3}$$

La Solución

$$\phi = v \tanh\left(\sqrt{\frac{\lambda v^2}{8}}\, x\right)$$

Satisface

$$\partial_x^2 \phi = \frac{\partial V}{\partial \phi} = \frac{\lambda}{4}(\phi^2 - v^2)\cdot\phi \quad \checkmark$$

Supongamos que existe una función $W(\phi)$ tal que

$$V = \frac{\lambda}{16}(\phi^2 - v^2)^2 = \frac{1}{2}\left(\frac{\partial W}{\partial \phi}\right)^2$$

$$\longrightarrow W(\phi) = \sqrt{\frac{\lambda}{8}}\left(\frac{\phi^3}{3} - v^2\phi\right) = \sqrt{\frac{\lambda}{8}}\left(\frac{\phi^2}{3} - v^2\right)\phi$$

Encontramos la Energía de esta solución

$$E = H = T_{00} = \frac{1}{2}\int_{-\infty}^{\infty} dx\, \left[(\partial_x \phi)^2 + 2V(\phi)\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx\, \left\{\left((\partial_x \phi) \pm \sqrt{2V}\right)^2 \mp 2\sqrt{2V}\,\partial_x\phi\right\}$$

Si minimizamos la energía $\underline{\partial_x\phi - \sqrt{2V} = 0}$ $\leadsto$

$$E = \frac{2}{2}\int dx\, \sqrt{2V}\,\partial_x\phi$$

The Energy

$$E = \int_{-\infty}^{\infty} dx \left[\frac{(\partial_x \phi)^2}{2} + V(\phi) \right] = \cancel{\int_{-\infty}^{\infty} dx}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{2} \left[\left((\partial_x \phi) \mp \sqrt{2V} \right)^2 \pm 2 \sqrt{2V} \cdot \partial_x \phi \right]$$

Notica now

$$\text{if } \partial_x \phi - \sqrt{2V} = 0 \rightsquigarrow$$

vor other

$$\sqrt{2V}\, \partial_x \phi = \sqrt{\frac{\lambda}{8}} (\phi^2 - v^2) \cdot \partial_x \phi = \partial_x \underbrace{\left[\sqrt{\lambda} (\phi^2 - v^2)^2 \right]}_{\text{evaluate on solution}}$$

$$\sqrt{2V}\, \partial_x \phi = \frac{\sqrt{\lambda}\, v^4}{2\sqrt{8}} \frac{1}{\cosh\left(\frac{v\sqrt{\lambda}}{\sqrt{8}} x \right)}$$

$$M \sim \int \sqrt{2V}\, \partial_x \phi \sim \left. \frac{v^3 \tanh\left(v\sqrt{\frac{\lambda}{8}} x \right)}{2 + \frac{1}{\cosh\left(v\sqrt{\frac{\lambda}{8}} x \right)}} \right]_{-\infty}^{\infty} \sim v^3$$