Deformed N=8 supergravity from IIA strings and its Chern-Simons duals

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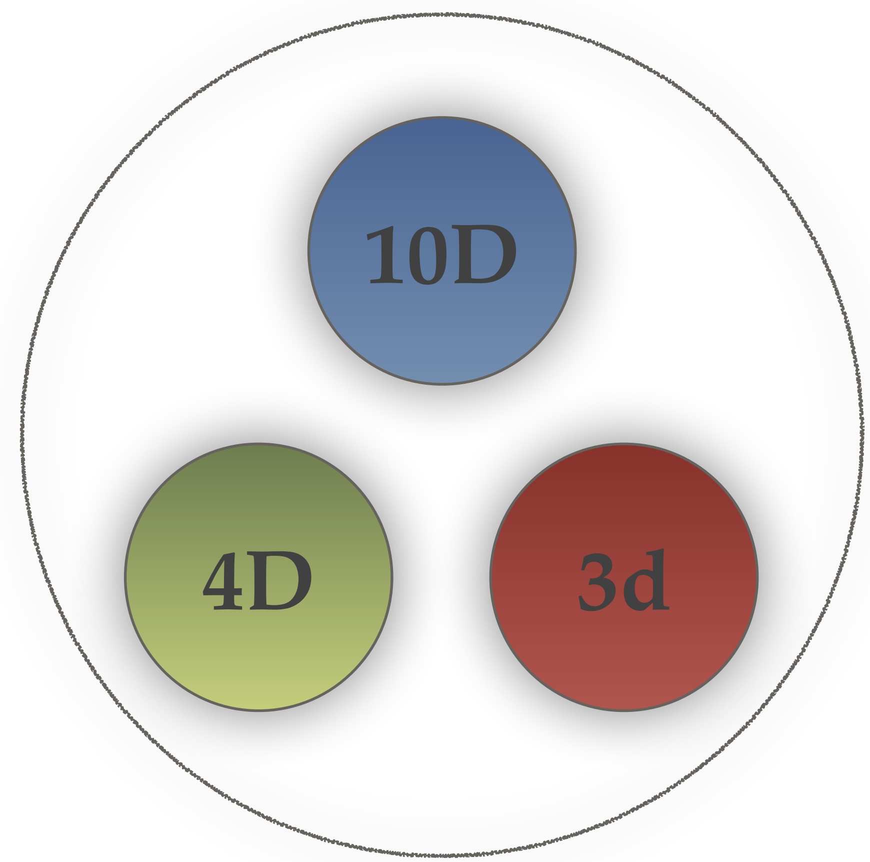


With Daniel Jafferis and Oscar Varela :

arXiv:1504.08009 , arXiv:1508.04432 , arXiv:1509.02526

Outlook

- ``Gong show'' motivation
- Deformed SO(8)-gauged supergravity
- Deformed ISO(7)-gauged supergravity
- Massive type IIA strings on S^6
- New AdS_4 $N=2$ solution in massive IIA and its CFT_3 dual

``Gong show" motivation

electric-magnetic deformations

• The uniqueness of the maximal supergravities is historically inherited from their connection to sphere reductions

$AdS_5 \times S^5$ (D3-brane) , $AdS_4 \times S^7$ (M2-brane) , $AdS_7 \times S^4$ (M5-brane)

• N=8 supergravity in 4D admits a deformation parameter c yielding inequivalent theories. It is an electric/magnetic deformation

$$\boxed{D = \partial - g\,(A^{\text{elec}} - c\,\tilde{A}_{\text{mag}})}$$

g = 4D gauge coupling
c = deformation param.

• There are two generic situations :

1) Family of $SO(8)_c$ theories : $c = [0, \sqrt{2} - 1]$ is a continuous param. [similar for $SO(p,q)_c$]

2) Family of $ISO(7)_c$ theories : $c = 0$ or 1 is an (on/off) param. [same for $ISO(p,q)_c$]

[Dall'Agata, Inverso, Trigiante '12]
[Dall'Agata, Inverso, Marrani '14]

The questions arise:

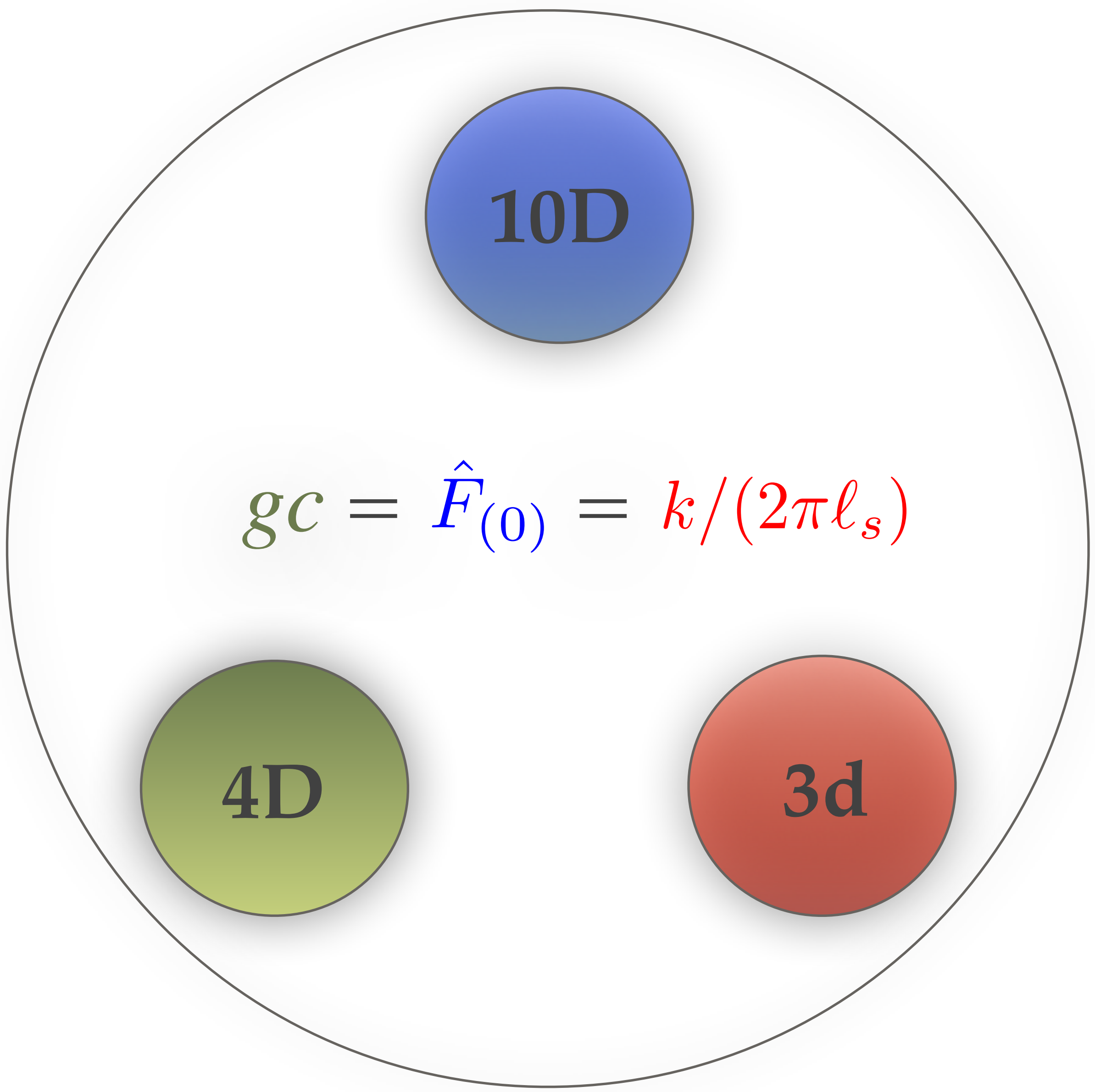
- Does such an electric/magnetic deformation of 4D maximal supergravity enjoy a string/M-theory origin, or is it just a 4D feature ?

Obstruction for $SO(8)_c$, *cf*. [Lee, Strickland-Constable, Waldram '15]

- For deformed 4D supergravities with supersymmetric AdS_4 vacua, are these AdS_4/CFT_3-dual to any identifiable 3d CFT ?

A new 10D / 4D / 3d correspondence

massive IIA on S^6 « ISO(7)$_c$-gauged sugra » SU(N)$_k$ C-S-M theory



gc = elec/mag deformation in 4D

$\hat{F}_{(0)}$ = Romans mass in 10D

k = Chern-Simons level in 3d

[Schwarz ’04]
[AG, Jafferis, Varela ’15]
[AG, Varela ’15]

Deformed SO(8)-gauged supergravity

$N = 8$ supergravities in 4D

- SUGRA : metric + 8 gravitini + 28 vectors + 56 dilatini + 70 scalars

  (s = 2) (s = 3/2) (s = 1) (s = 1/2) (s = 0)

Ungauged (abelian) supergravity: Reduction of M-theory on a *torus* T^7 down to 4D produces $N = 8$ supergravity with $G = U(1)^{28}$ [Cremmer & Julia '79]

Gauged (non-abelian) supergravity: Reduction of M-theory on a *sphere* S^7 down to 4D produces $N = 8$ supergravity with $G = SO(8)$ [de Wit & Nicolai '82]

✱ SO(8)-gauged supergravity believed to be unique for 30 years…

… but … is this true?

Framework to study $N = 8$ supergravities in 4D

[de Wit, Samtleben & Trigiante '03 , '07]

Gauging procedure : Part of the global E_7 symmetry group is promoted to a local symmetry group G (gauging)

Embedding tensor : It is a "selector" specifying which generators of E_7 (there are 133!!) become gauge symmetries G and then will have 28 associated gauge fields

$$\boxed{A_\mu = A_\mu^M \, \Theta_M{}^\alpha \, t_\alpha}$$

$M = 1 , \dots , 56 \; = \; 28$ (elec) + 28 (mag)

Redundancy!!

$$X_M = \Theta_M{}^\alpha \, t_\alpha \quad \Rightarrow \quad [\, X_M \, , \, X_N \,] = X_{MN}{}^P \, X_P \quad \text{with} \quad X_{MN}{}^P = \Theta_M{}^\alpha \, [t_\alpha]_N{}^P$$

$[\, \alpha = 1, \dots, 133 \,]$

✱ Closure of the gauge algebra : $\Omega^{MN} \, \Theta_M{}^\alpha \, \Theta_N{}^\beta = 0$

Only 28 physical l.c. of vectors!!

A family of G = SO(8) supergravities in 4D

- Choose G = SO(8)

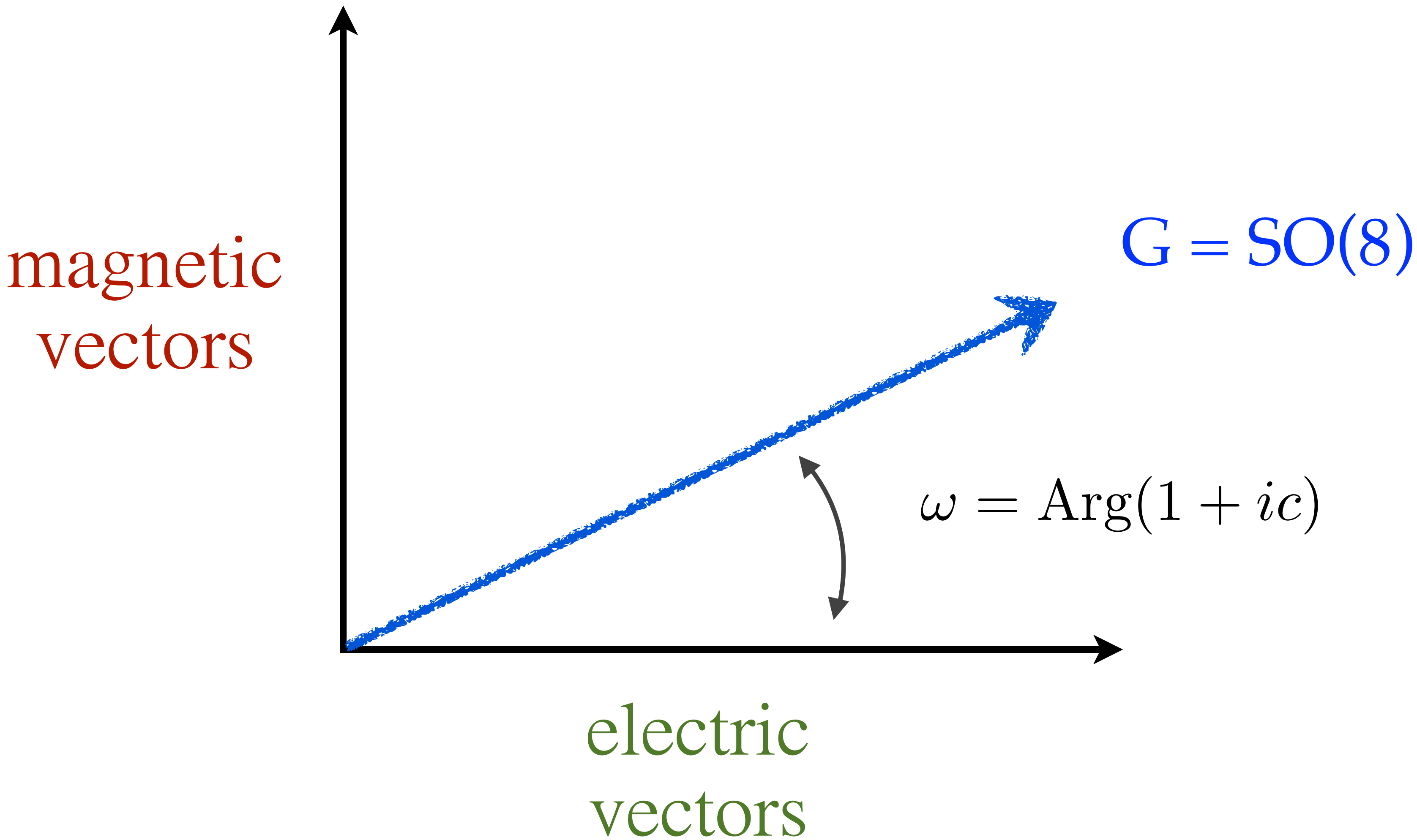
- Solve $\Omega^{MN}\,\Theta_M{}^{\alpha}\,\Theta_N{}^{\beta}=0$ ⇨ One-parameter (c) family of $SO(8)_c$ theories !!

[Dall' Agata, Inverso & Trigiante '12]

- Immediate questions :

1) What?	(Yes, surprising but true)
2) Are these c-theories equivalent?	(No)
3) Are there new AdS_4 solutions?	(Yes)
4) Higher-dimensional origin?	(Good question…)
5) AdS_4/CFT_3 dual?	(Good question too… ABJ?)

Physical meaning in 4D : electric / magnetic deformation



$$\boxed{D = \partial - g\,(A^{\mathrm{elec}} - c\,\tilde{A}_{\mathrm{mag}})}$$

Physical meaning 10D / 11D …

?

Holographic AdS_4/CFT_3 meaning …

?

In this talk we are going to investigate the electric/magnetic deformation of a different N=8 supergravity closely related to the $G = SO(8)$ theory …

… the $G = ISO(7) = SO(7) \ltimes \mathbb{R}^7$ supergravity !!

electric/magnetic deformation	higher-dimensional origin	Holographic AdS_4/CFT_3 dual ?
✓	✓	✓

Deformed ISO(7)-gauged supergravity

A family of G = ISO(7) supergravities in 4D

- Choose G = ISO(7)

- Solve $\Omega^{MN}\,\Theta_M{}^{\alpha}\,\Theta_N{}^{\beta} = 0$ ⇨ One-parameter (c) family of $\mathrm{ISO}(7)_c$ theories !!

[Dall'Agata, Inverso, Marrani '14]

- Immediate questions :

1) What? (Yes, and still surprising)

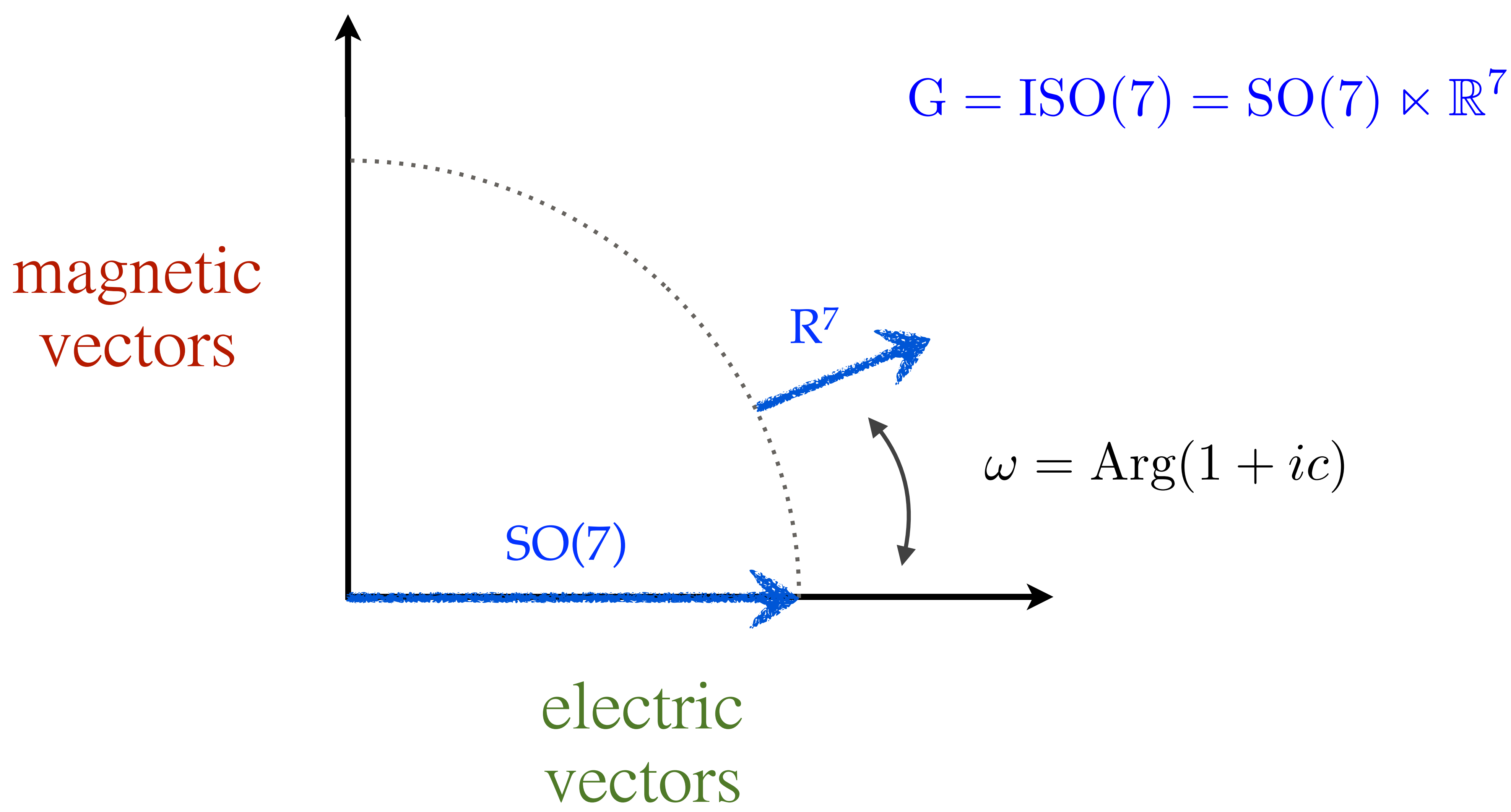
2) Are these c-theories equivalent? (No)

3) Are there new AdS_4 solutions? (Yes)

4) Higher-dimensional origin? (Yes)

5) $\mathrm{AdS}_4/\mathrm{CFT}_3$ dual? (Yes)

Physical meaning in 4D = electric / magnetic deformation



$$\boxed{D = \partial - g\, A^{\mathrm{elec}}_{\mathrm{SO}(7)} - g\,(A^{\mathrm{elec}}_{\mathbb{R}^7} - c\, \tilde{A}_{\mathbb{R}^7\,\mathrm{mag}})}$$

Deformed $\mathrm{ISO}(7)_c$ Lagrangian $(m = gc)$

$$\mathbb{M} = 1, \dots, 56 \qquad \Lambda = 1, \dots, 28 \qquad I = 1, \dots, 7$$

$$\begin{aligned}\mathcal{L}_{\text{bos}} \;=\;& (R - V)\,\text{vol}_4 - \tfrac{1}{48} D\mathcal{M}_{\mathbb{MN}} \wedge * D\mathcal{M}^{\mathbb{MN}} + \tfrac{1}{2}\mathcal{I}_{\Lambda\Sigma}\,\mathcal{H}^{\Lambda}_{(2)} \wedge *\mathcal{H}^{\Sigma}_{(2)} - \tfrac{1}{2}\mathcal{R}_{\Lambda\Sigma}\,\mathcal{H}^{\Lambda}_{(2)} \wedge \mathcal{H}^{\Sigma}_{(2)} \\ &+ m\left[\mathcal{B}^I \wedge \left(\tilde{\mathcal{H}}_{(2)I} - \tfrac{g}{2}\delta_{IJ}\mathcal{B}^J\right) - \tfrac{1}{4}\tilde{\mathcal{A}}_I \wedge \tilde{\mathcal{A}}_J \wedge \left(d\mathcal{A}^{IJ} + \tfrac{g}{2}\,\delta_{KL}\,\mathcal{A}^{IK} \wedge \mathcal{A}^{JL}\right)\right]\end{aligned}$$

✦ Setting $m = 0$, all the magnetic pieces in the Lagrangian disappear.

∗ *Ingredients :*

- Electric vectors $(21 + 7)$: $\mathcal{A}^{IJ} = \mathcal{A}^{[IJ]}$ [SO(7)] and $\mathcal{A}^I$ [$\mathbb{R}^7$] with $\mathcal{H}^{\Lambda}_{(2)} = (\mathcal{H}^{IJ}_{(2)}\,,\,\mathcal{H}^{I}_{(2)})$
- Auxiliary magnetic vectors (7) : $\tilde{\mathcal{A}}_I$ [$\mathbb{R}^7$] with $\tilde{\mathcal{H}}_{(2)I}$
- $\mathrm{E}_7/\mathrm{SU}(8)$ scalars : $\mathcal{M}_{\mathbb{MN}}$ with cov. deriv. $D = \partial - \tfrac{1}{2}\,g\,\mathcal{A}^{IJ}\,T_{IJ} - \left(g\,\mathcal{A}^I - m\,\delta^{IJ}\,\tilde{\mathcal{A}}_J\right)T_I$
- Auxiliary two-forms (7) : $\mathcal{B}^I$ [$\mathbb{R}^7$]
- Topological term : $m\,[\;\dots\;]$
- Scalar potential : $\tfrac{1}{4}V(\mathcal{M}) \;=\; \dfrac{g^2}{672}\left(X_{\mathbb{MN}}{}^{\mathbb{R}}X_{\mathbb{PQ}}{}^{\mathbb{S}}\mathcal{M}^{\mathbb{MP}}\mathcal{M}^{\mathbb{NQ}}\mathcal{M}_{\mathbb{RS}} + 7X_{\mathbb{MN}}{}^{\mathbb{Q}}X_{\mathbb{PQ}}{}^{\mathbb{N}}\mathcal{M}^{\mathbb{MP}}\right)$

A truncation : $G_0 = SU(3)$ invariant sector

[Warner '83]

- Truncation : Retaining the fields and couplings which are invariant under the action of a subgroup $G_0 \subset \mathrm{ISO}(7)$

  - SU(8) R-symmetry branching : gravitini $\mathbf{8} \rightarrow \mathbf{1} + \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}}$ ⇨ $\mathcal{N} = 2$ SUSY

  - Scalars fields : $\mathbf{70} \rightarrow \mathbf{1}\ (\times 6) + \text{non-singlets}$ ⇨ 6 real scalars $(\varphi\,,\,\chi\,;\,\phi\,,\,a\,,\,\zeta\,,\,\tilde{\zeta}\,)$

  - Vector fields : $\mathbf{56} \rightarrow \mathbf{1}\,(\times 4) + \text{non-singlets}$ ⇨ vectors $(\,A^0, A^1\,;\,\tilde{A}_0, \tilde{A}_1\,)$

- $N = 2$ gauged supergravity coupled to 1 vector + 1 hyper : $\mathcal{M}_{\text{scalar}} = \dfrac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \dfrac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}$

- Non-compact $G = U(1) \times SO(1,1)_m$ dyonic gauging in the hyper sector

- Scalar potential :

$$
\begin{aligned}
V \;=\;& \tfrac{1}{2}\, g^2 \Big[e^{4\phi-3\varphi}(1+e^{2\varphi}\chi^2)^3 - 12\, e^{2\phi-\varphi}(1+e^{2\varphi}\chi^2) - 12\, e^{2\phi+\varphi}\rho^2(1-3\, e^{2\varphi}\chi^2) \\
& - 24\, e^{\varphi} + 12\, e^{4\phi+\varphi}\,\chi^2\,\rho^2\,(1+e^{2\varphi}\chi^2) + 12\, e^{4\phi+\varphi}\rho^4\,(1+3\, e^{2\varphi}\chi^2)\Big] \\
& - \tfrac{1}{2}\, g m\, \chi\, e^{4\phi+3\varphi}\,(12\,\rho^2 + 2\chi^2) + \tfrac{1}{2}\, m^2\, e^{4\phi+3\varphi}\ ,
\end{aligned}
$$

AdS critical points !!

AdS$_4$ solutions

$\mathcal{N}$	G_0	$c^{-1/3}\,\chi$	$c^{-1/3}\,e^{-\varphi}$	$c^{-1/3}\,\rho$	$c^{-1/3}\,e^{-\phi}$	$\frac{1}{4}\,g^{-2}\,c^{1/3}\,V_0$	M^2L^2
$\mathcal{N}=1$	G_2	$-\frac{1}{2^{7/3}}$	$\frac{5^{1/2}\,3^{1/2}}{2^{7/3}}$	$-\frac{1}{2^{7/3}}$	$\frac{5^{1/2}\,3^{1/2}}{2^{7/3}}$	$-\frac{2^{22/3}\,3^{1/2}}{5^{5/2}}$	$4\pm\sqrt{6}\,,\ -\frac{1}{6}(11\pm\sqrt{6})$
$\mathcal{N}=2$	$\mathrm{U}(3)$	$-\frac{1}{2}$	$\frac{3^{1/2}}{2}$	0	$\frac{1}{2^{1/2}}$	$-3^{3/2}$	$3\pm\sqrt{17}\,,\ 2\,,\ 2$
$\mathcal{N}=1$	$\mathrm{SU}(3)$	$\frac{1}{2^2}$	$\frac{3^{1/2}\,5^{1/2}}{2^2}$	$-\frac{3^{1/2}}{2^2}$	$\frac{5^{1/2}}{2^2}$	$-\frac{2^6\,3^{3/2}}{5^{5/2}}$	$4\pm\sqrt{6}\,,\ 4\pm\sqrt{6}$
$\mathcal{N}=0$	$\mathrm{SO}(6)_+$	0	$2^{1/6}$	0	$\frac{1}{2^{5/6}}$	$-3\,2^{5/6}$	$6\,,\ 6\,,\ -\frac{3}{4}\,,\ 0$
$\mathcal{N}=0$	$\mathrm{SO}(7)_+$	0	$\frac{1}{5^{1/6}}$	0	$\frac{1}{5^{1/6}}$	$-\frac{3\,5^{7/6}}{2^2}$	$6\,,\ -\frac{12}{5}\,,\ -\frac{6}{5}\,,\ -\frac{6}{5}$
$\mathcal{N}=0$	G_2	$\frac{1}{2^{4/3}}$	$\frac{3^{1/2}}{2^{4/3}}$	$\frac{1}{2^{4/3}}$	$\frac{3^{1/2}}{2^{4/3}}$	$-\frac{2^{10/3}}{3^{1/2}}$	$6\,,\ 6\,,\ -1\,,\ -1$
$\mathcal{N}=0$	$\mathrm{SU}(3)$	0.455	0.838	0.335	0.601	-5.864	$6.214\,,\ 5.925\,,\ 1.145\,,\ -1.284$
$\mathcal{N}=0$	$\mathrm{SU}(3)$	0.270	0.733	0.491	0.662	-5.853	$6.230\,,\ 5.905\,,\ 1.130\,,\ -1.264$

- Relevant for holographic RG flows (in progress…) !!

The truncated Lagrangian (I)

- The Lagrangian still contains a tensor field *B^0* :

$$
\begin{aligned}
\mathcal{L} \;=\;& (R-V)\,\mathrm{vol}_4 + \tfrac{3}{2}\left[d\varphi\wedge * d\varphi + e^{2\varphi}\,d\chi\wedge * d\chi\right] \\
+\;& 2\,d\phi\wedge * d\phi + \tfrac{1}{2}\,e^{2\phi}\,[D\zeta\wedge * D\zeta + D\tilde{\zeta}\wedge * D\tilde{\zeta}] \\
+\;& \tfrac{1}{2}\,e^{4\phi}\,[Da+\tfrac{1}{2}(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta)]\wedge *[Da+\tfrac{1}{2}(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta)] \\
+\;& \tfrac{1}{2}\,\mathcal{I}_{\Lambda\Sigma}\,H^{\Lambda}_{(2)}\wedge * H^{\Sigma}_{(2)} - \tfrac{1}{2}\,\mathcal{R}_{\Lambda\Sigma}\,H^{\Lambda}_{(2)}\wedge H^{\Sigma}_{(2)} + m\,B^0\wedge d\tilde{A}_0 \;+\; \tfrac{1}{2}\,g\,m\,B^0\wedge B^0
\end{aligned}
$$

where

$$
Da = da \;+\; g\,A^0 \;-\; m\,\tilde{A}_0 \quad , \quad D\zeta = d\zeta - 3\,g\,A^1\,\tilde{\zeta} \quad , \quad D\tilde{\zeta} = d\tilde{\zeta} + 3\,g\,A^1\,\zeta
$$

and with

$$
H^0_{(2)} = dA^0 + m\,B^0 \qquad , \qquad H^1_{(2)} = dA^1
$$

- Vector kinetic & θ-term : $\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\,\mathcal{I}_{\Lambda\Sigma} = \dfrac{1}{(2\,e^{\varphi}\,\chi + i)}\begin{pmatrix} -\dfrac{e^{3\varphi}}{(e^{\varphi}\,\chi - i)^2} & \dfrac{3\,e^{2\varphi}\,\chi}{(e^{\varphi}\,\chi - i)} \\ \dfrac{3\,e^{2\varphi}\,\chi}{(e^{\varphi}\,\chi - i)} & 3\,(e^{\varphi}\,\chi^2 + e^{-\varphi}) \end{pmatrix}$

✦ Non-dynamical tensor & magnetic vector via *Da + topological term* !!

The truncated Lagrangian (II) : dual formulation

- Solving the EOM for the magnetic vector, one finds

$$H^0_{(3)} \;=\; e^{4\phi} * \left(Da + \tfrac{1}{2}\,(\zeta\, D\tilde{\zeta} - \tilde{\zeta}\, D\zeta)\right) \qquad \Rightarrow \qquad \text{scalar-tensor duality!!}$$

- The dual Lagrangian

$$\begin{aligned} \widetilde{\mathcal{L}} \;=\; & (R - V)\,\mathrm{vol}_4 + \tfrac{1}{2}\, e^{-4\phi}\, H^0_{(3)} \wedge * H^0_{(3)} + \tfrac{3}{2}\left[d\varphi \wedge * d\varphi + e^{2\varphi}\, d\chi \wedge * d\chi\right] \\ + \;& 2\, d\phi \wedge * d\phi + \tfrac{1}{2}\, e^{2\phi}\left[D\zeta \wedge * D\zeta + D\tilde{\zeta} \wedge * D\tilde{\zeta}\right] \\ + \;& \tfrac{1}{2}\,\mathcal{I}_{\Lambda\Sigma}\, H^\Lambda_{(2)} \wedge * H^\Sigma_{(2)} - \tfrac{1}{2}\,\mathcal{R}_{\Lambda\Sigma}\, H^\Lambda_{(2)} \wedge H^\Sigma_{(2)} \\ - \;& H^0_{(3)} \wedge \left[g\, A^0 + \tfrac{1}{2}\,(\zeta D\tilde{\zeta} - \tilde{\zeta} D\zeta)\right] + \tfrac{1}{2}\, gm\, B^0 \wedge B^0\ . \end{aligned}$$

✦ Dynamical tensor $H^0_{(3)} = dB^0$ & Da and $\tilde{A}_0$ have disappeared from the Lagrangian

✦ Natural duality frame to investigate possible higher-dimensional origin!! [Calabi-Yau red.]

[Louis & Micu '02]

Dual formulations seem to be crucial

to understand

higher-dimensional origins!!

… let's give up the Lagrangian.

SL(7)-covariant duality hierarchy

[de Wit, Nicolai & Samtleben '08]
[Bergshoeff, Hartong, Hohm, Huebscher & Ortin '09]

- Restricted SL(7)-covariant field content [index I]

$\mathbf{1}$	metric :	ds_4^2	
$\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7}$	coset representatives :	$\mathcal{V}^{IJ\,ij}$, $\mathcal{V}^{I8\,ij}$, $\tilde{\mathcal{V}}_{IJ}{}^{ij}$, $\tilde{\mathcal{V}}_{I8}{}^{ij}$,	
$\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7}$	vectors :	$\mathcal{A}^{IJ}$, $\mathcal{A}^{I}$, $\tilde{\mathcal{A}}_{IJ}$, $\tilde{\mathcal{A}}_{I}$,	
$\mathbf{48} + \mathbf{7}'$	two-forms :	$\mathcal{B}_I{}^J$, $\mathcal{B}^I$,	[out of 133]
$\mathbf{28}'$	three-forms :	$\mathcal{C}^{IJ}$,	[out of 912]

- Two-form field strengths [$\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7}$]

$$
\begin{aligned}
\mathcal{H}_{(2)}^{IJ} &= d\mathcal{A}^{IJ} - g\,\delta_{KL}\,\mathcal{A}^{IK}\wedge\mathcal{A}^{LJ} \;, \\
\mathcal{H}_{(2)}^{I} &= d\mathcal{A}^{I} - g\,\delta_{JK}\,\mathcal{A}^{IJ}\wedge\mathcal{A}^{K} + \tfrac{1}{2}m\,\mathcal{A}^{IJ}\wedge\tilde{\mathcal{A}}_J + m\,\mathcal{B}^I \;, \\
\tilde{\mathcal{H}}_{(2)IJ} &= d\tilde{\mathcal{A}}_{IJ} + g\,\delta_{K[I}\,\mathcal{A}^{KL}\wedge\tilde{\mathcal{A}}_{J]L} + g\,\delta_{K[I}\,\mathcal{A}^{K}\wedge\tilde{\mathcal{A}}_{J]} - m\,\tilde{\mathcal{A}}_I\wedge\tilde{\mathcal{A}}_J + 2g\,\delta_{K[I}\,\mathcal{B}_{J]}{}^K \;, \\
\tilde{\mathcal{H}}_{(2)I} &= d\tilde{\mathcal{A}}_{I} - \tfrac{1}{2}g\,\delta_{IJ}\,\mathcal{A}^{JK}\wedge\tilde{\mathcal{A}}_K + g\,\delta_{IJ}\,\mathcal{B}^J \;,
\end{aligned}
$$

[i , j are SU(8) indices relevant to couple fermions]

- Three-form field strengths **[48 + 7′]**

$$
\begin{aligned}
\mathcal{H}_{(3)I}{}^{J} &= D\mathcal{B}_I{}^J + \tfrac{1}{2}\mathcal{A}^{JK}\wedge d\tilde{\mathcal{A}}_{IK} + \tfrac{1}{2}\mathcal{A}^{J}\wedge d\tilde{\mathcal{A}}_{I} + \tfrac{1}{2}\tilde{\mathcal{A}}_{IK}\wedge d\mathcal{A}^{JK} + \tfrac{1}{2}\tilde{\mathcal{A}}_{I}\wedge d\mathcal{A}^{J} \\
&\quad -\tfrac{1}{2}g\,\delta_{KL}\,\mathcal{A}^{JK}\wedge\mathcal{A}^{LM}\wedge\tilde{\mathcal{A}}_{IM} - \tfrac{1}{2}g\,\delta_{KL}\,\mathcal{A}^{JK}\wedge\mathcal{A}^{L}\wedge\tilde{\mathcal{A}}_{I} \\
&\quad +\tfrac{1}{6}g\,\delta_{IK}\,\mathcal{A}^{JL}\wedge\mathcal{A}^{KM}\wedge\tilde{\mathcal{A}}_{LM} - \tfrac{1}{3}g\,\delta_{IK}\,\mathcal{A}^{(J}\wedge\mathcal{A}^{K)L}\wedge\tilde{\mathcal{A}}_{L} \\
&\quad -\tfrac{1}{2}m\,\mathcal{A}^{JK}\wedge\tilde{\mathcal{A}}_{I}\wedge\tilde{\mathcal{A}}_{K} - 2g\,\delta_{IK}\,\mathcal{C}^{JK} - \tfrac{1}{7}\,\delta_I^J\,(\text{trace})\ , \\
\mathcal{H}^{I}_{(3)} &= D\mathcal{B}^I - \tfrac{1}{2}\mathcal{A}^{IJ}\wedge d\tilde{\mathcal{A}}_{J} - \tfrac{1}{2}\tilde{\mathcal{A}}_{J}\wedge d\mathcal{A}^{IJ} + \tfrac{1}{2}g\,\delta_{JK}\,\mathcal{A}^{IJ}\wedge\mathcal{A}^{KL}\wedge\tilde{\mathcal{A}}_{L}\ ,
\end{aligned}
$$

- Four-form field strengths **[28′]**

$$
\begin{aligned}
\mathcal{H}^{IJ}_{(4)} &= D\mathcal{C}^{IJ} - \mathcal{H}^{K(I}_{(2)}\wedge\mathcal{B}_K{}^{J)} + \mathcal{H}^{(I}_{(2)}\wedge\mathcal{B}^{J)} - \tfrac{1}{2}m\,\mathcal{B}^I\wedge\mathcal{B}^J - \tfrac{1}{6}\mathcal{A}^{K(I}\wedge\tilde{\mathcal{A}}_{KL}\wedge d\mathcal{A}^{J)L} \\
&\quad +\tfrac{1}{6}\mathcal{A}^{IK}\wedge\mathcal{A}^{JL}\wedge d\tilde{\mathcal{A}}_{KL} - \tfrac{1}{6}\mathcal{A}^{K(I}\wedge\tilde{\mathcal{A}}_{K}\wedge d\mathcal{A}^{J)} - \tfrac{1}{3}\mathcal{A}^{K(I}\wedge\mathcal{A}^{J)}\wedge d\tilde{\mathcal{A}}_{K} \\
&\quad -\tfrac{1}{6}\mathcal{A}^{(I}\wedge\tilde{\mathcal{A}}_{K}\wedge d\mathcal{A}^{J)K} - \tfrac{1}{6}g\,\delta_{KL}\,\mathcal{A}^{K(I}\wedge\mathcal{A}^{J)M}\wedge\mathcal{A}^{LN}\wedge\tilde{\mathcal{A}}_{MN} \\
&\quad +\tfrac{1}{6}g\,\delta_{KL}\,\mathcal{A}^{K(I}\wedge\mathcal{A}^{J)}\wedge\mathcal{A}^{LM}\wedge\tilde{\mathcal{A}}_{M} - \tfrac{1}{6}g\,\delta_{KL}\,\mathcal{A}^{K(I}\wedge\mathcal{A}^{J)M}\wedge\mathcal{A}^{L}\wedge\tilde{\mathcal{A}}_{M} \\
&\quad -\tfrac{1}{8}m\,\mathcal{A}^{IK}\wedge\mathcal{A}^{JL}\wedge\tilde{\mathcal{A}}_{K}\wedge\tilde{\mathcal{A}}_{L}\ .
\end{aligned}
$$

- Closed set of Bianchi identities [consistency]

$$D\mathcal{H}^{IJ}_{(2)} = 0 \;\;,\;\; D\mathcal{H}^{I}_{(2)} = m\,\mathcal{H}^{I}_{(3)} \;\;,\;\; D\tilde{\mathcal{H}}_{(2)IJ} = -2\,g\,\mathcal{H}_{(3)[I}{}^{K}\,\delta_{J]K} \;\;,\;\; D\tilde{\mathcal{H}}_{(2)I} = g\,\delta_{IJ}\,\mathcal{H}^{J}_{(3)} \;,$$

$$D\mathcal{H}_{(3)I}{}^{J} = \mathcal{H}^{JK}_{(2)}\wedge\tilde{\mathcal{H}}_{(2)IK} + \mathcal{H}^{J}_{(2)}\wedge\tilde{\mathcal{H}}_{(2)I} - 2g\,\delta_{IK}\,\mathcal{H}^{JK}_{(4)} - \tfrac{1}{7}\,\delta^{J}_{I}\,(\text{trace}) \;,$$

$$D\mathcal{H}^{I}_{(3)} = -\mathcal{H}^{IJ}_{(2)}\wedge\tilde{\mathcal{H}}_{(2)J} \quad,\qquad D\mathcal{H}^{IJ}_{(4)} \equiv 0 \;.$$

- Closed set of duality relations [right number of d.o.f] [short-hand notation]

$$\begin{aligned}
\tilde{\mathcal{H}}_{(2)IJ} &= -\tfrac{1}{2}\mathcal{I}_{[IJ][KL]} * \mathcal{H}^{KL}_{(2)} - \mathcal{I}_{[IJ][K8]} * \mathcal{H}^{K}_{(2)} + \tfrac{1}{2}\mathcal{R}_{[IJ][KL]}\,\mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[IJ][K8]}\,\mathcal{H}^{K}_{(2)} \;,\\
\tilde{\mathcal{H}}_{(2)I} &= -\tfrac{1}{2}\mathcal{I}_{[I8][KL]} * \mathcal{H}^{KL}_{(2)} - \mathcal{I}_{[I8][K8]} * \mathcal{H}^{K}_{(2)} + \tfrac{1}{2}\mathcal{R}_{[I8][KL]}\,\mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[I8][K8]}\,\mathcal{H}^{K}_{(2)} \;,\\
\mathcal{H}_{(3)I}{}^{J} &= -\tfrac{1}{12}(t_I{}^{J})_{\mathbb{M}}{}^{\mathbb{P}}\,\mathcal{M}_{\mathbb{NP}} * D\mathcal{M}^{\mathbb{MN}} - \tfrac{1}{7}\,\delta^{J}_{I}\,(\text{trace}) \;,\\
\mathcal{H}_{(3)}{}^{I} &= -\tfrac{1}{12}(t_8{}^{I})_{\mathbb{M}}{}^{\mathbb{P}}\,\mathcal{M}_{\mathbb{NP}} * D\mathcal{M}^{\mathbb{MN}} \;,\\
\mathcal{H}^{IJ}_{(4)} &= \tfrac{1}{84}X_{\mathbb{NQ}}{}^{\mathbb{S}}\big((t_K{}^{(I|})_{\mathbb{P}}{}^{\mathbb{R}}\mathcal{M}^{|J)K\,\mathbb{N}} + (t_8{}^{(I|})_{\mathbb{P}}{}^{\mathbb{R}}\mathcal{M}^{|J)8\,\mathbb{N}}\big)\big(\mathcal{M}^{\mathbb{PQ}}\mathcal{M}_{\mathbb{RS}} + 7\,\delta^{\mathbb{P}}_{\mathbb{S}}\,\delta^{\mathbb{Q}}_{\mathbb{R}}\big)\text{vol}_4 \;.
\end{aligned}$$

$$\boxed{\text{tensor hierarchy + duality relations = duality hierarchy}}$$

[t's are $SL(7)\times R^7$ generators]

- Closed set of SUSY transformations [consistency]

[vielbein and scalars]

$$\delta e_\mu{}^\alpha = \tfrac{1}{4}\,\bar\epsilon_i\,\gamma^\alpha\,\psi_\mu{}^i + \tfrac{1}{4}\,\bar\epsilon^i\,\gamma^\alpha\,\psi_{\mu i}$$

$$\delta\mathcal{V}_{\mathbb{M}}{}^{ij} = \tfrac{1}{\sqrt{2}}\,\mathcal{V}_{\mathbb{M}\,kl}\,\left(\bar\epsilon^{[i}\,\chi^{jkl]} + \tfrac{1}{4!}\,\varepsilon^{ijklmnpq}\,\bar\epsilon_m\,\chi_{npq}\right)$$

[vectors]

$$\delta\mathcal{A}_\mu{}^{IJ} = i\,\mathcal{V}^{IJ}{}_{ij}\left(\bar\epsilon^i\psi_\mu{}^j + \tfrac{1}{2\sqrt{2}}\,\bar\epsilon_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}$$

$$\delta\mathcal{A}_\mu{}^{I} = i\,\mathcal{V}^{I8}{}_{ij}\left(\bar\epsilon^i\psi_\mu{}^j + \tfrac{1}{2\sqrt{2}}\,\bar\epsilon_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}$$

$$\delta\tilde{\mathcal{A}}_{\mu\,IJ} = -i\,\tilde{\mathcal{V}}_{IJ\,ij}\left(\bar\epsilon^i\psi_\mu{}^j + \tfrac{1}{2\sqrt{2}}\,\bar\epsilon_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}$$

$$\delta\tilde{\mathcal{A}}_{\mu\,I} = -i\,\tilde{\mathcal{V}}_{I8\,ij}\left(\bar\epsilon^i\psi_\mu{}^j + \tfrac{1}{2\sqrt{2}}\,\bar\epsilon_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}$$

[two-forms]

$$\begin{aligned}\delta\mathcal{B}_{\mu\nu\,J}{}^I = \Big[&-\tfrac{2}{3}\big(\mathcal{V}^{IK}{}_{jk}\,\tilde{\mathcal{V}}_{JK}{}^{ik} + \mathcal{V}^{I8}{}_{jk}\,\tilde{\mathcal{V}}_{J8}{}^{ik} + \tilde{\mathcal{V}}_{JK\,jk}\,\mathcal{V}^{IKik} + \tilde{\mathcal{V}}_{J8\,jk}\,\mathcal{V}^{I8ik}\big)\bar\epsilon_i\gamma_{[\mu}\psi^j_{\nu]} \\ &-\tfrac{\sqrt{2}}{3}\,\big(\mathcal{V}^{IK}{}_{ij}\,\tilde{\mathcal{V}}_{JK\,kl} + \mathcal{V}^{I8}{}_{ij}\,\tilde{\mathcal{V}}_{J8\,kl}\big)\,\bar\epsilon^{[i}\gamma_{\mu\nu}\chi^{jkl]} + \text{h.c.}\Big] \\ &+\big(\mathcal{A}^{IK}_{[\mu}\,\delta\tilde{\mathcal{A}}_{\nu]JK} + \mathcal{A}^{I}_{[\mu}\,\delta\tilde{\mathcal{A}}_{\nu]J} + \tilde{\mathcal{A}}_{[\mu|\,JK}\,\delta\mathcal{A}_{|\nu]}{}^{IK} + \tilde{\mathcal{A}}_{[\mu|\,J}\,\delta\mathcal{A}_{|\nu]}{}^{I}\big) - \tfrac{1}{7}\,\delta^I_J\,(\text{trace})\,,\end{aligned}$$

$$\begin{aligned}\delta\mathcal{B}_{\mu\nu}{}^I = \Big[&\tfrac{2}{3}\big(\mathcal{V}^{IJ}{}_{jk}\,\tilde{\mathcal{V}}_{J8}{}^{ik} + \tilde{\mathcal{V}}_{J8\,jk}\,\mathcal{V}^{IJik}\big)\,\bar\epsilon_i\gamma_{[\mu}\psi^j_{\nu]} + \tfrac{\sqrt{2}}{3}\,\mathcal{V}^{IJ}{}_{ij}\,\tilde{\mathcal{V}}_{J8\,kl}\,\bar\epsilon^{[i}\gamma_{\mu\nu}\chi^{jkl]} + \text{h.c.}\Big] \\ &-\big(\mathcal{A}^{IJ}_{[\mu}\,\delta\tilde{\mathcal{A}}_{\nu]J} + \tilde{\mathcal{A}}_{[\mu|\,J}\,\delta\mathcal{A}_{|\nu]}{}^{IJ}\big)\,.\end{aligned}$$

[three-forms]

$$
\begin{aligned}
\delta\mathcal{C}_{\mu\nu\rho}{}^{IJ} = \Big[- \tfrac{4i}{7} \Big(& \mathcal{V}^{K(I}{}_{jl}\big(\mathcal{V}^{J)L\,lk}\,\tilde{\mathcal{V}}_{KL\,ik} + \tilde{\mathcal{V}}_{KL}{}^{lk}\,\mathcal{V}^{J)L}{}_{ik}\big) \\
& + \mathcal{V}^{K(I}{}_{jl}\big(\mathcal{V}^{J)8\,lk}\,\tilde{\mathcal{V}}_{K8\,ik} + \tilde{\mathcal{V}}_{K8}{}^{lk}\,\mathcal{V}^{J)8}{}_{ik}\big) \\
& + \mathcal{V}^{(I|8}{}_{jl}\big(\mathcal{V}^{|J)K\,lk}\,\tilde{\mathcal{V}}_{K8\,ik} + \tilde{\mathcal{V}}_{K8}{}^{lk}\,\mathcal{V}^{|J)K}{}_{ik}\big)\Big)\,\bar{\epsilon}^{i}\gamma_{[\mu\nu}\psi^{j}_{\rho]} \\
+\, i\tfrac{\sqrt{2}}{3}\Big(& \mathcal{V}^{K(I|hi}\,\mathcal{V}^{|J)L}{}_{[ij|}\,\tilde{\mathcal{V}}_{KL|kl]} + \mathcal{V}^{K(Ihi}\,\mathcal{V}^{J)8}{}_{[ij|}\,\tilde{\mathcal{V}}_{K8|kl]} \\
& + \mathcal{V}^{(I|8hi}\,\mathcal{V}^{|J)K}{}_{[ij|}\,\tilde{\mathcal{V}}_{K8|kl]}\Big)\,\bar{\epsilon}_{h}\gamma_{\mu\nu\rho}\chi^{jkl} \; + \mathrm{h.c.}\Big] \\
-\, 3\Big(& \mathcal{B}_{[\mu\nu|K}{}^{(I}\,\delta\mathcal{A}^{J)K}_{|\rho]} + \mathcal{B}_{[\mu\nu}{}^{(I}\,\delta\mathcal{A}^{J)}_{\rho]}\Big) \\
+\, \mathcal{A}^{K(I}_{[\mu} & \big(\mathcal{A}^{J)L}_{\nu}\,\delta\tilde{\mathcal{A}}_{\rho]KL} + \tilde{\mathcal{A}}_{\nu KL}\,\delta\mathcal{A}^{J)L}_{\rho]}\big) + \mathcal{A}^{K(I}_{[\mu}\big(\mathcal{A}^{J)}_{\nu}\,\delta\tilde{\mathcal{A}}_{\rho]K} + \tilde{\mathcal{A}}_{\nu K}\,\delta\mathcal{A}^{J)}_{\rho]}\big) \\
& + \mathcal{A}^{(I}_{[\mu}\big(\mathcal{A}^{J)K}_{\nu}\,\delta\tilde{\mathcal{A}}_{\rho]K} + \tilde{\mathcal{A}}_{\nu K}\,\delta\mathcal{A}^{J)K}_{\rho]}\big)\;.
\end{aligned}
$$

… all scalars, vectors and the fermions should be kept !!

- How does the scalar potential potential V fit in the duality hierarchy ?

$$\boxed{\Theta_{\mathbb{M}}{}^{\alpha}\,\mathcal{H}_{(4)\alpha}{}^{\mathbb{M}} = -2\,V\,\mathrm{vol}_4}$$

- In our deformed ISO(7)$_c$ theory, one has four-form field strengths

$$g\,\delta_{IJ}\,\mathcal{H}^{IJ}_{(4)} + m\,\tilde{\mathcal{H}}_{(4)} = -2\,V\,\mathrm{vol}_4$$ [**28′** + **1** of SL(7)]

where we need the **SL(7)-*singlet*** four-form field strength $\tilde{\mathcal{H}}_{(4)}$ dual to the magnetic ET

- Consistency requires also the three-form field strength $\mathcal{H}_{(3)}$ rendering $\mathcal{H}_{(3)I}{}^{J}$ traceful

$$\begin{aligned}
\mathcal{H}_{(3)} &= \tfrac{1}{12}(t_8{}^8)_{\mathbb{M}}{}^{\mathbb{P}}\,\mathcal{M}_{\mathbb{N}\mathbb{P}} * D\mathcal{M}^{\mathbb{M}\mathbb{N}}\ ,\\
\tilde{\mathcal{H}}_{(4)} &= \tfrac{1}{84}\,X_{\mathbb{N}\mathbb{Q}}{}^{\mathbb{S}}(t_8{}^{K})_{\mathbb{P}}{}^{\mathbb{R}}\mathcal{M}_{8K}{}^{\mathbb{N}}\mathcal{M}^{\mathbb{P}\mathbb{Q}}\mathcal{M}_{\mathbb{R}\mathbb{S}}\,\mathrm{vol}_4
\end{aligned}$$

✱ Extended BI's :

$$\begin{aligned}
D\mathcal{H}_{(3)} &= \mathcal{H}^{IJ}_{(2)}\wedge\tilde{\mathcal{H}}_{(2)IJ} + \mathcal{H}^{I}_{(2)}\wedge\tilde{\mathcal{H}}_{(2)I} - 2g\,\delta_{IJ}\,\mathcal{H}^{IJ}_{(4)} - 14\,m\,\tilde{\mathcal{H}}_{(4)}\\
D\tilde{\mathcal{H}}_{(4)} &\equiv 0\ ,
\end{aligned}$$

Q : Why to bother with all these duality hierarchy issues?

A : Because the duality hierarchy allows us to derive simple embedding formulas to put the 4D *dynamics* into a higher-dimensional one.

Remember : *No index, no clue*. Good luck trying to embed V into higher dimensions…

Massive type IIA strings on S^6

Collecting clues

- The deformed $\mathrm{ISO}(7)_c$ gauging has its SO(7) piece untouched by the deformation. This points towards an undeformed S^6 description in higher dimension.

- If the higher-dimensional geometry is not affected, it should then be the higher-dimensional theory the one changing. The massive IIA theory by Romans proves a natural candidate. [Romans '86]

- The Romans mass parameter $\hat{F}_{(0)}$ is a discrete (on/off) deformation, exactly as the parameter c in the deformed $\mathrm{ISO}(7)_c$ theory.

- The SU(3)-invariant sector of the $\mathrm{ISO}(7)_c$ theory connects to CY_3 reductions of massive IIA upon the dualisation of some field. It is then natural to believe that the embedding of the full $\mathrm{ISO}(7)_c$ theory would require an enlarged set of duality relations… duality hierarchy!!

Derivation of the IIA embedding [4-step process]

* Step 1 : 10D KK expansion that leaves 4D spacetime symmetry manifest

$A(x,y)$'s and $B(x,y)$'s fields

* Step 2 : Redefinitions of the A's and B's fields to conform 4D SUSY transformations

$C(x,y)$'s fields

* Step 3 : Connection to 4D fields by dressing up with S^6 geometrical data

$$C(x,y) = \text{geometry}(y) \times \mathcal{C}(x)$$

* Step 4 : Plug and play

[Following prescription by de Wit & Nicolai '86, Godazgar, Godazgar & Nicolai '13, …]

✱ Step 1 : 10D redefinitions (KK expansion) that leave 4D spacetime symmetry manifest

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6)$$

Then one has :

$$\begin{aligned}
d\hat{s}^2_{10} &= \Delta^{-1}\, ds^2_4 + g_{mn}\big(dy^m + B^m\big)\big(dy^n + B^n\big) \ , \\
\hat{A}_{(3)} &= \tfrac{1}{6} A_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} A_{\mu\nu m}\, dx^\mu \wedge dx^\nu \wedge \big(dy^m + B^m\big) \\
&\quad + \tfrac{1}{2} A_{\mu mn}\, dx^\mu \wedge \big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \\
&\quad + \tfrac{1}{6} A_{mnp}\big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \wedge \big(dy^p + B^p\big) \ , \\
\hat{B}_{(2)} &= \tfrac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu + B_{\mu m}\, dx^\mu \wedge \big(dy^m + B^m\big) + \tfrac{1}{2} B_{mn}\big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \ , \\
\hat{A}_{(1)} &= A_\mu\, dx^\mu + A_m\big(dy^m + B^m\big) \ ,
\end{aligned}$$

In terms of representations of SL(6) [index m] :

$\mathbf{1}$	metric :	ds^2_4 ,
$\mathbf{21} + \mathbf{6} + \mathbf{1} + \mathbf{20} + \mathbf{15}$	scalars :	g_{mn} , A_m , $\hat{\phi}$, A_{mnp} , B_{mn} ,
$\mathbf{6}' + \mathbf{1} + \mathbf{15} + \mathbf{6}$	vectors :	$B_\mu{}^m$, A_μ , $A_{\mu mn}$, $B_{\mu m}$,
$\mathbf{6} + \mathbf{1}$	two-forms :	$A_{\mu\nu m}$, $B_{\mu\nu}$,
$\mathbf{1}$	three-form :	$A_{\mu\nu\rho}$.

$$\Delta^2 \equiv \frac{\det\ g_{mn}}{\det\ \mathring{g}_{mn}}$$

∗ Step 2 : Non-linear field redefinitions to conform 4D SUSY transformations

- Vectors : $C_\mu{}^{m8} \equiv B_\mu{}^m \quad , \quad C_\mu{}^{78} \equiv A_\mu \quad , \quad \tilde{C}_{\mu\, mn} \equiv A_{\mu mn} - A_\mu B_{mn} \quad , \quad \tilde{C}_{\mu\, m7} \equiv B_{\mu m}$
- Two-forms : $C_{\mu\nu\, m}{}^8 \equiv -A_{\mu\nu m} + C_{[\mu}{}^{n8}\, \tilde{C}_{\nu]nm} + C_{[\mu}{}^{78}\, \tilde{C}_{\nu]m7} \quad , \quad C_{\mu\nu\, 7}{}^8 \equiv -B_{\mu\nu} + C_{[\mu}{}^{m8}\, \tilde{C}_{\nu]m7}$
- Three-form : $C_{\mu\nu\rho}{}^{88} \equiv A_{\mu\nu\rho} - C_{[\mu}{}^{m8}\, C_\nu{}^{n8}\, \tilde{C}_{\rho]mn} + C_{[\mu}{}^{m8}\, C_\nu{}^{78}\, \tilde{C}_{\rho]m7} + 3\, C_{[\mu}{}^{78} C_{\nu\rho]7}{}^8$

These can be rearranged into representations of SL(7) [index *I*]

$$C_\mu{}^{I8} = (C_\mu{}^{m8}\, ,\, C_\mu{}^{78}) \qquad \tilde{C}_{\mu\, IJ} = (\tilde{C}_{\mu\, mn}\, ,\, \tilde{C}_{\mu\, m7}) \qquad C_{\mu\nu\, I}{}^8 = (C_{\mu\nu\, m}{}^8\, ,\, C_{\mu\nu\, 7}{}^8) \qquad C_{\mu\nu\rho}{}^{88}$$

with 10D SUSY transfs:

$$\delta C_\mu{}^{I8} = i\, V^{I8}{}_{AB} \left(\bar{\epsilon}^A \psi_\mu{}^B + \tfrac{1}{2\sqrt{2}}\, \bar{\epsilon}_C \gamma_\mu \chi^{ABC} \right) + \text{h.c.}\,,$$

$$\delta \tilde{C}_{\mu\, IJ} = -i\, V_{IJ\, AB} \left(\bar{\epsilon}^A \psi_\mu{}^B + \tfrac{1}{2\sqrt{2}}\, \bar{\epsilon}_C \gamma_\mu \chi^{ABC} \right) + \text{h.c.}\,,$$

$$\begin{aligned} \delta C_{\mu\nu\, I}{}^8 = & \Big[\tfrac{2}{3} \big(V^{J8}{}_{BC}\, \tilde{V}_{IJ}{}^{AC} + \tilde{V}_{IJ\, BC}\, V^{J8\, AC} \big)\, \bar{\epsilon}_A \gamma_{[\mu} \psi^B_{\nu]} \\ & + \tfrac{\sqrt{2}}{3}\, V^{J8}{}_{AB}\, \tilde{V}_{IJ\, CD}\, \bar{\epsilon}^{[A} \gamma_{\mu\nu} \chi^{BCD]} + \text{h.c.} \Big] - C_{[\mu}{}^{J8}\, \delta \tilde{C}_{\nu]IJ} - \tilde{C}_{[\mu|\, IJ}\, \delta C_{|\nu]}{}^{J8}\,, \end{aligned}$$

$$\begin{aligned} \delta C_{\mu\nu\rho}{}^{88} = & \Big[\tfrac{4i}{7}\, V^{I8}{}_{BD} \big(V^{J8DC}\, \tilde{V}_{IJ\, AC} + \tilde{V}_{IJ}{}^{DC}\, V^{J8}{}_{AC} \big)\, \bar{\epsilon}^A \gamma_{[\mu\nu} \psi^B_{\rho]} \\ & - i \tfrac{\sqrt{2}}{3}\, V^{I8\, AE}\, V^{J8}{}_{[EB|}\, \tilde{V}_{IJ\, |CD]}\, \bar{\epsilon}_A \gamma_{\mu\nu\rho} \chi^{BCD} + \text{h.c.} \Big] \\ & + 3\, C_{[\mu\nu|\, I}{}^8\, \delta C_{|\rho]}{}^{I8} - C_{[\mu}{}^{I8} \big(C_\nu{}^{J8}\, \delta \tilde{C}_{\rho]IJ} + \tilde{C}_{\nu|\, IJ}\, \delta C_{|\rho]}{}^{J8} \big)\,. \end{aligned}$$

Like those of the tensor hierarchy !!

… given in terms of a generalised vielbein

$$V^{I8}{}_{AB} = (V^{m8}{}_{AB}\,,\, V^{78}{}_{AB}) \qquad , \qquad \tilde{V}_{IJ\,AB} = (\tilde{V}_{mn\,AB}\,,\, \tilde{V}_{m7\,AB}) \,.$$

with components

$$\begin{aligned}
V^{m8\,AB} &= -\tfrac{1}{4}\,\Delta^{-\frac{1}{2}}\,e_a{}^m(\Gamma^a C^{-1})^{AB}\,,\\
V^{78\,AB} &= -\tfrac{1}{4}\,e^{-\frac{3}{4}\hat{\phi}}\,\Delta^{-\frac{1}{2}}\,(\Gamma_7 C^{-1})^{AB} - V^{m8\,AB}A_m\,,\\
\tilde{V}_{m7}{}^{AB} &= \tfrac{1}{4}\,e^{\frac{1}{2}\hat{\phi}}\,\Delta^{-\frac{1}{2}}\,e_m{}^a(\Gamma_a\Gamma_7 C^{-1})^{AB} + V^{n8\,AB}B_{nm}\,,\\
\tilde{V}_{mn}{}^{AB} &= \tfrac{1}{4}\,e^{-\frac{1}{4}\hat{\phi}}\,\Delta^{-\frac{1}{2}}\,e_m{}^a e_n{}^b(\Gamma_{ab}C^{-1})^{AB} + V^{p8\,AB}(A_{pmn} - 2B_{p[m}A_{n]})\\
&\quad + V^{78\,AB}B_{mn} + 2\,\tilde{V}_{[m|7}{}^{AB}\,A_{|n]}
\end{aligned}$$

and

$$\begin{aligned}
V^{m8}{}_{AB} &= \tfrac{1}{4}\,\Delta^{-\frac{1}{2}}\,e_a{}^m(C\Gamma^a)_{AB}\,,\\
V^{78}{}_{AB} &= \tfrac{1}{4}\,e^{-\frac{3}{4}\hat{\phi}}\,\Delta^{-\frac{1}{2}}\,(C\Gamma_7)_{AB} - V^{m8}{}_{AB}\,A_m\,,\\
\tilde{V}_{m7\,AB} &= \tfrac{1}{4}\,e^{\frac{1}{2}\hat{\phi}}\,\Delta^{-\frac{1}{2}}\,e_m{}^a(C\Gamma_a\Gamma_7)_{AB} + V^{n8}{}_{AB}\,B_{nm}\,,\\
\tilde{V}_{mn\,AB} &= \tfrac{1}{4}\,e^{-\frac{1}{4}\hat{\phi}}\,\Delta^{-\frac{1}{2}}\,e_m{}^a e_n{}^b(C\Gamma_{ab})_{AB} + V^{p8}{}_{AB}(A_{pmn} - 2B_{p[m}A_{n]})\\
&\quad + V^{78}{}_{AB}\,B_{mn} + 2\,\tilde{V}_{[m|7\,AB}A_{|n]}
\end{aligned}$$

The result is then a set of SL(7)-covariant 10D fields :

$$
\begin{array}{rcl}
\mathbf{1} & \text{metric :} & ds_4^2\,(x,y)\ , \\
\mathbf{7}' + \mathbf{21} & \text{generalised vielbeine :} & V^{I8}{}_{AB}(x,y)\ ,\ \tilde{V}_{IJ\,AB}(x,y)\ , \\
\mathbf{7}' + \mathbf{21} & \text{vectors :} & C_\mu{}^{I8}(x,y)\ ,\ \tilde{C}_{\mu\,IJ}(x,y)\ , \\
\mathbf{7} & \text{two-forms :} & C_{\mu\nu\,I}{}^8(x,y)\ , \\
\mathbf{1} & \text{three-form :} & C_{\mu\nu\rho}{}^{88}(x,y)\ .
\end{array}
$$

that is to be connected with the SL(7)-covariant 4D fields of the tensor hierarchy :

$$
\begin{array}{rcl}
\mathbf{1} & \text{metric :} & ds_4^2\,(x)\ , \\
\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7} & \text{coset representatives :} & \mathcal{V}^{IJ\,ij}(x)\ ,\ \mathcal{V}^{I8\,ij}(x)\ ,\ \tilde{\mathcal{V}}_{IJ}{}^{ij}(x)\ ,\ \tilde{\mathcal{V}}_{I8}{}^{ij}(x)\ , \\
\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7} & \text{vectors :} & \mathcal{A}_\mu{}^{IJ}(x)\ ,\ \mathcal{A}_\mu{}^{I}(x)\ ,\ \tilde{\mathcal{A}}_{\mu\,IJ}(x)\ ,\ \tilde{\mathcal{A}}_{\mu\,I}(x)\ , \\
\mathbf{48} + \mathbf{7}' & \text{two-forms :} & \mathcal{B}_{\mu\nu\,I}{}^{J}(x)\ ,\ \mathcal{B}_{\mu\nu}{}^{I}(x)\ , \\
\mathbf{28}' & \text{three-forms :} & \mathcal{C}_{\mu\nu\rho}{}^{IJ}(x)\ ,
\end{array}
$$

> This connection is established by using geometrical data of the S^6 !!

∗ Step 3 : Connecting 4D [SL(7)] and 10D [SL(6)] fields using the S^6 geometrical data in the dressing up process

[vielbein and scalars]

$$ds_4^2(x,y) = ds_4^2(x)$$

$$\begin{aligned} V^{m8\,AB}(x,y) &= \tfrac{1}{2}\, g\, K^m_{IJ}(y)\, \eta_i^A(y)\, \eta_j^B(y)\, \mathcal{V}^{IJ\,ij}(x) \,, \\ V^{78\,AB}(x,y) &= -\mu_I(y)\, \eta_i^A(y)\, \eta_j^B(y)\, \mathcal{V}^{I8\,ij}(x) \,, \\ \tilde{V}_{mn}{}^{AB}(x,y) &= \tfrac{1}{4}\, K^{IJ}_{mn}(y)\, \eta_i^A(y)\, \eta_j^B(y)\, \tilde{\mathcal{V}}_{IJ}{}^{ij}(x) \,, \\ \tilde{V}_{m7}{}^{AB}(x,y) &= -g^{-1}\, (\partial_m \mu^I)(y)\, \eta_i^A(y)\, \eta_j^B(y)\, \tilde{\mathcal{V}}_{I8}{}^{ij}(x) \,, \end{aligned}$$

[two-forms]

$$C_{\mu\nu\, m}{}^{8}(x,y) = -g^{-1}\, (\mu_I \partial_m \mu^J)(y)\, \mathcal{B}_{\mu\nu\, J}{}^{I}(x)$$

$$C_{\mu\nu\, 7}{}^{8}(x,y) = \mu_I(y)\, \mathcal{B}_{\mu\nu}{}^{I}(x)$$

[three-form]

$$C_{\mu\nu\rho}{}^{88}(x,y) = (\mu_I \mu_J)(y)\, \mathcal{C}_{\mu\nu\rho}{}^{IJ}(x)$$

[vectors]

$$C_\mu{}^{m8}(x,y) = \tfrac{1}{2}\, g\, K^m_{IJ}(y)\, \mathcal{A}_\mu{}^{IJ}(x) \quad , \quad C_\mu{}^{78}(x,y) = -\mu_I(y)\, \mathcal{A}_\mu{}^{I}(x) \,,$$

$$\tilde{C}_{\mu\, mn}(x,y) = \tfrac{1}{4}\, K^{IJ}_{mn}(y)\, \tilde{\mathcal{A}}_{\mu\, IJ}(x) \quad , \quad \tilde{C}_{\mu\, m7}(x,y) = -g^{-1}\, (\partial_m \mu^I)(y)\, \tilde{\mathcal{A}}_{\mu\, I}(x)$$

✦ S^6 geometrical data : embedding coordinates μ^I, Killing vectors K^m_{IJ} and tensors K^{mn}_{IJ}

∗ Step 4 : Plug and play… so that the final embedding of ISO(7)$_c$ into type IIA is

$$
\boxed{
\begin{aligned}
d\hat{s}_{10}^2 &= \Delta^{-1}\, ds_4^2 + g_{mn}\, Dy^m\, Dy^n \ , \\
\hat{A}_{(3)} &= \mu_I \mu_J \left(\mathcal{C}^{IJ} + \mathcal{A}^I \wedge \mathcal{B}^J + \tfrac{1}{6} \mathcal{A}^{IK} \wedge \mathcal{A}^{JL} \wedge \tilde{\mathcal{A}}_{KL} + \tfrac{1}{6} \mathcal{A}^I \wedge \mathcal{A}^{JK} \wedge \tilde{\mathcal{A}}_K \right) \\
&\quad + g^{-1} \left(\mathcal{B}_J{}^I + \tfrac{1}{2} \mathcal{A}^{IK} \wedge \tilde{\mathcal{A}}_{KJ} + \tfrac{1}{2} \mathcal{A}^I \wedge \tilde{\mathcal{A}}_J \right) \wedge \mu_I D\mu^J + \tfrac{1}{2} g^{-2}\, \tilde{\mathcal{A}}_{IJ} \wedge D\mu^I \wedge D\mu^J \\
&\quad - \tfrac{1}{2}\, \mu_I\, B_{mn}\, \mathcal{A}^I \wedge Dy^m \wedge Dy^n + \tfrac{1}{6} A_{mnp}\, Dy^m \wedge Dy^n \wedge Dy^p \ , \\
\hat{B}_{(2)} &= -\mu_I \left(\mathcal{B}^I + \tfrac{1}{2} \mathcal{A}^{IJ} \wedge \tilde{\mathcal{A}}_J \right) - g^{-1}\, \tilde{\mathcal{A}}_I \wedge D\mu^I + \tfrac{1}{2} B_{mn}\, Dy^m \wedge Dy^n \ , \\
\hat{A}_{(1)} &= -\mu_I\, \mathcal{A}^I + A_m\, Dy^m \ .
\end{aligned}
}
$$

where we have defined : $\quad Dy^m \equiv dy^m + \tfrac{1}{2}\, g\, K^m_{IJ}\, \mathcal{A}^{IJ} \quad , \quad D\mu^I \equiv d\mu^I - g\, \mathcal{A}^{IJ} \mu_J$

The scalars are embedded as

$$
\begin{aligned}
g^{mn} &= \tfrac{1}{4}\, g^2\, \Delta\, \mathcal{M}^{IJ\,KL}\, K^m_{IJ}\, K^n_{KL} \quad , \quad & B_{mn} &= -\tfrac{1}{2}\, \Delta\, g_{mp}\, K^p_{IJ}\, \partial_n \mu^K\, \mathcal{M}^{IJ}{}_{K8} \ , \\
A_m &= \tfrac{1}{2}\, g\, \Delta\, g_{mn}\, K^n_{IJ}\, \mu_K\, \mathcal{M}^{IJ\,K8} \quad , \quad & A_{mnp} &= \tfrac{1}{8}\, g\, \Delta\, g_{mq}\, K^q_{IJ}\, K^{KL}_{np}\, \mathcal{M}^{IJ}{}_{KL} + A_m B_{np} \ .
\end{aligned}
$$

Remarks and consistency checks

- 10D (bosonic) SUSY transformations exactly reduce to those of the 4D tensor hierarchy.

- Computing the 10D field strengths $\hat{F}_{(2)} = d\hat{A}_{(1)} + m\,\hat{B}_{(2)}$, etc. one finds

$$\hat{F}_{(4)} = \mu_I \mu_J\,\mathcal{H}_{(4)}^{IJ} + g^{-1}\,\mathcal{H}_{(3)\,J}{}^{I} \wedge \mu_I D\mu^J + \tfrac{1}{2}\,g^{-2}\,\tilde{\mathcal{H}}_{(2)IJ} \wedge D\mu^I \wedge D\mu^J + \dots\ ,$$
$$\hat{H}_{(3)} = -\mu_I\,\mathcal{H}_{(3)}^{I} - g^{-1}\,\tilde{\mathcal{H}}_{(2)I} \wedge D\mu^I + \dots\ ,$$
$$\hat{F}_{(2)} = -\mu_I \mathcal{H}_{(2)}^{I} + g^{-1}\left(g\,\delta_{IJ}\,\mathcal{A}^J - m\,\tilde{\mathcal{A}}_I\right) \wedge D\mu^I + \dots\ ,$$

which are expressed in terms of the 4D tensor hierarchy. The parameter m only appears through standard Romans' redefinitions of $\hat{F}_{(p)}$ in 10D (formulas also valid for massless IIA)

- The set of Bianchi identities of the above 10D field strengths reduces to

$$D\mathcal{H}_{(2)}^{IJ} = 0\ ,\quad D\mathcal{H}_{(2)}^{I} = m\,\mathcal{H}_{(3)}^{I}\ ,\quad D\tilde{\mathcal{H}}_{(2)IJ} = -2\,g\,\mathcal{H}_{(3)[I}{}^{K}\,\delta_{J]K}\ ,\quad D\tilde{\mathcal{H}}_{(2)I} = g\,\delta_{IJ}\,\mathcal{H}_{(3)}^{J}\ ,$$
$$D\mathcal{H}_{(3)I}{}^{J} = \mathcal{H}_{(2)}^{JK} \wedge \tilde{\mathcal{H}}_{(2)IK} + \mathcal{H}_{(2)}^{J} \wedge \tilde{\mathcal{H}}_{(2)I} - 2g\,\delta_{IK}\,\mathcal{H}_{(4)}^{JK} - \tfrac{1}{7}\,\delta_I^J\,(\text{trace})\ ,$$
$$D\mathcal{H}_{(3)}^{I} = -\mathcal{H}_{(2)}^{IJ} \wedge \tilde{\mathcal{H}}_{(2)J}\ ,\qquad D\mathcal{H}_{(4)}^{IJ} \equiv 0\ .$$

which exactly matches the one of the 4D tensor hierarchy.

Freund-Rubin term

[Freund & Rubin '80]

- By looking at the RR field strength $\hat{F}_{(4)} = \mathcal{H}^{IJ}_{(4)}\, \mu_I\, \mu_J + \ldots$, one immediately identifies the Freund-Rubin term

$$\mathcal{H}^{IJ}_{(4)}\, \mu_I\, \mu_J = -\tfrac{1}{3}\, g^{-1}\, V\, \mathrm{vol}_4 + \tfrac{1}{84}\, g^{-1} \left(D\mathcal{H}_{(3)} - 7\, \mathcal{H}^{IJ}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)IJ} - 7\, \mathcal{H}^{I}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)I} \right)$$
$$-\tfrac{1}{2}\, g^{-1} \left(D\mathcal{H}_{(3)I}{}^{J} - \mathcal{H}^{JK}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)IK} - \mathcal{H}^{J}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)I} \right) \mu^I \mu_J \ ,$$

NOTE: We have expressed the EOMs for the scalars as BI for the three-form field strengths of the tensor hierarchy.

- At a critical point of V one has $\hat{F}_{(4)} = -\dfrac{1}{3g} V \mathrm{vol}_4 + \ldots$, and the S^6 dependence drops out

[see also Godazgar, Godazgar, Krueger & Nicolai '15]

New AdS$_4$ N=2 solution in massive IIA and its CFT$_3$ dual

A new N=2 solution of massive type IIA

- Using the uplifting formulae, it is a straightforward exercise to obtain the 10D embedding of a 4D critical point. An example is the N=2&U(3) AdS_4 point of the $ISO(7)_c$ theory

$$d\hat{s}^2_{10} = L^2 \frac{(3+\cos 2\alpha)^{\frac{1}{2}}}{(5+\cos 2\alpha)^{-\frac{1}{8}}} \left[ds^2(\mathrm{AdS}_4) + \frac{3}{2}\, d\alpha^2 + \frac{6\sin^2\alpha}{3+\cos 2\alpha}\, ds^2(\mathbb{CP}^2) + \frac{9\sin^2\alpha}{5+\cos 2\alpha}\, \boldsymbol{\eta}^2 \right] ,$$

$$e^{\hat{\phi}} = e^{\phi_0} \frac{(5+\cos 2\alpha)^{3/4}}{3+\cos 2\alpha} \quad , \qquad \hat{H}_{(3)} = 24\sqrt{2}\, L^2\, e^{\frac{1}{2}\phi_0} \frac{\sin^3\alpha}{(3+\cos 2\alpha)^2}\, \boldsymbol{J} \wedge d\alpha \ ,$$

$$L^{-1}\, e^{\frac{3}{4}\phi_0}\, \hat{F}_{(2)} = -4\sqrt{6}\, \frac{\sin^2\alpha\cos\alpha}{(3+\cos 2\alpha)(5+\cos 2\alpha)}\, \boldsymbol{J} - 3\sqrt{6}\, \frac{(3-\cos 2\alpha)}{(5+\cos 2\alpha)^2}\, \sin\alpha\ d\alpha \wedge \boldsymbol{\eta} \ ,$$

$$\begin{aligned} L^{-3}\, e^{\frac{1}{4}\phi_0}\, \hat{F}_{(4)} &= 6\,\mathrm{vol}_4 \\ &\quad + 12\sqrt{3}\, \frac{7+3\cos 2\alpha}{(3+\cos 2\alpha)^2}\, \sin^4\alpha\ \mathrm{vol}_{\mathbb{CP}^2} + 18\sqrt{3}\, \frac{(9+\cos 2\alpha)\sin^3\alpha\cos\alpha}{(3+\cos 2\alpha)(5+\cos 2\alpha)}\, \boldsymbol{J} \wedge d\alpha \wedge \boldsymbol{\eta} \ , \end{aligned}$$

where we have introduced the quantities $L^2 \equiv 2^{-\frac{5}{8}}\, 3^{-1}\, g^{-2}\, c^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 2^{\frac{1}{4}}\, c^{-\frac{5}{6}}$

✦ The angle $0 \leq \alpha \leq \pi$ locally foliates S^6 with S^5 regarded as Hopf fibrations over $\mathbb{CP}^2$

CFT$_3$ candidate and matching of free energies

[Schwarz '04]

- We propose and N=2 Chern-Simons-matter theory with simple gauge group SU(N), level k and only adjoint matter, as the CFT of the N=2 massive IIA solution.

- The 3d free energy F = -Log(Z), where Z is the partition function on the CFT on a Euclidean S^3 can be computed via localisation over supersymmetric configurations [Pestun '07] [Jafferis '10] [Jafferis, Klebanov, Pufu & Safdi '11]

$$Z = \int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \prod_{i<j=1}^{N} \left(2\sinh^2\left(\frac{\lambda_i - \lambda_j}{2}\right)\right) \times \prod_{i,j=1}^{N} \left(\exp\left(\ell\left(\frac{1}{3} + \frac{i}{2\pi}(\lambda_i - \lambda_j)\right)\right)\right)^3 e^{\frac{ik}{4\pi}\sum \lambda_i^2}$$

where λ_i are the Coulomb branch parameters. In the $N \gg k$ limit, the result is given by

$$F = \frac{3^{13/6}\pi}{40}\left(\frac{32}{27}\right)^{2/3} k^{1/3} N^{5/3}$$

- The gravitational free energy can be computed from the warp factor in the N=2 massive IIA solution. Using the charge quantisation condition $N = -(2\pi\ell_s)^{-5}\int_{S^6} e^{\frac{1}{2}\phi}\hat{*}\hat{F}_{(4)} + \hat{B}_{(2)} \wedge d\hat{A}_{(3)} + \frac{1}{6}\hat{F}_{(0)}\hat{B}^3_{(2)}$ for the D2-brane, one finds

$$F = \frac{16\pi^3}{(2\pi\ell_s)^8}\int_{S^6} e^{8A}\mathrm{vol}_6 = \frac{\pi}{5} 2^{1/3} 3^{1/6} k^{1/3} N^{5/3}$$

provided

$$\boxed{g\,c = \hat{F}_{(0)} = k/(2\pi\ell_s)}$$

Summary

1) We propose a new method to embed 4D theories into 10D that makes extensive use of the 4D duality hierarchy. Using this method, we have connected the electric-magnetic deformed N=8 supergravity with $ISO(7)_c$ gauging to massive IIA reductions on S^6.

2) Any 4D configuration is embedded into 10D via the uplifting formulas. As an example, we have found a new $AdS_4 \times S^6$ solution of massive IIA based on an N=2&U(3) AdS_4 vacua in 4D.

3) We propose CFT_3 duals for the $AdS_4 \times S^6$ solutions of massive IIA based on the D2-brane field theory. In the massive IIA case, there is a CS-term and a superpotential that make the theory flowing to a conformal phase (IR). This translates into the appearance of supersymmetric AdS_4 vacua in the deformed $ISO(7)_c$ supergravity theory.

4) For the new N=2 massive IIA solution, the gravitational and FT free energies do match provided

$$\boxed{g\,c \;=\; \hat{F}_{(0)} \;=\; k/(2\pi\ell_s)}$$

dank u wel !!

Extra material

More AdS$_4$ critical points

SUSY	bos. sym.	M^2L^2	stability	ref.
$\mathcal{N}=3$	SO(4)	$3(1\pm\sqrt{3})^{(1)}\ ,\ (1\pm\sqrt{3})^{(6)}\ ,\ -\frac{9}{4}^{(4)}\ ,\ -2^{(18)}\ ,\ -\frac{5}{4}^{(12)}\ ,\ 0^{(22)}$ $(3\pm\sqrt{3})^{(3)}\ ,\ \frac{15}{4}^{(4)}\ ,\ \frac{3}{4}^{(12)}\ ,\ 0^{(6)}$	yes	[30]
$\mathcal{N}=2$	U(3)	$(3\pm\sqrt{17})^{(1)}\ ,\ -\frac{20}{9}^{(12)}\ ,\ -2^{(16)}\ ,\ -\frac{14}{9}^{(18)}\ ,\ 2^{(3)}\ ,\ 0^{(19)}$ $4^{(1)}\ ,\ \frac{28}{9}^{(6)}\ ,\ \frac{4}{9}^{(12)}\ ,\ 0^{(9)}$	yes	[15] , [here]
$\mathcal{N}=1$	G$_2$	$(4\pm\sqrt{6})^{(1)}\ ,\ -\frac{1}{6}(11\pm\sqrt{6})^{(27)}\ ,\ 0^{(14)}$ $\frac{1}{2}(3\pm\sqrt{6})^{(7)}\ ,\ 0^{(14)}$	yes	[4]
$\mathcal{N}=1$	SU(3)	$(4\pm\sqrt{6})^{(2)}\ ,\ -\frac{20}{9}^{(12)}\ ,\ -2^{(8)}\ ,\ -\frac{8}{9}^{(12)}\ ,\ \frac{7}{9}^{(6)}\ ,\ 0^{(28)}$ $6^{(1)}\ ,\ \frac{28}{9}^{(6)}\ ,\ \frac{25}{9}^{(6)}\ ,\ 2^{(1)}\ ,\ \frac{4}{9}^{(6)}\ ,\ 0^{(8)}$	yes	[here]
$\mathcal{N}=0$	SO(7)$_+$	$6^{(1)}\ ,\ -\frac{12}{5}^{(27)}\ ,\ -\frac{6}{5}^{(35)}\ ,\ 0^{(7)}$ $\frac{12}{5}^{(7)}\ ,\ 0^{(21)}$	no	[3]
$\mathcal{N}=0$	SO(6)$_+$	$6^{(2)}\ ,\ -3^{(20)}\ ,\ -\frac{3}{4}^{(20)}\ ,\ 0^{(28)}$ $6^{(1)}\ ,\ \frac{9}{4}^{(12)}\ ,\ 0^{(15)}$	no	[3]
$\mathcal{N}=0$	G$_2$	$6^{(2)}\ ,\ -1^{(54)}\ ,\ 0^{(14)}$ $3^{(14)}\ ,\ 0^{(14)}$	yes	[4]
$\mathcal{N}=0$	SU(3)	see (3.44) see (3.45)	yes	[here]
$\mathcal{N}=0$	SU(3)	see (3.46) see (3.47)	yes	[here]
$\mathcal{N}=0$	SO(4)	see (5.12) see (5.13)	yes	[here]