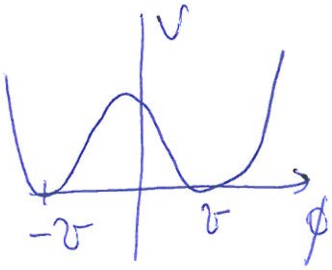
Summary (of Lectures #1)

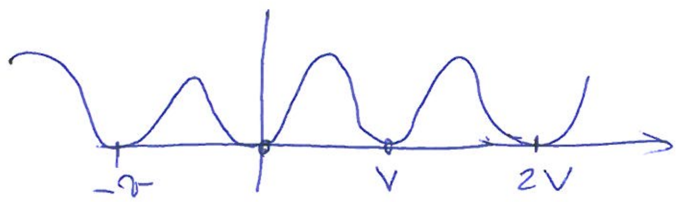
We studied a model in $1+1$ dimensions t, x

$$\mathcal{L} = +\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \qquad \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

we studied perturbative configurations for

$$V(\phi) = \frac{\lambda}{16}(\phi^2 - v^2)^2 \qquad \text{or} \qquad V(\phi) = \lambda\left(1 - \cos\frac{2\pi\phi}{v}\right)$$





$$\phi = v + \eta \qquad\qquad \phi = kv + \eta \quad k \in \mathbb{Z}$$

Expanded for small η

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^2 + \underbrace{\left(\frac{v^2 \lambda}{4}\right)} \eta^2 + \underbrace{\left(\frac{\lambda v}{4}\right)} \eta^3 + \frac{\lambda}{16}\eta^4 + \dots$$

$$m^2 = \frac{\lambda v^2}{2} \qquad g_3 = 3\lambda v \qquad \text{etc}$$

Perturbatively $(\lambda \to 0)$ the excitation η is

almost massless $m^2 = \frac{\lambda v^2}{2}$

almost free $g_3 = 3\lambda v$

Then we studied a different configuration

- Static $\phi(x,t) = \phi(x)$
  $\downarrow$
  Energy conserved

we rewrite the eq of motion $\partial_t^2\phi - \partial_x^2\phi = -\frac{\partial V}{\partial\phi}$

(static) $\left[\partial_x^2\phi = \frac{\partial V}{\partial\phi}\right] \longrightarrow \frac{\partial}{\partial x}\left\{\frac{1}{2}(\partial_x\phi)^2 - V(\phi)\right\} = 0$

$\rightsquigarrow \frac{1}{2}(\partial_x\phi)^2 - V(\phi) = \text{constant} = C_1$

$\downarrow$ ~~for consta~~

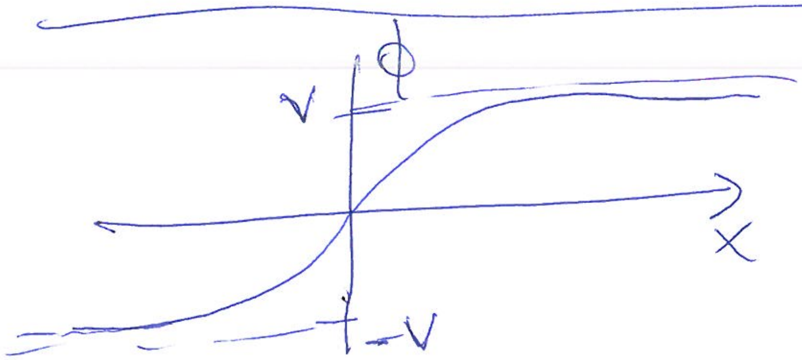
$(\partial_x\phi)^2 = 2(C+V) \rightsquigarrow \frac{d\phi}{\sqrt{2(C+V(\phi))}} = dx$

we integrated for

$$V(\phi) = \frac{\lambda}{16}(\phi^2 - v^2)^2$$

$$\int \frac{d\phi}{\sqrt{2(c+V(\phi))}} = (x - x_0) \qquad \text{for } c=0$$

$$\rightsquigarrow \boxed{\phi = v \tanh\left(\sqrt{\frac{\lambda v^2}{8}}(x-x_0)\right)}$$



$$\Delta x = \sqrt{\frac{8}{\lambda v^2}}$$

$\lambda \to 0$ very spread

$\lambda \to \infty$ a particle

we calculated the $\boxed{T_{\mu\nu} = \left(\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}\right)\frac{2}{\sqrt{g}}}$

used $\boxed{\frac{\delta\sqrt{g}}{\delta g^{\mu\nu}} = -\sqrt{g}\, g_{\mu\nu}}$

$$\boxed{T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{g_{\mu\nu}}{2}(\partial_\alpha\phi)^2 + g_{\mu\nu} V(\phi)}$$

Computed

$$T_{00} = \frac{(\partial_x \phi)^2}{2} + V(\phi)$$

$$E = \int dx \; T_{00} = \int dx \left\{ \frac{(\partial_x \phi)^2}{2} + V(\phi) \right\} =$$

$$E = \int dx \left\{ \left(\partial_x \phi \pm \sqrt{2V(\phi)} \right)^2 \mp 2\sqrt{2V} \cdot \partial_x \phi \right\}$$

minimize energy $\boxed{\partial_x \phi = \mp \sqrt{2V(\phi)}}$

on which

$$\rightarrow \boxed{E = \mp \int \sqrt{2V} \; \partial_x \phi} = \cancel{\int \partial_x \left[\sqrt{\lambda} (\phi^2 - v^2) \right]}$$

$$\text{if } \boxed{V = \frac{1}{2} \left(\frac{\partial W}{\partial \phi} \right)^2} = \int \frac{\partial W}{\partial \phi} \frac{\partial \phi}{\partial x} = \int \frac{\partial}{\partial x} W$$

$$= W(+\infty) - W(-\infty)$$

$= \#$ topological.

Today, we will study a different solution

- it lives in 2+1 dimensions t, x_1, x_2
- It contains a scalar field and a ~~vector~~ gauge field

Gauge fields

You know

$$\oiint_\Sigma \bar{E}\, d\bar{S} = \frac{q}{\epsilon_0} \rightsquigarrow \bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0}$$

$$\oiint \bar{B}\, d\bar{S} = 0 \rightsquigarrow \bar{\nabla}\cdot\bar{B} = 0$$

$$\frac{d}{dt}\iint_\Sigma \bar{B}\, d\bar{S} = -\oint_C \bar{E}\, d\bar{l} \iff \frac{d}{dt}\iint_\Sigma \bar{B}\, d\bar{S} - \iint_\Sigma (\bar{\nabla}\times\bar{E})\, dl$$

Faraday's law $\boxed{\frac{\partial}{\partial t}\bar{B} = -\bar{\nabla}\times\bar{E}}$

$$\oint_C \bar{B}\, d\bar{l} = \mu_0 I + \frac{d}{dt}\iint_\Sigma \bar{E}\cdot d\bar{S} \; ; \quad I = \iint_\Sigma \bar{J}\cdot d\bar{S}$$

$$\int_\Sigma \bar{\nabla}\times\bar{B} = \mu_0\iint \bar{J}\, d\bar{S} + \frac{d}{dt}\iint \bar{E}\cdot d\bar{S} \rightsquigarrow \boxed{\nabla\times B = \mu_0\bar{J} + \frac{\partial\bar{E}}{\partial t}}$$

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0 \qquad \bar{\nabla}\cdot\bar{B} = 0$$

1 $$\bar{\nabla}\times\bar{E} = -\partial_t \bar{B} \qquad \bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \frac{\partial}{\partial t}\bar{E}$$

Notice if $\underline{\vec{J} = \rho = 0}$

$$\vec{E} \longleftrightarrow -\bar{B}$$ is a $\underline{\text{Symmetry}}$

The eq $\bar{\nabla}\cdot\bar{B} = 0 \rightsquigarrow \boxed{\bar{B} = \bar{\nabla}\times A}$

Using $\bar{\nabla}\times\bar{E} = -\frac{\partial}{\partial t}\bar{\nabla}\times\bar{A} \rightsquigarrow \boxed{\bar{E} = \frac{\partial}{\partial t}\bar{A} - \bar{\nabla}\phi}$

$\boxed{\text{static case } \bar{E} = -\bar{\nabla}\phi}$

$\boxed{-\bar{\nabla}^2\phi = \rho/\epsilon}$

Electrostatics

We can change
$$\vec{A} \longrightarrow \bar{A} + \vec{\nabla}f$$
$$\phi \longrightarrow \phi + \partial_t f$$

$\bar{B} = \bar{\nabla}\times\bar{A}$ is invariant since $\underline{\vec{\nabla}\times(\bar{\nabla}\cdot f) = 0}$

invariant

$$E = \frac{\partial}{\partial t}(\bar{A}+\vec{\nabla}f) - \bar{\nabla}\phi - \bar{\nabla}\partial_t f = \partial_t\bar{A} - \bar{\nabla}\phi + \cancel{\frac{\partial}{\partial t}\bar{\nabla}f} - \cancel{\bar{\nabla}\partial_t f}$$

To accomodate the notation

$$A_\mu = (\phi, \vec{A})$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu f \quad \text{is an invariance}$$

We can Define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow \left(\begin{array}{l} \text{antisymmetric} \\ F_{00} = F_{xx} = F_{yy} = F_{zz} = 0 \end{array} \right)$$

$$F_{tx} = \partial_t A_x - \partial_x A_t = E_x \qquad F_{ty} = \partial_t A_y - \partial_y A_t = E_y$$

$$F_{xt} = -E_x \qquad F_{yt} = -E_y$$

$$F_{tz} = -F_{zt} = E_z$$

$$\longrightarrow \boxed{F_{0i} = E_i}$$

$$F_{xy} = \partial_x A_y - \partial_y A_x = B_z \quad ; \quad F_{zx} = \partial_z A_x - \partial_x A_z = B_y$$

$$F_{yz} = \partial_y A_z - \partial_z A_y = B_x$$

$$F_{\mu\nu} = \begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & E_x & E_y & E_z \\ x & -E_x & 0 & B_z & -B_y \\ y & -E_y & -B_z & 0 & B_x \\ z & -E_z & B_y & -B_x & 0 \end{array}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} \left(F_{\mu\nu} F_{\alpha\beta} \right) = \frac{E^2 - B^2}{2}$$

$$\partial_p \frac{\partial \mathcal{L}}{\partial(\partial_p A_q)} = \frac{\partial \mathcal{L}}{\partial A_q} \qquad = -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)$$

$$\downarrow$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_p A_q)} = -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} \left\{ \left(\delta_{p\mu}\delta_{\nu q} - \delta_{p\nu}\delta_{\mu q}\right)^{F_{\alpha\beta}} + \left(\delta_{p\alpha}\delta_{\beta q} - \delta_{p\beta}\delta_{q\alpha}\right)^{F_{\mu\nu}} \right\}$$

$$= F^{pq}$$

$$\partial_p \frac{\partial \mathcal{L}}{\partial(\partial_p A_q)} = \partial_p \left[F^{pq} \right]$$

If the Lagrangian were

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^{\mu} A_{\mu}$$

$$\partial_p \frac{\partial \mathcal{L}}{\partial(\partial_p A_q)} = \partial_p \left[F^{pq} \right]$$

$$\frac{\partial \mathcal{L}}{\partial A_q} = J^q$$

$$\longrightarrow \boxed{\partial_p F^{pq} = J^q}$$

$q = t$

$$\partial_x F^{xt} + \partial_y F^{yt} + \partial_z F^{zt} = -\vec{\nabla} \cdot \vec{E} = J^0$$

$$\boxed{\bar{\nabla} \bar{E} = \rho} \quad J_0 = -\rho$$

Similarly $q = x, y, z$

$$\boxed{\partial_t \bar{E} + \vec{J} = \bar{\nabla} \times \bar{B}}$$

the eqs

$$\begin{cases} \bar{\nabla} \times \bar{E} = -\partial_t \bar{B} \\ \bar{\nabla} \cdot \bar{B} = 0 \end{cases}$$

appears as a consequence

$$\boxed{\partial_p F_{\mu\nu} + \partial_\nu F_{p\mu} + \partial_\mu F_{\nu p} = 0}$$

Now, let us mix a gauge field with

a Scalar

Consider this to be a complex Scalar $\equiv$ 2 real Scalars

$$\phi = |\phi| e^{i\,\text{phase}(\varphi)} = \phi_1 + i\phi_2$$

$$|\phi_1| = \phi_1^2 + \phi_2^2$$

$$\text{phase}\ \varphi = \arctan\left(\frac{\phi_2}{\phi_1}\right)$$

We observe $\phi \to e^{i\alpha}\phi$ is a symmetry of

$$\boxed{\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi \cdot \partial_\nu\phi^* - V(\phi\phi^*)}$$

if α = constant

global symmetry

instead if $\alpha = \alpha(x)$ the term

$$\partial_\mu\phi = \partial_\mu\phi\, e^{i\alpha(x)} + i(\partial_\mu\alpha)e^{i\alpha}\phi$$

$$\partial_\mu\phi^* = \partial_\mu\phi^* e^{-i\alpha} - i\partial_\mu\alpha\, e^{-i\alpha}\phi^*$$

$\partial_\mu\phi\,\partial_\mu\phi^*$ is NOT invariant

What can we do?

using that $\phi \rightarrow e^{i\alpha}\phi$

$$\phi^* \rightarrow e^{-i\alpha}\phi^*$$

$$A_\mu \rightarrow A_\mu - \partial_\mu \alpha \frac{1}{e}$$

If we construct

$$D_\mu \phi = \partial_\mu \phi + i e A_\mu \phi$$

$$D_\mu \phi^* = \partial_\mu \phi^* - i e A_\mu \phi^*$$

$$\phi \rightarrow e^{i\alpha(x)}\phi$$

$$\left.\begin{array}{l} \partial_\mu \phi \longrightarrow e^{i\alpha}\partial_\mu\phi + i e^{i\alpha}\phi\,\partial_\mu\alpha \\ i A_\mu \phi \longrightarrow i e^{i\alpha}(A_\mu - \partial_\mu\alpha)\phi \end{array}\right] \; D_\mu\phi \longrightarrow e^{i\alpha}\left[\partial_\mu\phi + i\partial_\mu\alpha\,\phi + iA_\mu\phi - i\partial_\mu\alpha\,\phi\right]$$

$$\rightarrow e^{i\alpha} D_\mu \phi$$

$$D_\mu \phi^* \rightarrow e^{-i\alpha} D_\mu \phi^*$$

$$\boxed{D_\mu \phi\, D_\mu \phi^* \rightarrow \text{invariant}}$$

It is then natural to write [illegible]

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi^* - V(\phi)}$$

The Vortex

Motivation $\begin{cases} \text{- Strings in QCD} \\ \text{- Higgs Mechanism} \end{cases}$

Consider the model (Nielsen-Olesen 1973)

$\boxed{\Phi = \text{complex Scalar field}}$

$\boxed{D_\mu\phi = \partial_\mu\phi + ig A_\mu \phi}$

$$\mathcal{L} = -|D_\mu\phi|^2 - \frac{\lambda}{2}(|\phi|^2 - v^2)^2 - \frac{1}{4}F_{\mu\nu}^2.$$

in $\underline{\text{dim } 2+1}$ (t, x, y).

$\lambda \to 8\lambda_{\text{canonical}}$

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\left(\frac{2}{\sqrt{g}}\right)$$

Aside on $T_{\mu\nu}$

$$\mathcal{L} = \sqrt{g}\left[-g^{\mu\nu}D_\mu\phi D_\nu\phi^* - \frac{\lambda}{2}(\phi^2 - v^2)^2 - \frac{1}{4}g^{\rho q}g^{\alpha\beta}F_{\rho\alpha}F_{q\beta}\right]$$

$$\frac{2}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{g}}\frac{\delta\sqrt{g}}{\delta g^{\mu\nu}}\mathcal{L} + \frac{2\sqrt{g}}{\sqrt{g}}\left[-D_\mu\phi D_\nu\phi^* - \frac{1}{2}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right]$$

$$= -g_{\mu\nu}\left[-|D_\mu\phi|^2 - \frac{\lambda}{2}(\phi^2-v^2)^2 - \frac{1}{4}F^2\right] + \left[-2D_\mu\phi D_\nu\phi^* - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right]$$

$$\boxed{T_{\mu\nu} = -2\left(D_\mu\phi D_\nu\phi^* - g_{\mu\nu}\frac{|D_\alpha\phi|^2}{2}\right) - \left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right) + \frac{g_{\mu\nu}}{2}\lambda(\phi^2-v^2)^2}$$

$$T_{\mu\nu} = -2\left(D_\mu \phi D_\nu \phi^* - \frac{g_{\mu\nu}}{2}|D\phi|^2\right) - \left(F_{\mu\circ} F_\nu^{\circ} - \frac{1}{4} g_{\mu\nu} F^2\right)$$

$$\mp g_{\mu\nu} \frac{\lambda}{2}(\phi^2 - v^2)^2.$$

$$T_{00} = -2\left[D_0\phi D_0\phi^* - \frac{g_{00}}{2}\left(g_{00} D_0\phi D_0\phi^* + g_{ij} D_i\phi D_j\phi^*\right)\right]$$

$$-\left(F_{0i} F_{0j} g^{ij} - \frac{1}{4} g_{00}\left(F_{0i}F_{0j} g^{ij} + g^{il} g^{pq} F_{ip} F_{lq}\right)\right)$$

$$\mp g_{00} \frac{\lambda}{2}(\phi^2 - v^2)^2$$

Consider Static configurations

$\partial_t = 0$ and choose the gauge $A_0 = 0$

$$F_{ti} = \partial_t A_i - \partial_i A_t = 0$$

$$D_t \phi = \partial_t \phi - i g A_t \phi = 0$$

$$T_{00} = +2\left[\frac{g^{ij}}{2} D_i\phi D_j\phi^*\right] + \frac{1}{4} F_{ik} F_{pq} g^{ip} g^{kq} - \frac{\lambda}{2}(\phi^2 - v^2)^2$$

$$\boxed{T_{00} = D_i\phi D_i\phi^* + \frac{F_{12}^2}{2} + \frac{\lambda}{2}(\phi^2 - v^2)^2}$$

Now one can write the Energy

$$E = \int d^2x \left\{ D_i\phi D^i\phi^* + \frac{F_{12}^2}{2} + \frac{\lambda}{2}(\phi^2 - v^2)^2 \right\}$$

that can be written as

$$E = \int d^2x \left\{ \left(D_i\phi \pm i\varepsilon_{ij} D_j \phi\right)^2 + \frac{1}{2}\left(F_{12} \pm \sqrt{\lambda}(\phi^2 - v^2)\right)^2 \mp \sqrt{\lambda}(\phi^2 - v^2) F_{12} \pm \underbrace{e\phi^2 F_{12}}_{\text{integration by parts}} \right\}$$

Vor pogue symmetric

If $\boxed{\lambda = e^2} \longrightarrow$

$\lambda \to \frac{\lambda}{8}$ for correct normalization

$$E = \int d^2x \left\{ \left(D_i\phi \pm i\varepsilon_{ij} D_j \phi\right)^2 + \left(F_{12} \pm g(\phi^2 - v^2)\right)^2 \pm g v^2 F_{12} \right\}.$$

Cuenta explícita de los puntos de BPS

$$\frac{1}{2}\left(D_i\phi + i\epsilon_{ij}D_j\phi\right)\left(D_i\phi^* - i\epsilon_{il}D_l\phi^*\right)$$

$$\frac{1}{2}\left\{D_i\phi D_i\phi^* - i\epsilon_{il}D_i\phi D_l\phi^* + i\epsilon_{ij}D_j\phi D_i\phi^* + \underbrace{\epsilon_{ij}\epsilon_{il}}_{\delta_{jl}}D_j\phi D_l\phi^*\right\}$$

$$|D_i\phi|^2$$

$$D_i\phi D_i\phi^* + i\epsilon_{ik}\left(-D_i\phi D_k\phi^* + D_i\phi^* D_k\phi\right)$$

$$\left.\begin{aligned} D_i\phi &= \partial_i\phi + igA_i\phi \\ D_k\phi^* &= \partial_k\phi^* - igA_k\phi^* \end{aligned}\right\} D_i\phi D_k\phi^* = \partial_i\phi\partial_k\phi^* - igA_k\phi^*\partial_i\phi + igA_i\phi\partial_k\phi^* + g^2A_iA_k|\phi|^2$$

$$i\epsilon_{ik}\left(D_i\phi D_k\phi^* - D_i\phi^* D_k\phi\right) = \left(2\,\mathrm{Im}\, D_i\phi D_k\phi^*\right)\epsilon_{ik}$$

$$= 2g\,\epsilon_{ik}\left[A_k\phi^*\partial_i\phi + A_i\phi\partial_k\phi^*\right]$$

$$= \cancel{4g\,\epsilon_{ik}\,A_i\phi\partial_k\phi^*}$$

Integrate by parts

$$2g\,\epsilon_{ik}\left\{A_i\phi\partial_k\phi^* - \underbrace{A_k\phi^*\partial_i\phi}_{+A_i\phi^*\partial_k\phi}\right\} =$$

$$= 2g\int \epsilon_{ik}A_i\partial_k(\phi\phi^*)dx = -2g\int \epsilon_{ik}\partial_k A_i|\phi|^2\,dx$$

$$= -2g\int F_{ki}|\phi|^2 + 2g|\phi|^2\oint A_i\,dl^i + \oint A_i|\phi|^2 dl^i$$

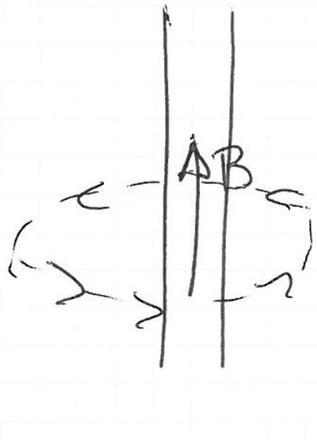
Now, one can write this Energy

in the case in which

$$\boxed{\begin{aligned} &D_i \phi \pm i \varepsilon_{ij} D_j \phi^* = 0 \\ \text{and} \quad &F_{12} \pm g(\phi^2 - v^2) = 0 \end{aligned}} \to \text{1st order}$$

$$E = \pm g v^2 \int d^2x \, F_{12} = \pm g v^2 \oint_C \bar{A} \, d\bar{l}$$

$$\oint A_\mu dx^\mu = \oint A_\varphi \, r \, d\varphi \equiv \text{Magnetic flux}$$



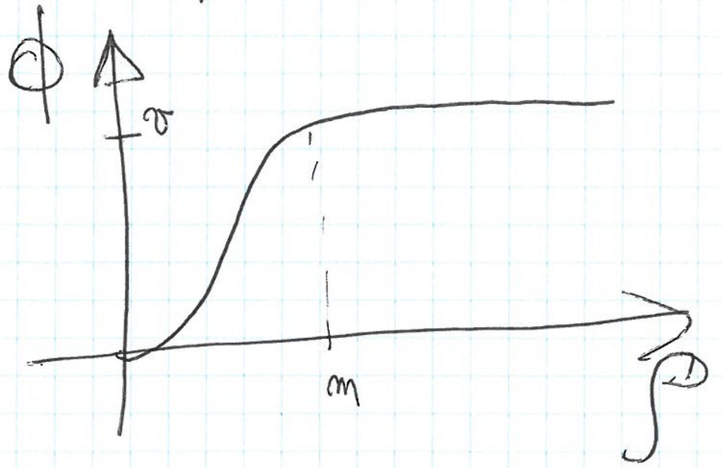
when one solves the

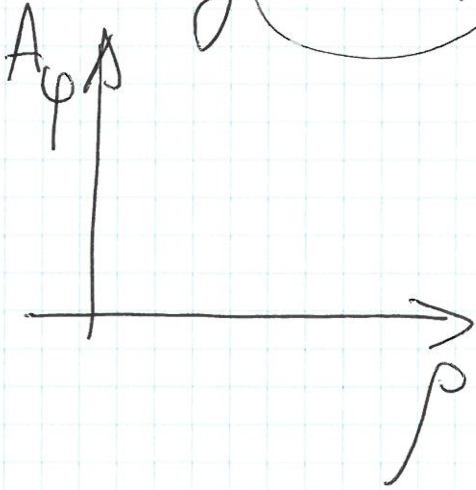
eqs $\quad D_1 \phi + D_2 \phi^* = 0$

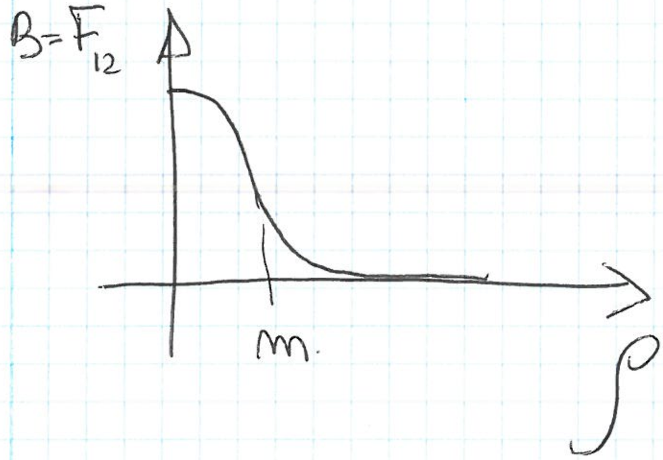
$$F_{12} = g(\phi^2 - v^2)$$

K4

the eqs can be solved numerically. (1st order)







The characteristic scale m ~ mass of vortex

$$\boxed{M_{vortex} \sim \frac{1}{g^2}}$$

→ at weak coupling → heavy

at strong coupling → light