[School on Holography and Supergravity 2021 : July 2021]

Lectures on Supergravity and Dualities

I. Space-time, frames and connections

II. Actions and symmetries

III.1. $\mathcal{N}=1$ supergravity with $\Lambda = 0$

III.2. $\mathcal{N}=1$ supergravity with $\Lambda \neq 0$

IV. Coupling $\mathcal{N}=1$ supergravity to SYM and matter fields

* Scalar kinetic terms and coset spaces

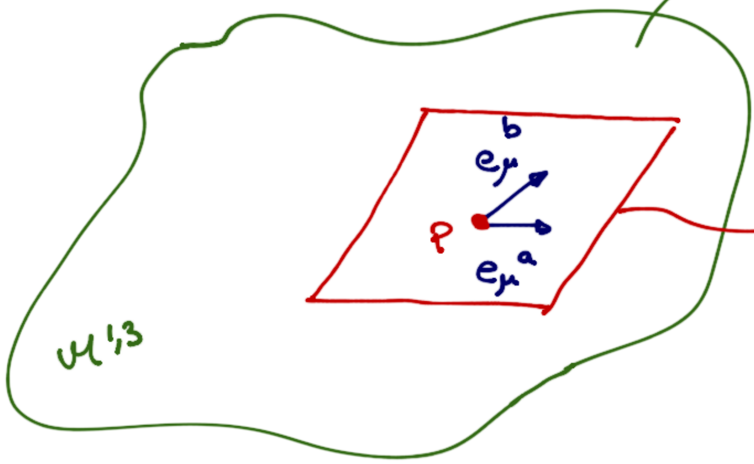
V. Extended $\mathcal{N} \geq 2$ supergravity

VI. Maximal $\mathcal{N}=8$ ungauged supergravity

* Electromagnetic duality

VII. Maximal $\mathcal{N}=8$ gauged supergravity : gaugings and embedding tensor

I. Space-time : frames and connections



space-time $\mathcal{M}^{1,3}$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Tangent space (at point p)

$$ds^2 = \eta_{ab}\, e^a e^b$$

with $e^a = \underbrace{e_\mu{}^a}_{\text{frame field}} dx^\mu$

- Locally one can define a "tetrad" $e_\mu{}^a(x)$ such that

$$g_{\mu\nu}(x) = e_\mu{}^a(x)\; e_\nu{}^b(x)\; \eta_{ab}$$

- GCT vs Local Lorentz transf. $\begin{cases} \text{GCT} : & T'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu}\, T^\nu(x) \\ \text{LLT} : & T'^a(x) = \Lambda^a{}_b(x)\, T^b(x) \\ [SO(1,3)] & \text{with } \Lambda^{a'}{}_a \Lambda^{b'}{}_b\, \eta_{a'b'} = \eta_{ab} \end{cases}$

Important : Note that frames are not unique (redundancy) as

$$e'_\mu{}^a = \Lambda^a{}_b(x)\; e_\mu{}^b \;\Rightarrow\; g_{\mu\nu}' = e'_\mu{}^a\, e'_\nu{}^b\, \eta_{ab}$$

$$= e_\mu{}^a\, e_\nu{}^b\, \eta_{ab} = g_{\mu\nu}$$

This local (gauge) symmetry will introduce a gauge field for the particles (spinors) transforming under the local Lorentz group

$\Rightarrow$ spin connection $\omega_\mu{}^{ab}$!!

- Connections and covariant derivatives

→ World tensors : $\underbrace{\text{Christoffel}}$: $\Gamma_{\mu\sigma}{}^{\rho}(g) = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$

$\Rightarrow \Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$

(no torsion)

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho$$
$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^{\rho} V_\rho$$

→ Tangent space tensors : $\underbrace{\text{Lorentz}}$ $SO(1,3)$ generators $(M_{cd})^a{}_b = 2\delta_{[c}^a \eta_{d]b}$

$$\left.\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^{cd} (M_{cd})^a{}_b V^b = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \\ D_\mu V_a &= \partial_\mu V_a - \omega_\mu{}^{cd} (M_{cd})^b{}_a V_b = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b \end{aligned}\right\} \text{Vector}$$

$$D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \omega_\mu{}^{cd} \underbrace{(\gamma_{cd})_\alpha{}^\beta}_{M_{cd} = \frac{1}{4}\gamma_{cd}} \psi_\alpha \quad \} \text{ Spinor}$$

- Vielbein postulate : $\nabla_\mu e_\nu{}^a = 0 \Rightarrow D_\mu V^a = e_\nu{}^a \nabla_\mu V^\nu$

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^{\rho} e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \qquad (\times e_a{}^\lambda)$$

$$\Rightarrow e_a{}^\lambda \left(\partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_{\nu b} \right) = \Gamma_{\mu\nu}{}^{\lambda} \Rightarrow \text{One indep connection !!}$$

$\Longrightarrow$ The spin connection is a composite field

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) \qquad [\text{no torsion}]$$

$$T_{\mu\nu}{}^a \equiv D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b \quad \equiv \text{Torsion}$$

Important: $T_{\mu\nu}{}^a \overset{!}{=} 0 \Rightarrow \underbrace{\omega_\mu{}^{ab}(e)} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c$

Levi-Civita connection [torsion free]

II. Actions and symmetries

Symmetries $\begin{cases} \text{space-time symmetries} \\ \text{gauge internal symmetries} \end{cases}$ $\begin{cases} \text{GCT} \longrightarrow \text{Diff}: x'^\mu = x^\mu - \xi^\mu(x) \\ \text{Local Lorentz} \longrightarrow \psi'_\alpha = \psi_\alpha - \lambda^{ab}[\gamma_{ab}]_\alpha{}^\beta \psi_\beta \end{cases}$

$\boxed{s=0}$ Scalar field ϕ [no world or tangent space index]

- G.C.T : $\phi'(x') = \phi(x) \Rightarrow \delta\phi = \mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi$
- gauge : No gauge symmetry (matter field)
- action : $S_\phi = \int d^4x \sqrt{-|g|}\left(-\frac{1}{2} g^{\mu\nu} \underbrace{\partial_\mu\phi}_{\nabla_\mu\phi = \partial_\mu\phi} \partial_\nu\phi\right)$
- EOM : $\Box\phi = 0$

$\boxed{s=1/2}$ Fermion field ψ_α [$\alpha \equiv$ tangent space spinorial index]

- Local Lorentz : $\psi'_\alpha(x) = \Lambda_\alpha{}^\beta \psi_\beta(x) \Rightarrow \delta_\lambda \psi_\alpha = \lambda^{ab}[\gamma_{ab}]_\alpha{}^\beta \psi_\beta$
- gauge : No gauge symmetry (matter field)
- action : $S_\psi = \int d^4x \sqrt{-|g|}\left(+\frac{1}{2}\bar{\psi}\gamma^\mu \nabla_\mu \psi - \frac{1}{2}\bar{\psi}\overleftarrow{\nabla}_\mu \gamma^\mu \psi\right)$

where $\nabla_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4}\underbrace{\omega_\mu{}^{ab}}_{\text{spin connection}}[\gamma_{ab}]_\alpha{}^\beta \psi_\beta \equiv D_\mu \psi_\alpha$

$\bar{\psi} \equiv \psi^t C \overset{!}{=} i\psi^\dagger\gamma^0$

↳ Majorana condition with $(\gamma^\mu)^t = t_0 t_1 C\gamma^\mu C^{-1}$

NOTE: In the absence of torsion one can integrate by parts S_ψ to write it as

$$S_\psi = \int d^4x \sqrt{-|g|}\; \bar{\psi}\, \gamma^\mu D_\mu \psi \qquad \text{[Dirac action]}$$

- EOM : $\gamma^\mu D_\mu \psi_\alpha = 0$

$\boxed{s=1}$ Abelian [Maxwell] vector field A_μ [$\mu \equiv$ world index]

- GCT : $A'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \Rightarrow \delta_\xi A_\mu = \mathcal{L}_\xi A_\mu = \xi^\rho \partial_\rho A_\mu + (\partial_\mu \xi^\rho) A_\rho$
- gauge : $A'_\mu(x) = A_\mu(x) + \nabla_\mu \theta(x) \Rightarrow \delta_\theta A_\mu = \nabla_\mu \theta(x) = \partial_\mu \theta(x)$

  ↳ scalar function $\theta(x)$
- action : $S_A = \int d^4x \sqrt{-|g|} \left(-\frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu}} \right)$

  $\underbrace{F_{\mu\nu}}_{\text{gauge invariant}} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

  ↳ $\Gamma_{[\mu\nu]}^{\rho} = 0$ [No torsion]
- EOM : $\nabla_\mu F^{\mu\nu} = 0$

$\boxed{s = 3/2}$ Gravitino (vector-spinor) field $\Psi_{\mu\alpha}$ [$\mu \equiv$ world index, $\alpha \equiv$ tangent space index]

- GCT : $\Psi'_{\mu\alpha}(x') = \dfrac{\partial x^\nu}{\partial x^\mu} \Psi_{\nu\alpha}(x) \Rightarrow \delta_\xi \Psi_{\mu\alpha} = \mathcal{L}_\xi \Psi_{\mu\alpha} = \xi^\rho \partial_\rho \Psi_{\mu\alpha} + (\partial_\mu \xi^\rho) \Psi_{\rho\alpha}$
- Local Lorentz : $\Psi'_{\mu\alpha}(x) = \Lambda_\alpha{}^\beta \Psi_{\mu\beta}(x) \Rightarrow \delta_\Lambda \Psi_{\mu\alpha} = \lambda^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\mu\beta}$
- gauge : $\Psi'_{\mu\alpha}(x) = \Psi_{\mu\alpha}(x) + \nabla_\mu \epsilon_\alpha(x) \Rightarrow \delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$
- action : $S_\Psi = \int d^4x \sqrt{-|g|}\; \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \underbrace{\nabla_\nu \Psi_\rho}$ [Rarita-Schwinger]

$$\nabla_\nu \Psi_{\rho\alpha} = \partial_\nu \Psi_{\rho\alpha} + \frac{1}{4} \omega_\nu{}^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\rho\beta} - \Gamma_{\nu\rho}^{\lambda} \Psi_{\lambda\alpha} = D_\nu \Psi_{\rho\alpha} \underbrace{- \Gamma_{\nu\rho}^{\lambda} \Psi_{\lambda\alpha}}$$

No torsion ⇒ Irrelevant due to $\gamma^{\mu\nu\rho}$

Important : $\delta_\epsilon S^{r}_{\Psi} = \int d^4x \sqrt{-|g|} \left(-\nabla_\sigma \bar\Psi_\mu \gamma^{\mu\sigma\rho} \delta_\epsilon \Psi_\rho \right) \times 2$ ← variation w.r.t $\bar\Psi$

$$= \int d^4x \sqrt{-|g|} \left(-\nabla_\sigma \bar\Psi_\mu \gamma^{\mu\sigma\rho} \nabla_\rho \epsilon \right) \times 2$$

$$= \int d^4x \sqrt{-|g|} \Big[\underbrace{\nabla_\rho \left(-\nabla_\sigma \bar\Psi_\mu \gamma^{\mu\sigma\rho} \epsilon \right)}_{\text{boundary term}} + \nabla_\rho \nabla_\sigma \bar\Psi_\mu \gamma^{\mu\sigma\rho} \epsilon \Big] \times 2$$

(no torsion)

$$= -\int d^4x \sqrt{-|g|}\, \bar\epsilon \gamma^{\mu\sigma\rho} \nabla_{[\sigma} \nabla_{\rho]} \Psi_\mu \times 2$$

$$= -\int d^4x \sqrt{-|g|}\, \bar\epsilon \gamma^{\mu\sigma\rho} \tfrac{1}{2} \left(\tfrac{1}{4} R_{\sigma\rho ab} [\gamma^{ab}] \right) \Psi_\mu \times 2$$

$$= -\tfrac{1}{4} \int d^4x \sqrt{-|g|}\, \bar\epsilon \gamma^{\mu\sigma\rho} \gamma_{ab} R_{\mu\sigma}{}^{ab} \Psi_\rho = (*)$$

NOTE: $\gamma^{\mu\sigma\rho}\gamma_{ab} = \gamma^{\mu\sigma\rho}{}_{ab} + 6\, \gamma^{[\mu\sigma}{}_b\, \delta^{\rho]}_a + 6\, \gamma^{[\mu}\, \delta^{\sigma\rho]}_{ba}$

- $\gamma^{\mu\sigma\rho}{}_{ab} R_{\mu\sigma}{}^{ab} = 0$ [only in D=4]
- $6\gamma^{[\mu\sigma}{}_b \delta^{\rho]}_a R_{\mu\sigma}{}^{ab} = 2\gamma^{\mu\sigma}{}_b \delta^\rho_a R_{\mu\sigma}{}^{ab} + 4\gamma^{\sigma\rho}{}_b \delta^\mu_a R_{\mu\sigma}{}^{ab}$

Torsion-free Bianchi id.

$R_{[\mu\sigma\rho]}{}^a = 0$

$$= 2\gamma^{\mu\sigma b} R_{\mu\sigma}{}^\rho{}_b + 4\gamma^{\sigma\rho b} R_{\mu\sigma}{}^\mu{}_b$$

$$= 2\gamma^{\mu\sigma b} R_{\mu\sigma}{}^\rho{}_b + 4\gamma^{\sigma\rho b} R_{\sigma b} \to 0 \text{ [sym]}$$

- $6\gamma^{[\mu}\delta^{\sigma\rho]}_{ba} R_{\mu\sigma}{}^{ab} = 4\gamma^\mu \delta^{\sigma\rho}_{ba} R_{\mu\sigma}{}^{ab} + 2\gamma^\rho \delta^{\mu\sigma}_{ba} R_{\mu\sigma}{}^{ab}$

$$= 4\gamma^\mu R_{\mu\sigma}{}^{\rho\sigma} + 2\gamma^\rho R_{\mu\sigma}{}^{\sigma\mu}$$

$$= 4\gamma^\mu R_\mu{}^\rho - 2\gamma^\rho R$$

$$= 4\gamma^\mu \left(R_\mu{}^\rho - \tfrac{1}{2} \delta^\rho_\mu R \right)$$

NOTE: $\bar\chi \gamma_{\mu_1 \dots \mu_n} \lambda = t_n \bar\lambda \gamma_{\mu_1 \dots \mu_n} \chi$ with $t_0 = t_3 = +1$, $t_1 = t_2 = -1$
(4D)

$$(*) = -\int d^4x \sqrt{-|g|} \left(R_\mu{}^\rho - \frac{1}{2} \delta_\mu^\rho R \right) \bar{\epsilon} \gamma^\mu \Psi_\rho$$

$$= -\int d^4x \sqrt{-|g|} \underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)}_{\text{Einstein tensor } G_{\mu\nu} \text{ !!}} \bar{\epsilon} \gamma^\mu \Psi^\nu$$

Therefore:

i) $\delta_\epsilon S_\Psi = 0$ in Minkowski space-time where $\nabla_\mu \Psi_\rho = \partial_\mu \Psi_\rho$ and $\nabla_\mu \epsilon_\alpha = \partial_\mu \epsilon_\alpha$.
This is the work by Rarita-Schwinger
(free $s=3/2$ field has ϵ_α-gauge invariance)

ii) $\delta_\epsilon S_\Psi \propto \underbrace{G_{\mu\nu}}_{\text{gravity}} \bar{\epsilon} \gamma^\mu \Psi^\nu \neq 0 \Rightarrow$ This suggests that gravity has something to say about general ϵ_α-gauge invariance !!

- EOM : $\gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho = 0$

  or $\gamma^{\mu\nu\rho} D_\nu \Psi_\rho = 0$ (no torsion)

$\boxed{s=2}$ Metric field $g_{\mu\nu}$ [$\mu,\nu \equiv$ world indices)

- GCT : $g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \Rightarrow \delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2\, \partial_{(\mu} \xi^\rho g_{\nu)\rho}$
- gauge : No gauge symmetry in GR, but ... ϵ_α-gauge symmetry?
- action : $S_g = \frac{1}{2\kappa^2} \int d^4x \sqrt{-|g|}\, g^{\mu\nu} R_{\mu\nu}$ [Einstein - Hilbert]
  (2nd order formalism)
- EOM : $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$

$\kappa^2 = 8\pi G_N$ = grav coupling cte

Proposal : Can gravity compensate for the lack of ϵ_α-invariance of S'_Ψ ?

(i) The fact that $\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$ appears in $\delta_\epsilon S_\Psi$ is highly remarkable.

(ii) We know from supersymmetry that $s=0$ $(\phi \in \mathbb{C})$ and $s=1/2$ (Weyl ψ) can furnish a susy chiral multiplet transforming as

$$\boxed{\begin{aligned} \delta_\epsilon \phi &\propto \bar{\epsilon}\,\psi \\ \delta_\epsilon \psi &\propto \bar{\epsilon}\,\sigma^\mu (\partial_\mu \phi) \end{aligned}}$$

} Free theory or Wess-Zumino model in flat space {
- $\delta \mathcal{L}$ = boundary terms
- SUSY algebra on-shell

(iii) We also know from supersymmetry that $s=1/2$ (Majorana λ_α) and $s=1$ (A_μ) can furnish a susy vector multiplet transforming as

$$\boxed{\begin{aligned} \delta_\epsilon A_\mu &\propto \bar{\epsilon}\,\gamma_\mu \lambda \\ \delta_\epsilon \lambda &\propto \bar{\epsilon}\,\gamma_{\mu\nu} F^{\mu\nu} \end{aligned}}$$

} Super Yang-Mills theory in flat space

(iv) Could there be a similar story involving $s=3/2$ (Ψ) and $s=2$ (e) ??

Beautiful idea !!

III.1 $\mathcal{N}=1$ Supergravity with $\Lambda=0$

We have collected already quite some clues for the local ϵ_α-invariance to actually be local supersymmetry involving not only the $s=3/2$ field but also the $s=2$ metric field. We are going to work this idea out now.

Let's start from:

$$S_g = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\; g^{\mu\nu} R_{\mu\nu} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\; e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(e)$$

$$S_\Psi = -\underbrace{\frac{1}{2\kappa^2}}_{\substack{\text{we have}\\ \text{introduced this constant}}}\int d^4x\,\sqrt{-|g|}\;\bar\Psi_\mu \gamma^{\mu\nu\rho} \underbrace{\nabla_\nu \Psi_\rho}_{D_\nu[\omega]\Psi_\rho}$$

Important: We are considering a space-time without torsion

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$$

so integration by parts works normally: $\int dx\,\sqrt{-|g|}\,\nabla_\mu V^\mu = \int dx\,\partial_\mu\left(\sqrt{-|g|}\,V^\mu\right)$

We already know that $\delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$. But the question is: what should be $\delta_\epsilon e_\mu{}^a$?

$$\delta_\epsilon e_\mu{}^a = c\,\bar\epsilon\,\gamma^a \Psi_\mu \qquad \text{(what else could it be?)}$$

with c being a constant to be fixed. Note that this combination already appeared in $\delta_\epsilon S_\Psi \Rightarrow$ ✓

Let's compute things explicitly

$$S = S_g(e) + S_\psi(e,\psi)$$

$$\delta_e S = \Big(\underbrace{\frac{\delta S_g}{\delta e}}_{A.1} + \underbrace{\frac{\delta S_\psi}{\delta e}}_{A.2} \Big)_{A} \delta_e e + \underbrace{\frac{\delta S_\psi}{\delta \Psi} \delta_e \Psi}_{B \to \text{This we already computed}}$$

<u>B</u> : $\displaystyle \frac{\delta S_\psi}{\delta \Psi_{\mu\alpha}} \delta_e \Psi_{\mu\alpha} = \frac{1}{2K^2} \int d^4x \underbrace{e}_{\sqrt{|-g|}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \bar{\epsilon} \gamma^\mu \Psi^\nu$

<u>A.1</u>: This is, by definition, the computation of the Einstein equations in terms of frames $e_\mu{}^a$:

$$\frac{\delta S_g}{\delta e_\mu{}^a} \delta e_\mu{}^a = \frac{1}{2K^2} \int d^4x \left[\delta(e) R + 2e\, \delta(e_a{}^\mu) e_b{}^\nu R_{\mu\nu}{}^{ab} + e\, e_a{}^\mu e_b{}^\nu \delta(R_{\mu\nu}{}^{ab}) \right] = (*)$$

- $\delta(|M|) = |M| \operatorname{Tr}(M^{-1} \delta M) \Rightarrow \delta(e) = e\, e_a{}^\mu \delta e_\mu{}^a$
- $e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu \Rightarrow \delta e_\mu{}^a e_a{}^\nu + e_\mu{}^a \delta e_a{}^\nu = 0$

$$\Rightarrow \delta e_a{}^\nu e_\mu{}^a = - e_a{}^\nu \delta e_\mu{}^a \qquad (\times e_b{}^\mu)$$

$$\Rightarrow \delta e_b{}^\nu = - e_a{}^\nu e_b{}^\mu \delta e_\mu{}^a$$

- $\delta(R_{\mu\nu}{}^{ab}) = D_\mu \delta\omega_\nu{}^{ab} - D_\nu \delta\omega_\mu{}^{ab} \underset{\text{no torsion}}{=} \nabla_\mu \delta\omega_\nu{}^{ab} - \nabla_\nu \omega_\mu{}^{ab} \underset{\text{no torsion}}{\Longrightarrow}$ Total derivative

$$(*) = \frac{1}{2K^2} \int d^4x\, e \left(e_a{}^\mu R\, \delta e_\mu{}^a - 2\, e_b{}^\nu e_c{}^\mu e_a{}^\rho R_{\mu\nu}{}^{ab} \delta e_\rho{}^c \right)$$

$$= \frac{1}{2K^2} \int d^4x\, e \left(e_a{}^\mu R\, \delta e_\mu{}^a - 2 R_\mu{}^\rho e_c{}^\mu \delta e_\rho{}^c \right) = \text{(relabeling)}$$

$$= \frac{1}{2K^2} \int d^4x \, e \left(e_a{}^\rho R - 2 R_\mu{}^\rho e_a{}^\mu \right) \delta e_\rho{}^a = (*)$$

Now we plug: $\delta_\epsilon e_\rho{}^a = c \, \bar{\epsilon} \gamma^a \Psi_\rho$

$$(*) = -\frac{c}{K^2} \int d^4x \, e \left(R_\mu{}^\rho e_a{}^\mu - \frac{1}{2} e_a{}^\rho R \right) \bar{\epsilon} \gamma^a \Psi_\rho$$

$$= -\frac{c}{K^2} \int d^4x \, e \left(R_\mu{}^\rho - \frac{1}{2} \delta_\mu^\rho R \right) \bar{\epsilon} \gamma^\mu \Psi_\rho$$

$$= -\frac{c}{K^2} \int d^4x \, e \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \bar{\epsilon} \gamma^\mu \Psi^\nu$$

lowest order Ψ

<u>A.2</u>: $\frac{\delta S_{RS}}{\delta e_\mu{}^a} \delta e_\mu{}^a = -\frac{1}{2K^2} \int d^4x \, \delta \left[e \, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho \right] = 0$

NOTE: $\delta e = e \, e_a{}^\mu \delta e_\mu{}^a = c \, e \, e_a{}^\mu \bar{\epsilon} \gamma^a \Psi_\mu \Rightarrow \epsilon \Psi^3\text{-terms}$

$$\delta(\gamma^{\mu\nu\rho}) = \delta(e_a{}^\mu e_b{}^\nu e_c{}^\rho) \gamma^{abc}$$
$$= \delta e_a{}^\mu e_b{}^\nu e_c{}^\rho \gamma^{abc} + \dots$$
$$= - e_d{}^\mu e_a{}^\lambda \delta e_\lambda{}^d \, e_b{}^\nu e_c{}^\rho \gamma^{abc} + \dots$$
$$= - e_d{}^\mu \delta e_\lambda{}^d \gamma^{\lambda\nu\rho} + \dots$$
$$= - e_d{}^\mu \gamma^{\lambda\nu\rho} (\bar{\epsilon} \gamma^d \Psi_\lambda) \Rightarrow \epsilon \Psi^3\text{-terms}$$

$$\delta(\nabla_\nu \Psi_\rho) = \delta(D_\nu \Psi_\rho) = \delta(\partial_\nu \Psi_\rho + \tfrac{1}{4} \omega_\nu{}^{ab} [\gamma_a, \gamma_b] \Psi_\rho)$$
$$= \frac{1}{4} \delta(\omega_\nu{}^{ab}) [\gamma_a, \gamma_b] \Psi_\rho$$
$$\Rightarrow \epsilon \Psi^3\text{-terms}$$

schematically: $\omega = e^{-1} \partial e - e^{-1} e^{-1} e \, \partial e$

Wrapping up B, (A.1) and (A.2) we obtain at lowest order ($\bar{\epsilon}\Psi$-terms) in fermions the following result:

$$\delta_\epsilon S = \delta_\epsilon (S_g + S_\Psi) = \underbrace{\left(\tfrac{1}{2} - c\right)}_{c = \frac{1}{2}\,!!} \frac{1}{\kappa^2} \int d^4x\, e \left(R_{\mu\sigma} - \tfrac{1}{2} g_{\mu\sigma} R\right) \bar{\epsilon}\gamma^\mu \Psi^\sigma$$

Therefore we have an ϵ_α-invariant action at lowest order in fermionic fields with transformation rules:

$$\text{(i)} \quad \boxed{\begin{aligned} \delta_\epsilon e_\mu{}^a &= \tfrac{1}{2}\, \bar{\epsilon}\gamma^a \Psi_\mu \\ \delta_\epsilon \Psi_\mu &= D_\mu \epsilon \end{aligned}} \quad \Rightarrow \text{SUGRA !!}$$

Remark : It was crucial that $\nabla_\sigma \Psi_\rho$ in the action R-S action appears with a $\gamma^{\mu\sigma\rho}$. This allowed us to replace $\nabla_\sigma \Psi_\rho = D_\sigma \Psi_\rho$ fitting well with $D_\mu \epsilon$ and

$$D_{[\mu} D_{\sigma]} = R_{\mu\sigma} \Rightarrow \text{GR !!}$$

However, there is an easier way to see how GR emerges from the SUSY algebra of supersymmetry:

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a &= \delta_{\epsilon_1}\left(\tfrac{1}{2} \bar{\epsilon}_2 \gamma^a \Psi_\mu\right) - (1 \leftrightarrow 2) \\ &= \tfrac{1}{2} \bar{\epsilon}_2 \gamma^a \nabla_\mu \epsilon_1 - (1 \leftrightarrow 2) = \tfrac{1}{2} \bar{\epsilon}_2 \gamma^a D_\mu \epsilon_1 - (1 \leftrightarrow 2) \\ &= \tfrac{1}{2} \bar{\epsilon}_2 \gamma^a D_\mu \epsilon_1 - \tfrac{1}{2} \bar{\epsilon}_1 \gamma^a D_\mu \epsilon_2 \\ \overset{t_1 = -1}{=}\; &\tfrac{1}{2}\left(\bar{\epsilon}_2 \gamma^a D_\mu \epsilon_1 + D_\mu \bar{\epsilon}_2 \gamma^a \epsilon_1\right) = D_\mu \underbrace{\left(\tfrac{1}{2} \bar{\epsilon}_2 \gamma^a \epsilon_1\right)}_{\equiv \xi^a} \\ &= D_\mu \xi^a \qquad \text{with } \xi^a = \tfrac{1}{2} \bar{\epsilon}_2 \gamma^a \epsilon_1 \end{aligned}$$

Therefore we have found that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = D_\mu \xi^a$

Moreover $D_\mu \xi^a = \nabla_\mu \xi^a$ and

$$\delta_\xi e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a = \xi^\rho \partial_\rho e_\mu{}^a + \partial_\mu \xi^\rho e_\rho{}^a = \text{(explicit covariantisation)}$$

$$= \xi^\rho \underbrace{\nabla_\rho e_\mu{}^a}_{0} + \cancel{\xi^\rho \Gamma_{\rho\mu}^\lambda e_\lambda{}^a} - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b + \underbrace{\nabla_\mu \xi^\rho e_\rho{}^a}_{\nabla_\mu \xi^a} - \cancel{\Gamma_{\mu\lambda}{}^\rho \xi^\lambda e_\rho{}^a} \quad \text{/ = no torsion}$$

$$= \nabla_\mu \xi^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b$$

Then : $\nabla_\mu \xi^a = \delta_\xi e_\mu{}^a + \underbrace{\xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b}_{\delta_\Lambda e_\mu{}^a} = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$ with $\Lambda^a{}_b = \xi^\rho \omega_\rho{}^a{}_b$

So we have shown that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$

$\Rightarrow$ Local SUSY algebra implies $\left\{ \begin{array}{l} \text{GCT} \\ \text{Local Lorentz} \end{array} \right. \Rightarrow$ General Relativity !!

→ Next question : Are the transformations (i) a symmetry of the action to all orders in fermions ?

Unfortunately the answer is no ... ☹ ...

It had been surprising otherwise as the Rarita - Schwinger action describes a free $s = 3/2$ field Ψ_ρ whereas the E-H action describing gravity is highly interacting !!

IMPORTANT : The computation we have performed did not rely on the dimension D or any special feature but having a real Ψ. Therefore, it works in the same way in any D. What makes certain dimensions special is that local susy can be stated at all orders in fermions. For example: $\mathcal{N}=1$ D=4 or D=11

To have $\mathcal{N}=1$ and $D=4$ sugra to all orders in fermions one has to introduce Ψ^4-terms both in the R-S action and in the transformation rules :

$$S_\Psi = -\frac{1}{2\kappa^2}\int d^4x\, e\left\{ \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{16}\left[(\bar{\Psi}^\rho \gamma^\mu \Psi^\nu)(\bar{\Psi}_\rho \gamma_\mu \Psi_\nu + 2\bar{\Psi}_\rho \gamma_\nu \Psi_\mu) - 4(\bar{\Psi}_\mu \gamma^\rho \Psi_\rho)(\bar{\Psi}^\mu \gamma^\rho \Psi_\rho) \right]\right\}$$

with a torsion-free $D_\mu \Psi_\rho = \partial_\mu \Psi_\rho + \frac{1}{4}\omega_\mu{}^{ab}(e)\,[\gamma_{ab}]\,\Psi_\rho$

The SUSY transformation rules (i) must be also modified with higher-order fermion terms :

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\epsilon}\gamma^a \Psi_\mu$$
$$\delta \Psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\left(\omega_\mu{}^{ab}(e) + K_\mu{}^{ab}\right)[\gamma_{ab}]\,\epsilon$$

with $K_{\mu\nu\rho} = -\frac{1}{4}\left(\bar{\Psi}_\mu \gamma_\rho \Psi_\nu - \bar{\Psi}_\nu \gamma_\mu \Psi_\rho + \bar{\Psi}_\rho \gamma_\nu \Psi_\mu\right)$

IMPORTANT : We see that local SUSY at the full fermion level can be very conveniently described using gravitino torsion. This is an example of how torsion appears in Physics. Some 1st and 1.5 order formulations of supergravity make all these structures manifest and render the problem of full local supersymmetry tractable !!

Some involved fermionic manipulations "Fierzing" are also required in the process.

III.2 $\mathcal{N}=1$ Supergravity with $\Lambda \neq 0$

Question: Is local supersymmetry compatible with a cosmological constant Λ? What modifications are required?

$$S_g = S_{E-H} + S_\Lambda = \frac{1}{2\kappa^2}\int d^4x \, e \left(e_a{}^\mu e_a{}^\nu R_{\mu\nu}{}^{ab} - \Lambda \right)$$

$$S_\Psi = S_{R-S} + S_{mass} = -\frac{1}{2\kappa^2}\int d^4x \, e \left(\bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho + m \, \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu \right)$$

NOTE: A mass term of the form $g^{\mu\nu} \bar{\Psi}_\mu \Psi_\nu$ does not describe the right number of d.o.f.

SUSY transformation rules:

(ii)
$$\boxed{\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \Psi_\mu \\ \delta \Psi_\mu &= D_\mu \epsilon - g \gamma_\mu \epsilon \end{aligned}} \Rightarrow \text{SUGRA} + \Lambda \; !!$$

Let's repeat the computation we performed in the case $\Lambda = m = g = 0$

The new pieces to compute are:

- $\dfrac{\delta S_{EH}}{\delta e_\mu{}^a}$ = same computation $= -\dfrac{1}{2\kappa^2}\int d^4x \, e \, G_{\mu\nu} \, \bar{\epsilon} \gamma^\mu \Psi_\nu$

- $\dfrac{\delta S_\Lambda}{\partial e_\mu{}^a} = \dfrac{-1}{2\kappa^2}\int d^4x \, \Lambda \, \delta(e) = \dfrac{-1}{2\kappa^2}\int d^4x \, e \Lambda e_a{}^\mu \delta e_\mu{}^a =$
$$= \frac{-\Lambda}{4\kappa^2}\int d^4x \, e \, \bar{\epsilon} \gamma^\mu \Psi_\mu$$

- $\dfrac{\delta S_{R-S}}{\delta e_\mu{}^a} = 0$ (same computation at lowest order in fermions)

$$\bullet\quad \frac{\delta S_{R-S}}{\delta \Psi_\mu} = -\frac{1}{2\kappa^2}\int d^4x\, e\left(-\nu_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\, \overbrace{\delta_\epsilon \Psi_\rho}^{\text{new transf (ii)}}\right)\times 2 = \text{same computation}$$

$$= \underbrace{\frac{1}{2\kappa^2}\int d^4x\, e\, G_{\mu\nu}\, \bar{\epsilon}\gamma^\mu \Psi^\nu}_{\text{Piece from } D_\rho\epsilon} - \underbrace{\frac{g}{\kappa^2}\int d^4x\, e\, \nu_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\gamma_\rho \epsilon}_{A.3} = (*)$$

$\underline{\underline{A.3}}$: $$-\frac{g}{\kappa^2}\int d^4x\, e\, \nabla_\nu \bar{\Psi}_\mu \underbrace{\gamma^{\mu\nu\rho}\gamma_\rho}_{2\gamma^{\mu\nu}} \epsilon = +\frac{2g}{\kappa^2}\int d^4x\, e\, \nu_\mu \bar{\Psi}_\nu \gamma^{\mu\nu}\epsilon$$

$$\overset{t_2=-1}{=} -\frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu \Psi_\nu = -\frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu} D_\mu \Psi_\nu$$

$$(*) = \frac{1}{2\kappa^2}\int d^4x\, e\, G_{\mu\nu}\, \bar{\epsilon}\gamma^\mu \Psi^\nu - \frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu} D_\mu \Psi_\nu$$

$$\bullet\quad \frac{\delta S_{mass}}{\delta e_a{}^\mu} = -\frac{m}{2\kappa^2}\int d^4x\, \delta\left[e\, e_a{}^\mu e_b{}^\nu \bar{\Psi}_\mu \gamma^{ab}\Psi_\nu\right] \Rightarrow \epsilon\Psi^3\text{-terms}$$

$$\bullet\quad \frac{\delta S_{mass}}{\delta \Psi_\nu} = -\frac{m}{2\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \gamma^{\mu\nu}\delta\Psi_\nu \times 2 =$$

$$= -\frac{m}{\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \gamma^{\mu\nu} D_\nu \epsilon + \frac{mg}{\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \overbrace{\gamma^{\mu\nu}\gamma_\nu}^{3\gamma^\mu}\epsilon$$

$t_1 = t_2 = -1$ $$= +\frac{m}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu} D_\mu \Psi_\nu - \frac{3mg}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^\mu \Psi_\mu$$

For the various terms to cancel one needs:

$m = 2g$, $\frac{\Lambda}{4} = -3mg \Rightarrow \Lambda = -12mg = -24g^2 < 0$

$\underline{\text{IMPORTANT}}$: Invariance to all order in fermions is achieved by adding the torsion terms to $\delta\Psi_\mu$ as in the $\Lambda = 0$ case.

IV. Coupling $\mathcal{N}=1$ Supergravity to SYM and Matter fields

We are now going to couple the $\mathcal{N}=1$ supergravity multiplet to other multiplets of $\mathcal{N}=1$ supersymmetry: $\underbrace{\text{vector mult.}}_{\text{Super Yang-Mills (SYM)}}$ & $\underbrace{\text{chiral mult.}}_{\text{Matter}}$

	MULTIPLET	-2	-3/2	-1	1/2	0	1/2	1	3/2	2
Gravity	$g_{\mu\nu}$	1								1
	Ψ_μ		1						1	
Vector	A_μ			1				1		
	λ_α				1		1			
Chiral	ψ_α				1		1			
	$\phi \in \mathbb{C}$					11				

Table: $\mathcal{N}=1$ multiplets, fields and helicity states

* Free theory: Local susy transformations [$\epsilon_\alpha(x)$ parameter]

$s = 2, \frac{3}{2}$

$$\boxed{\begin{aligned} \delta_\epsilon e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\Psi_\mu \\ \delta_\epsilon \Psi_\mu &= D_\mu \epsilon \end{aligned}}$$

SUGRA Multiplet

$s = 1, \frac{1}{2}$

$$\boxed{\begin{aligned} \delta_\epsilon A_\mu &= -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda \\ \delta_\epsilon \lambda &= \tfrac{1}{4}\bar{\epsilon}\gamma_{\mu\nu}F^{\mu\nu} \end{aligned}}$$

Vector multiplet

$s = \frac{1}{2}, 0$

$$\boxed{\begin{aligned} \delta_\epsilon \phi &= \tfrac{1}{\sqrt{2}}\bar{\epsilon}\psi \\ \delta_\epsilon \psi &= \tfrac{1}{\sqrt{2}}\bar{\epsilon}\sigma^\mu(\partial_\mu\phi) \end{aligned}}$$

Chiral Multiplet

The bosonic Lagrangian

We focus on the bosonic terms in the action S. The fermionic terms then follow from requiring $\mathcal{N}=1$ local supersymmetry.

$$S = \frac{1}{2\kappa^2}\int d^4x\, e\Big[R - \frac{1}{4}\underbrace{\mathrm{Im}\, N_{ab}(\phi)}_{\text{"}g^{-2}\delta_{ab}\text{-like"}} F_{\mu\nu}{}^a F^{\mu\nu b} - \frac{1}{4}\frac{1}{e}\underbrace{\mathrm{Re}\, N_{ab}(\phi)}_{\text{"}\frac{1}{8\pi^2}\theta\delta_{ab}\text{-like"}}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^a F_{\rho\sigma}{}^b$$

$$- \underbrace{K_{i\bar{j}}(\phi,\bar{\phi})}_{\text{"scalar geometry"}}\nabla_\mu\phi^i\nabla^\mu\bar{\phi}^{\bar{j}} - V(\phi,\bar{\phi}) + \text{fermionic terms.}$$

with $\phi^i \in \mathbb{C}$ and $A_\mu^a \in \mathbb{R}$ spanning an internal gauge symmetry G_0

- $i = 1,\dots,n_c$ chiral multiplets
- $a = 1,\dots,n_v$ vector multiplets $\Rightarrow$ $n_v = \dim(G_0)$

We will consider matter charged under G_0:

$$\nabla_\mu\phi^i = \partial_\mu\phi^i - \underbrace{A_\mu{}^a\ \delta_a\phi^i}_{} = \partial_\mu\phi^i - A_\mu{}^a\, k_a{}^i(\phi)$$

$$\nabla_\mu\bar{\phi}^{\bar{\imath}} = (\nabla_\mu\phi^i)^*$$

generators of $R[G_0]$

linear sym $\Rightarrow$ $\delta_a\phi^i = \overbrace{(t_a)}^{}{}^i{}_j\,\phi^j \equiv k_a{}^i(\phi)$

* Scalar geometry and $V(\phi,\bar{\phi})$

$\mathcal{N}=1$ supersymmetry requires the scalar geometry to be a complex Kähler manifold. This implies

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi^i \, \partial \bar{\phi}^{\bar{j}}} \qquad \equiv \text{"Kähler metric"}$$

in terms of a "Kähler potential" $K(\phi, \bar{\phi}) \in \mathbb{R}$.

NOTE: Demanding gauge invariance of $K_{i\bar{j}}(\phi,\bar{\phi})$ under a

a gauge G_0 transformation with parameters Θ^a $(a = 1, \dots, \dim G_0)$

- $\delta_\Theta \phi^i = \Theta^a(x) \, (t_a)^i{}_j \, \phi^j = \Theta^a \; \mathcal{K}_a{}^i(\phi)$
- $\delta_\Theta \bar{\phi}^{\bar{i}} = \Theta^a(x) \, (t_a^*)^{\bar{i}}{}_{\bar{j}} \, \bar{\phi}^{\bar{j}} = \Theta^a \; \bar{\mathcal{K}}_a{}^{\bar{i}}(\bar{\phi})$ [conjugate]

$\delta_\Theta K_{i\bar{j}} = 0 \implies$ $\mathcal{K}_a{}^i(\phi)$ are a subset (labelled by a) of the Killing vectors of the scalar geometry determined by the $K_{i\bar{j}}$ metric !!

↳ isometry of the scalar geometry !!

For cosets $\frac{G}{H} \implies G_0 \subset G$

gauge symmetry ↲ ↳ scalar geometry

Moreover

$$\mathcal{K}_a{}^i(\phi) = -i \; K^{i\bar{j}}(\phi,\bar{\phi}) \; \frac{\partial P_a(\phi,\bar{\phi})}{\partial \bar{\phi}^{\bar{j}}}$$

where $P_a(\phi,\bar{\phi})$ are called "moment maps" or "Killing prepotentials"

and enter
$$\nabla_\mu \phi^i = \partial_\mu \phi^i - A_\mu{}^a \; \mathcal{K}_a{}^i(\phi)$$

The moment maps P_a are then expressed as :

$$P_a = -\frac{i}{2}\left[\mathcal{K}_a{}^i \partial_i K - \bar{\mathcal{K}}_a{}^{\bar{j}} \partial_{\bar{j}} K\right] - r_a$$

with $r_a = \delta_{a\,U(1)} \, \underbrace{\xi_{FI}}$ only for Abelian factors in G_0.

Fayet-Iliopoulos $\equiv$ arbitrary parameters

Lastly the scalar potential $V(\phi,\bar\phi)$ is given by

$$V = e^{\kappa^2 K}\Big[\underbrace{\overbrace{D_i W}^{e^{-\frac{1}{2}\kappa^2 K}F^i} K^{i\bar j} D_{\bar j}\overline{W} - 3\kappa^2 |W|^2}_{\text{F-terms}>0}\Big] + \underbrace{\overbrace{\tfrac{1}{2}\mathcal{P}_a \,\mathrm{Im}(N^{-1})^{ab}\,\mathcal{P}_b}^{D^a}}_{\text{D-terms}>0}$$

$\kappa^2 = 8\pi G_N$

in terms of an arbitrary "holomorphic superpotential" $W(\phi)$ where the Kähler derivatives read:

$$D_i W = \partial_i W + \kappa^2 (\partial_i K) W$$

$$D_{\bar i}\overline{W} = \partial_{\bar i}\overline{W} + \kappa^2 (\partial_{\bar i} K)\overline{W}$$

Important: Theory defined by: $\underbrace{G_0}_{\text{gauge group + representation } R[G_0] \text{ for } \phi^i} \oplus N_{ab}(\phi), K(\phi,\bar\phi), W(\phi) \oplus \mathcal{P}_a(\phi,\bar\phi)$

* Local susy transformations & susy breaking.

The local susy transformations take the generic form

$$\delta_\epsilon \text{ Fermion} \sim \bar\epsilon \text{ Boson}, \qquad \delta_\epsilon \text{ Boson} \sim \bar\epsilon \text{ Fermion}$$

=> Lorentz invariance at the vacuum requires $\langle \text{Fermion} \rangle = 0$ and consequently $\langle \delta_\epsilon \text{ Boson} \rangle = 0$ always.

=> Lorentz invariance at the vacuum permits $\langle \text{Boson} \rangle \neq 0$ and consequently $\langle \delta_\epsilon \text{ Fermion} \rangle$ $= 0 \rightarrow$ susy preserved; $\neq 0 \rightarrow$ susy broken (spontaneously)

Let us look at the δ_ϵ Fermions in the interacting theory:

Gravitino : $\delta_\epsilon \Psi_\mu = D_\mu \epsilon + \frac{1}{2}\kappa^2 e^{\frac{1}{2}\kappa^2 K} W \quad \bar{\epsilon}\gamma_\mu$

Chiralini : $\delta_\epsilon \psi^i = \frac{1}{\sqrt{2}}\left[\bar{\epsilon}\,\sigma^\mu(\partial_\mu \phi^i) - F^i \epsilon\right]$

Gaugini : $\delta_\epsilon \lambda^a = \frac{1}{4}\bar{\epsilon}\,\gamma_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} i\, \bar{\epsilon}\,\underbrace{\gamma_*}\, D^a$

$\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3$

with

- $F^i = e^{\frac{1}{2}\kappa^2 K} K^{i\bar{j}} D_{\bar{j}} \overline{W}$ => F-term in $V(\phi,\bar{\phi})$
- $D^a = \mathrm{Im}(N^{-1})^{ab} P_b$ => D-term in $V(\phi,\bar{\phi})$

As a result one has that

$$\text{SUSY}\begin{cases}\delta_\epsilon \psi^i = 0 \\ \delta_\epsilon \lambda^a = 0\end{cases} \Rightarrow \begin{matrix}\langle F^i\rangle = 0 \\ \langle D^a\rangle = 0\end{matrix} \Rightarrow V = -3e^{\kappa^2 K}\kappa^2 |W|^2 < 0$$

AdS vacuum !!

NOTE: If SUSY is broken the gravitino $\Psi_{\mu\alpha}$ gets a mass and acquires a longitudinal mode $\partial_\mu \eta_\alpha$ by eating up the goldstino associated to the direction of ~~SUSY~~.

Ex: F-term breaking => $\eta \propto F_i \psi^i$ and $m_{3/2}^2 = \kappa^4 e^{\kappa^2 K}|W|^2$.

NOTE: The existence of a de Sitter vacuum requires susy to be broken => ϕ^i scalars relevant for cosmology !!

[late-time cosmic acceleration, inflation, ...]

* Global susy : Switching off Gravity [$\kappa^2 \to 0$]

In order to go from local susy (supergravity) to global susy (supersymmetric field theory) one switches off gravity by setting $\kappa^2 \to 0$. As a result :

- $g_{\mu\nu} \to \eta_{\mu\nu}$ (Minkowski space-time)
- $\Psi_{\mu\alpha} \to 0$ (no gauge field for a global symmetry)
- $F_i \to F_i = \partial_i W$
- $D^a \to D^a$ (gauge structure unaffected)

$$\Rightarrow \quad V = K^{i\bar{j}} F_i \bar{F}_{\bar{j}} + \frac{1}{2} \mathrm{Im} N_{ab} D^a D^b \geq 0$$

Therefore : $V > 0 \Rightarrow F_i \neq 0$ and/or $D^a \neq 0 \Rightarrow$ Susy breaking !!

NOTE: Susy field theories are very interesting playgrounds where to discover universality classes of phenomena both classically and also at the quantum level.

* Scalar kinetic terms and "coset" spaces

The scalar kinetic terms can be understood geometrically from a "fictitious" (or auxiliary) scalar space perspective where scalar fields $\phi_i \in \mathbb{R}$ $(i=1,\dots,N)$ play the role of coordinates :

$$S_\phi = \int d^4x \sqrt{-|g|} \left[- \underbrace{K_{ij}(\phi)}_{\text{"metric" in fied space}} \partial_\mu \phi^i \partial^\mu \phi^j \right]$$

- One canonically normalised scalar :

$$K_{\phi\phi} = \frac{1}{2}$$

- N canonically normalised scalars :

$$K_{ij} = \frac{1}{2} \delta_{ij}$$

The geometrical interpretation becomes obvious when writing the kinetic terms as :

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = - K_{ij}(\phi) \underbrace{\partial_\mu \phi^i}_{d\phi^i} \underbrace{\partial^\mu \phi^j}_{d\phi^j} \quad \Rightarrow \quad [\sigma\text{-model}]$$

$$\Rightarrow \quad ds_\phi^2 = K_{ij}(\phi)\, d\phi^i\, d\phi^j \quad \Rightarrow \quad \text{"Scalar geometry"}$$

If $\mathcal{L}_{\text{kin}}$ (or the theory) has some Lie group symmetry G then :

- Linear action on ϕ^i : Then $K_{ij} = \text{cte}$ and also invariant under the linear action of G

- Non-linear action on ϕ^i : Then the scalar geometry is described by a coset G/H and K_{ij} is the corresponding G-invariant metric

Important : In supergravity the scalar geometries are often coset spaces $\frac{G}{H}$ as global symmetries G are non-linearly realised on the scalar sector.

Important: Every two points connected by $g \in G$ $\Rightarrow$ "Homogeous space"

* Coset space $\mathcal{M} = \frac{G}{H}$: Coordinates on $\mathcal{M}$ (fields ϕ^i) correspond to an element of G not being an element of its maximal compact subgroup $H \subset G$ (Isotropy group) :

- generators of G : $\{ \underbrace{h_1, \dots, h_{\dim H}}_{\substack{H \subset G \\ \text{max. compact}}} ; \underbrace{t_1, \dots, t_{\dim G - \dim H}}_{\substack{M = A \oplus N \to \text{Nilpotent} \\ \text{max abelian (Cartan)}}} \}$

[in a given representation]

"Iwasawa decomposition"

NOTE: Split real form of a complex Lie algebra is a maximally non-compact version whose Cartan abelian subalgebra A can be chosen along non-compact directions.

- G algebra structure: Let us denote $h \in H$ and $m \in M$ so that

$$\underbrace{[h,h] = h}_{H \subset G \text{ (subgroup)}} \quad , \quad [h,m] = \underbrace{\cancel{h}}_{\text{Reductive}} \oplus m \quad , \quad [m,m] = h \oplus \underbrace{\cancel{m}}_{\text{Symmetric}}$$

- Coset representative (Borel or triangular gauge) ↱ suitable for higher-dim origin

$$V(\phi) = \underbrace{e^{\sum_{i=1}^{\dim N} \chi_i E_i}}_{g_N} \cdot \underbrace{e^{\sum_{i=1}^{\dim A} \phi_i T_i}}_{g_A}$$

Positive roots (E_i) Cartan (T_i)

- Maurer-Cartan one-form:

$$J_\mu = \mathcal{V}^{-1}\partial_\mu \mathcal{V} = \underbrace{Q_\mu}_{\in H} + \underbrace{P_\mu}_{\in M} \in \mathrm{Lie}(G)$$

NOTE: Q_μ plays the role of a composite connection for the local H symmetry. It enters the covariant derivatives of the fermions transforming linearly under H:

$$D_\mu \Psi_\upsilon \equiv \partial_\mu \Psi_\upsilon + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Psi_\upsilon - \overbrace{Q_\mu}^{Q_\mu(\phi)}\Psi_\upsilon$$

$\to h \in H$

Moreover: $D_\mu \mathcal{V} \equiv \partial_\mu \mathcal{V} - \mathcal{V} Q_\mu \Rightarrow \mathcal{V}^{-1} D_\mu \mathcal{V} = \underbrace{\mathcal{V}^{-1}\partial_\mu \mathcal{V}}_{Q_\mu + P_\mu} - Q_\mu$
$= P_\mu$

- Scalar matrix: $M(\phi) = \mathcal{V}\underbrace{\Delta}\mathcal{V}^t$

H-invariant positive def. matrix

- Scalar kinetic terms:

$$\frac{1}{\sqrt{-|g|}}\mathcal{L}_{kin} = -\mathrm{Tr}\left[D_\mu \mathcal{V}\, D^\mu \mathcal{V}^{-1}\right] = -\mathrm{Tr}\left[P_\mu P^\mu\right] = \frac{1}{4}\mathrm{Tr}\left[\partial_\mu M\, \partial^\mu M^{-1}\right]$$

$$= -K_{ij}(\phi)\,\partial_\mu \phi^i\, \partial^\mu \phi^j$$

Important: Coset representatives transform as

$$\mathcal{V}' = g\,\mathcal{V}\,h(x) \qquad \text{with } g \in G,\ h(x) \in H$$

$\hookrightarrow$ global $\quad \hookrightarrow$ local

$$\Rightarrow \quad M' = \mathcal{V}' \Delta \mathcal{V}'^t = g\,\mathcal{V} h \overbrace{\Delta}^{\Delta} h^t \mathcal{V}^t g^t = g \overbrace{\mathcal{V} \Delta \mathcal{V}^t}^{M} g^t = g M g^t$$

As a result, $\mathcal{L}_{kin}$ is invariant under the action of $g \in G$

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}' = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M' \, \partial^\mu M'^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g \, \partial_\mu M \underbrace{g^t g^{-t}}_{\mathbb{I}} \partial^\mu M^{-1} g^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g \, \partial_\mu M \, \partial^\mu M^{-1} g^{-1}\right]$$

cyclicity

$$= \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1}\right] = \frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}$$

Example 3 : $\mathcal{M} = \dfrac{SL(2)}{SO(2)} \Rightarrow G = SL(2)$, $H = \overbrace{SO(2)}^{\Delta = \mathbb{I}} \subset SL(2)$

- Generators of G : $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
  [fundamental represent]

  $\Rightarrow$ Commutators: $[T, E_\pm] = \pm 2 E_\pm$
  $[E_+, E_-] = T$

- Some examples of group elements of $G = SL(2)$

$$g_T = e^{-\frac{1}{2}\phi T} = \begin{pmatrix} e^{-\frac{\phi}{2}} & 0 \\ 0 & e^{\frac{\phi}{2}} \end{pmatrix} \quad , \quad g_{E_+} = e^{\chi E_+} = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}$$

$$g_h = e^{\theta \underbrace{(E_+ - E_-)}_{h \text{ generator}}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2) = H$$

- When constructing $\mathcal{V} \in \frac{SL(2)}{SO(2)}$ one must be careful for not to exponentiate $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using Borel gauge

$$\mathcal{V}(\phi, \chi) = g_{E_+} \, g_T = \begin{bmatrix} e^{-\frac{\phi}{2}} & e^{\frac{\phi}{2}} \chi \\ 0 & e^{\frac{\phi}{2}} \end{bmatrix} \in \frac{SL(2)}{SO(2)}$$

so that

$$M(\phi, \chi) = \mathcal{V} \, \mathbb{I} \, \mathcal{V}^t = \begin{bmatrix} e^{-\phi} + \chi e^{\phi} \chi & e^{\phi} \chi \\ e^{\phi} \chi & e^{\phi} \end{bmatrix}$$

and

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = \frac{1}{4} \operatorname{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right]$$

$$= -\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \chi \, \partial^\mu \chi$$

$$\Rightarrow \quad K_{ij}(\phi, \chi) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\phi} \end{bmatrix}$$

NOTE: Coset spaces of the form $\frac{G}{H}$ with H being the maximal compact subgroup of G $\left(\text{like } \frac{SL(2)}{SO(2)} \right)$ are important when describing the scalar geometries arising from Kaluza-Klein reductions.

V Extended $\mathcal{N} \geq 2$ supergravity

- There are theories of supergravity in D=4 with more than one ($\mathcal{N}=1$) gravitino fields : the so called "extended SUGRA's"

- The field content of the SUGRA multiplet includes :

$$\underbrace{e_\mu{}^a}_{\substack{\text{metric} \\ (s=2)}} \oplus \underbrace{\begin{matrix}\Psi_{\mu\alpha}^1 \\ \vdots \\ \Psi_{\mu\alpha}^{\mathcal{N}}\end{matrix}}_{\substack{\mathcal{N}\text{ gravitini} \\ (s=3/2)}} \oplus \text{EXTRA FIELDS (bosonic and fermionic)}$$

- The most studied cases are $\mathcal{N} = 2 \ , \ 4 \ , \ 8$ (Minimally extended, Half maximal, Maximally extended)

	$\mathcal{N}=2$	$\mathcal{N}=4$	$\mathcal{N}=8$
Number of supercharges in the susy algebra :	8	16	32

* $\mathcal{N}=2$: $\underbrace{e_\mu{}^a}_{s=2}, \underbrace{A_\mu}_{\substack{s=1 \\ \text{(graviphoton)}}} \oplus \underbrace{\Psi_{\mu\alpha}^{1,2}}_{s=3/2}$

* $\mathcal{N}=4$: $\underbrace{e_\mu{}^a}_{s=2}, \underbrace{A_\mu^{1,\dots,6}}_{s=1}, \underbrace{\tau}_{s=0} \overset{\in \mathbb{C}}{} \in \frac{SL(2)}{SO(2)} \oplus \underbrace{\Psi_{\mu\alpha}^{1,\dots,4}}_{s=3/2}, \underbrace{\psi_\alpha^{1,\dots,4}}_{s=1/2}$; "coset space"

* $\mathcal{N}=8$: $\underbrace{e_\mu{}^a}_{s=2}, \underbrace{A_\mu^{1,\dots,28}}_{s=1}, \underbrace{\phi^{1,\dots,70}}_{\substack{s=0 \\ \in\mathbb{R}}} \in \underbrace{\frac{E_{7(7)}}{SU(8)}}_{\text{"coset space"}} \oplus \underbrace{\Psi_{\mu\alpha}^{1,\dots,8}}_{s=3/2}, \underbrace{\psi_\alpha^{1,\dots,56}}_{s=1/2}$

- Depending on the number $\mathcal{N}$ of supersymmetries, the SUGRA multiplet can be coupled to various types of matter multiplets

$\ast$ $\mathcal{N}=2$: Vector multiplet : $(A_\mu,\ z\in\mathbb{C},\ \lambda_{1,2})$ ↱ Majorana

Hypermultiplet : $(q_{1,\dots,4}\in\mathbb{R},\ \zeta^{1,2})$ ↳ Majorana

$\ast$ $\mathcal{N}=4$: Vector multiplet : $(A_\mu,\ \phi_{1,\dots,6}\in\mathbb{R},\ \lambda_{1,\dots,4})$ ↱ Majorana

$\ast$ $\mathcal{N}=8$: No matter multiplets

- The various scalar fields parameterise different classes of scalar manifolds

[Sp$(2n_v+2)$ vector bundle]

$\ast$ $\mathcal{N}=2$: Vector multiplets $z^{i=1,\dots,n_v}$ span $\underbrace{\mathcal{M}_{SK}}_{\text{Special Kähler}}$

$\mathcal{M}=\mathcal{M}_{SK}\times\mathcal{M}_{QK}$ Hypermultiplets $q_{u=1,\dots,4n_h}$ span $\underbrace{\mathcal{M}_{QK}}_{\text{Quaternionic Kähler}}$

[Holonomy : $SU(2)\times Sp(2n_h)$]

n = # vector multiplets

$\ast$ $\mathcal{N}=4$: $\mathcal{M}=\dfrac{SL(2)}{SO(2)}\times\dfrac{SO(6,n)}{SO(6)\times SO(n)}$

$\ast$ $\mathcal{N}=8$: $\mathcal{M}=\dfrac{E_{7(7)}}{SU(8)}$

VI. Maximal $\mathcal{N}=8$ ungauged supergravity

* Action : ungauged theory $\approx>$ $\nabla_\mu = \partial_\mu$

$$F_{\mu\nu}{}^{\Lambda} = \partial_\mu A_\nu{}^{\Lambda} - \partial_\nu A_\mu{}^{\Lambda}$$

$[\Lambda = 1, \dots, 28]$

$$
\begin{aligned}
S^{\text{ungauged}}_{\mathcal{N}=8} = \int d^4x \sqrt{-|g|} \Big\{ & \frac{R}{2} \\
& + \frac{1}{96} \operatorname{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right] \\
& + \frac{1}{4} \underbrace{\mathcal{I}_{\Lambda\Sigma}(\phi)}_{\text{"}g^{-2}\delta_{\Lambda\Sigma}\text{ like"}} F_{\mu\nu}{}^{\Lambda} F^{\mu\nu\,\Sigma} \\
& + \frac{1}{4} \frac{1}{2\sqrt{-|g|}} \underbrace{\mathcal{R}_{\Lambda\Sigma}(\phi)}_{\text{"}\frac{1}{8\pi^2}\theta\,\delta_{\Lambda\Sigma}\text{ like"}} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma} \\
& + \text{fermi-terms} \Big\} = \int d^4x \sqrt{-|g|} \, \mathcal{L}
\end{aligned}
$$

$M_{MN} \in \dfrac{E_{7(7)}}{SU(8)} = \dfrac{G}{H}$

$\hookrightarrow M = 1, \dots, 56$ [fund. rep. $E_{7(7)}$]

$\mathcal{L}$ ↝ Lagrangian

NOTE : KK reduction of 11D / Type II SUGRA on T^7 / T^6 yields ungauged $\mathcal{N}=8$ (maximal) supergravity in 4D

Symmetries :

* Global $G = E_{7(7)}$ of the scalar sector $M \in \frac{G}{H}$
* Local $H = SU(8)$ R-symmetry linearly acting on fermions
* $U(1)^{28}$ gauge theory with $\underbrace{\text{uncharged matter}}_{\nabla_\mu M_{MN} = \partial_\mu M_{MN}}$

Electric-magnetic Sp(56) duality

As in classical electromagnetism we can associate with the electric $F_{\mu\nu}{}^{\Lambda}$ their magnetic duals $G_{\mu\nu\Lambda}$

$$G_{\mu\nu\Lambda} \equiv -\varepsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}{}^{\Lambda}} = R_{\Lambda\Sigma}(\phi)\, F_{\mu\nu}{}^{\Sigma} - I_{\Lambda\Sigma}(\phi)\, \underbrace{* F_{\mu\nu}{}^{\Sigma}}_{\text{4D Hodge dual}}$$

with

$$* F_{\mu\nu}{}^{\Sigma} \equiv \frac{\sqrt{-|g|}}{2!}\, \varepsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma\Sigma}$$

$$\hookrightarrow \; ** = -1$$

NOTE: In ordinary Maxwell theory without scalars ($\phi^i = 0$) one has $I_{\Lambda\Sigma} = -\delta_{\Lambda\Sigma}$, $R_{\Lambda\Sigma} = 0$.

In terms of $(F_{\mu\nu}{}^{\Lambda}, G_{\mu\nu\Lambda})$ the $\underbrace{\text{vacuum}}_{\text{no charged matter}}$ Maxwell equations are

$$\nabla^{\mu}(* F_{\mu\nu}{}^{\Lambda}) = 0 \quad , \quad \nabla^{\mu}(* G_{\mu\nu\Lambda}) = 0$$

which can be expressed as

$$d\, \mathcal{G}_{\mu\nu}{}^{M} = 0 \quad \text{with} \quad \mathcal{G}_{\mu\nu}{}^{\overbrace{M}^{M=1,\dots,56}} = \begin{pmatrix} F_{\mu\nu}{}^{\Lambda} \\ G_{\mu\nu\Lambda} \end{pmatrix}$$

Using $G_{\mu\nu}{}^{\mathbb{M}}$ the vector sector of the Lagrangian can be expressed as

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}\sqrt{-|g|}\; M_{\mathbb{M}\mathbb{N}}(\phi)\, G_{\mu\nu}{}^{\mathbb{M}}\, G^{\mu\nu\,\mathbb{N}}$$

with

$$\underbrace{M_{\mathbb{M}\mathbb{N}}}_{\text{symmetric}} = \begin{bmatrix} M_{\Lambda\Sigma} & M_{\Lambda}{}^{\Sigma} \\ M^{\Lambda}{}_{\Sigma} & M^{\Lambda\Sigma} \end{bmatrix} = \begin{bmatrix} -(\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma} \\ (\mathcal{I}^{-1}\mathcal{R})^{\Lambda}{}_{\Sigma} & -(\mathcal{I}^{-1})^{\Lambda\Sigma} \end{bmatrix}$$

Importantly the electric $F_{\mu\nu}{}^{\Lambda}$ and magnetic $G_{\mu\nu\,\Lambda}$ field strengths do NOT carry independent dynamics as they obey (by construction) twisted self-duality conditions

$$* G^{\mathbb{M}} = - \overbrace{\Omega^{\mathbb{M}\mathbb{N}}}^{Sp(56)\text{-inv matrix } \Omega_{\mathbb{M}\mathbb{N}} = \begin{bmatrix} 0 & \mathbb{I}_{28} \\ -\mathbb{I}_{28} & 0 \end{bmatrix}} M_{\mathbb{N}\mathbb{P}}(\phi)\, G^{\mathbb{P}}$$

NOTE : The scalar matrix satisfies $M(\phi)\,\Omega\, M(\phi) = \Omega$

Important : The reformulation of the vector sector in terms of $G_{\mu\nu}{}^{\mathbb{M}}$ allows to elevate the $G = E_{7(7)}$ global symmetry of the scalar sector to global symmetries of field equations and Bianchi identities. [on-shell]

More concretely

$$g \in G = E_{7(7)} \quad \begin{cases} \phi \to g \circ \phi & \text{(non-linear action)} \\ \mathcal{G}^{M} \to [R(g)]^{M}{}_{N}\, \mathcal{G}^{N} & \text{(linear action)} \end{cases}$$

action on scalars

and invariance of $d\mathcal{G} = 0$ and $* \mathcal{G} = -\Omega M \mathcal{G}$ imposes (sufficient conditions)

i) $R(g) \in Sp(56) \Rightarrow R(g)^{t}\, \Omega\, R(g) = \Omega$

ii) $M(g \circ \phi) = R(g)^{-t}\, M(\phi)\, R(g)^{-1}$

non-linear action linear action

These two conditions are verified by virtue of supersymmetry.

Important: At the level of the action [off-shell] the symmetry is reduced to $G_{elec} \subset G$

$$g \in G_{elec} \; : \; R(g) = \begin{bmatrix} A & 0 \\ C & D = A^{-t} \end{bmatrix} \qquad \text{Boundary term}$$

$$\Rightarrow \delta \mathcal{L}_{bos} = \frac{1}{8} (CA)_{\Lambda\Sigma}\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma}$$

Symplectic frame: It is a choice of embedding of $R(g) \subset Sp(56)$

$\Rightarrow$ NOT UNIQUE $\Rightarrow$ Important consequences when having a "gauging"

VII. Maximal $\mathcal{N}=8$ gauged supergravity

The ungauged maximal supergravity can be deformed by means of the so-called gauging procedure.

Gauging : Promote a subgroup $G_0 \subset G = E_{7(7)}$ from global to local (gauge)

gauge coupling $\quad$ $\alpha = 1, \dots, 133$ [Adj rep $E_{7(7)}$]

$$\nabla_\mu = \partial_\mu - g \underbrace{A_\mu^{\mathbb{P}}}_{\text{gauge fields}} \underbrace{\Theta_{\mathbb{P}}{}^{\alpha}}_{\text{embedding tensor}} \underbrace{t_\alpha}_{E_{7(7)} \text{ generators } t_{\alpha=1,\dots,133}}$$

Both electric/magnetic [dyonic gaugings]

$\Downarrow$

"Selector"

Ex :
$$\nabla_\mu M_{MN} = \partial_\mu M_{MN} - g A^{\mathbb{P}} \underbrace{\Theta_{\mathbb{P}}{}^{\alpha} [t_\alpha]_{(M}{}^{Q}}_{X_{\mathbb{P}M}{}^{Q} \Leftrightarrow \text{"Charges"}} M_{N)Q}$$

Important : The constant embedding tensor charges $X_{MN}{}^{\mathbb{P}}$ encodes all the information about the 4D theory !!

* Action : gauged theory $\Rightarrow$ $\partial_\mu \to \nabla_\mu$

$F_{\mu\nu}{}^{\Lambda} \to H_{\mu\nu}{}^{\Lambda}$ (non-Abelian)

[$\Lambda = 1, \dots, 28$]

$$S^{\text{gauged}}_{\mathcal{N}=8} = \int d^4x \sqrt{-|g|} \Big\{ \frac{R}{2} + \frac{1}{96} \operatorname{Tr}\Big[\underbrace{\nabla_\mu M \; \nabla^\mu M^{-1}}_{\text{charged fields}} \Big] - \overbrace{V(\phi, \underbrace{X}_{\text{couplings}})}^{\text{Scalar potential}}$$

$$+ \frac{1}{4} \mathcal{I}_{\Lambda\Sigma}(\phi) \, \underbrace{H_{\mu\nu}{}^{\Lambda} H^{\mu\nu\,\Sigma}}_{\text{non-abelian vectors}}$$

$$+ \frac{1}{4} \frac{1}{2\sqrt{-|g|}} \mathcal{R}_{\Lambda\Sigma}(\phi) \, \varepsilon^{\mu\nu\rho\sigma} H_{\mu\nu}{}^{\Lambda} H_{\rho\sigma}{}^{\Sigma}$$

$$+ \underbrace{\mathcal{L}_{top}}_{\text{topological terms}} + \text{fermi-terms} + \underbrace{\text{fermi masses}}_{\text{to restore susy}} \Big\}$$

NOTE: Fermi masses are required to restore susy as when we saw in the context of $\mathcal{N}=1$ SUGRA in the presence of a C.C $\Lambda \neq 0$.

* Consistency conditions on $X_{MN}{}^{P}$

i) Linear constraint $\equiv$ Representation constraint

$\mathcal{N}=8$ susy
[tensor hierarchy] $\Rightarrow$ $X_{MN}{}^{P} \in \underbrace{912}$ of $E_{7(7)}$

$X_{M[NP]}=0 \;, \quad X_{(MNP)}=0$
$X_{PN}{}^{P} = X_{MP}{}^{P} = 0$

ii) Quadratic constraints

Closure of the gauge group G_0 $\Rightarrow$ $\Omega^{MN} X_{MP}{}^{Q} X_{NR}{}^{S} = 0$

Then i) and ii) imply

$$[X_M , X_N] = - X_{MN}{}^{P} X_P$$

↳ close gauge algebra in the 4D theory

* Vector-tensor sector : Vectors fields $A_\mu{}^M$ span now a non-abelian gauge group $G_0 \subset G = E_{7(7)}$

$$\mathcal{G}_{\mu\nu}{}^{M} \longrightarrow H_{\mu\nu}{}^{M} = 2\partial_{[\mu}A_{\nu]}{}^{M} + g\, X_{[PQ]}{}^{M} A_\mu{}^{P} A_\nu{}^{Q}$$
$$+ g\, \underbrace{\tfrac{1}{2}\, \Omega^{MN}\, \Theta_N{}^{\alpha}}_{Z^{M\alpha}}\, B_{\mu\nu\,\alpha}$$

Required for $H_{\mu\nu}{}^{M}$ to be gauge covariant !!

↳ with tensor gauge param. $\Xi_{\mu\alpha}$

NOTE: Jacobi identity for $X_{MN}{}^P$ does not hold $\Rightarrow$ $\delta A_\mu^M = D_\mu \Lambda^M - g\, Z^{M\alpha}\, \Xi_{\mu\alpha}$

To gauge away vectors in the sector where Jacobi fails

Auxiliary two-forms $B_{\mu\nu\alpha}$ dual to scalars

$\Rightarrow$ Non-dynamical

[They also enter $\mathcal{L}_{top}$]

relevant when magnetic charges are present

NOTE: Duality relation:

$$H_{(3)\alpha} = \frac{1}{12} (t_\alpha)_M{}^P \underbrace{M_{NP} * \nabla M^{MN}}_{\text{scalar current (adjoint)}}$$

* Scalar potential: This is probably the most distinctive feature of a gauged supergravity. It takes the form:

$$V(M,X) = \frac{g^2}{672} \Big[X_{MN}{}^R X_{PQ}{}^S M^{MP} M^{NQ} M_{RS} + 7 X_{MN}{}^Q X_{PQ}{}^N M^{MP} \Big]$$

NOTE: The scalar potential makes the gauged theory more interesting for Phenomenology: dS vacua, ...

Important: Maximal gauged supergravities in D=4 appear when compactifying 11D / Type II supergravity on tori and spheres possibly with fluxes, ...

$$\text{Geometry} \oplus \text{fluxes} \oplus \dots = X_{MN}{}^P$$

VIII. Kaluza-Klein reduction on S^1

In this section we are working out the dimensional reduction of gravity in D+1 dimension down to D dimensions. As we will see, this provides a unification of the form:

D+1 Gravity $\Rightarrow$ Gravity + Maxwell + scalar in D

We will describe gravity in D+1 dimensions:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \sqrt{-|\hat{g}|}\ \hat{R}$$

with $\hat{g}_{MN}$ and $\hat{R}_{MN}$ being the metric and Ricci scalar in a (D+1) dimensional space-time $\underbrace{M}_{x^M} = \underbrace{0, 1, \ldots, D-1}_{x^\mu}, \underbrace{z}_{z}$.

Let's take the z-coordinate to be S^1 $\Rightarrow$ Fourier expansion

$$\hat{g}_{MN}(x, z) = \sum_{n=0}^{\infty} \hat{g}^{(n)}_{MN}(x)\, e^{i \frac{n}{L} z}$$

S^1

$(z \to z + 2\pi L)$

n ⇑ Fourier mode

$\Rightarrow$ The zero-mode $(n=0)$ is a massless mode whereas $n \neq 0$ corresponds to a tower of massive modes (KK tower).

Example: Scalar field $\hat{\phi}$ in D+1 dimensional flat space-time

$$S_\phi = \int d^{D+1}x\ \partial_M \hat{\phi}\, \partial^M \hat{\phi} \underset{\text{E.O.M}}{\Rightarrow} \hat{\Box} \hat{\phi} = \partial_M \partial^M \hat{\phi} = 0$$

Fourier expansion along S^1: $\hat{\phi}(x,z) = \sum_{n=0}^{\infty} \phi^{(n)}(x)\, e^{i\frac{n}{L}z}$

so that

$$\hat{\Box}\hat{\phi} = (\underbrace{\partial_\mu \partial^\mu}_{\Box} + \partial_z^2)\hat{\phi} = \sum_{n=0}^{\infty} \underbrace{\left[\Box\phi^{(n)} - \frac{n^2}{L^2}\phi^{(n)}\right]}_{} e^{i\frac{n}{L}z} = 0$$

Each mode must satisfy

$$\Box\phi^{(n)} - \underbrace{\frac{n^2}{L^2}}_{}\phi^{(n)} = 0$$

$m^2 \equiv \frac{n^2}{L^2}$ => Massive modes !! $m = \frac{|n|}{L}$

Important : The KK phylosophy is to assume a very small L (we don't observe S^1) so that all the modes with $n \neq 0$ are very massive $m = \frac{|n|}{L}$ and we have to reach very high energies in order to produce such KK modes :

$$L \sim m_p \sim 10^{-33}\text{cm} \quad \Rightarrow \quad m \sim 10^{-5}\text{ gr}$$

(Higgs mass $\sim m_{top} \sim 10^{-22}$ gr.)

Important : KK reduction = Truncation to $n=0$ massless modes

=> z - independence !!

$$\boxed{\hat{g}_{MN}(x) = \hat{g}^{(0)}_{MN}(x)}$$

The first step to perform a KK reduction is to split the (D+1)-metric in D-dimensional blocks:

$$\hat{g}_{MN}(x) = \begin{bmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{z\nu} & g_{zz} \end{bmatrix} \equiv \begin{bmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix} \Rightarrow \text{Too naive (big mess)}$$

$$\equiv \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix}$$

$\Downarrow$

Much more convenient !!

(see discussion on symmetries)

Therefore we parameterise the (D+1) metric $\hat{g}_{MN}$ as

$$\boxed{d\hat{s}^2_{D+1} = e^{2\alpha\underbrace{\phi}_{\phi \equiv \text{"Dilaton"}}} ds^2_D + e^{2\beta\phi}(dz + A_\mu dx^\mu)^2}$$

with α and β being constants to be chosen later on.

The associated (D+1) frame then reads:

$$\hat{e}_M{}^A = \begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix}$$

$$\boxed{\begin{matrix} M = \mu, z \\ A = a, \underline{z} \end{matrix}}$$

Equivalently : $\hat{e}^a = e^{\alpha\phi} \underbrace{e^a}_{e_\mu{}^a dx^\mu}$ and $\hat{e}^{\underline{z}} = e^{\beta\phi}(dz + A)$ with $A \equiv A_\mu dx^\mu$

Ex: Check that $\hat{e}_M{}^A \, \hat{e}_N{}^B \, \hat{\eta}_{AB} = \hat{g}_{MN}$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \underbrace{\begin{bmatrix} \eta_{ab} & 0_{4\times 1} \\ 0_{1\times 4} & +1 \end{bmatrix} \begin{bmatrix} e^{\alpha\phi} e_\nu{}^b & 0_{4\times 1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}_{\begin{bmatrix} e^{\alpha\phi} e_{\nu a} & 0_{4\times 1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}$$

$$= \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix} = \hat{g}_{MN}(x)$$

In the following our goal will be to compute S_{D+1} using the (D+1)-dimensional frame $\hat{e}_M{}^A$ given above:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \; \hat{e} \; \hat{e}_A{}^M \, \hat{e}_B{}^N \, \hat{R}_{MN}{}^{AB}(\hat{e})$$

- $\hat{e} = e^{(\alpha D + \beta)\phi} \, e$

$A_a = e_a{}^\nu A_\nu$

- We need the inverse (D+1)-dim frame $\hat{e}_A{}^N$

$$\hat{e}_M{}^A \cdot \hat{e}_A{}^N = \delta_M{}^N \Rightarrow \hat{e}_A{}^N = \begin{bmatrix} e^{-\alpha\phi} e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{bmatrix}$$

Ex: check that $\hat{e}_M{}^A \, \hat{e}_A{}^N = \delta_M{}^N$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \begin{bmatrix} e^{-\alpha\phi} e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{bmatrix} = \begin{bmatrix} \delta_\mu{}^\nu & 0_{4\times 1} \\ 0_{1\times 4} & 1 \end{bmatrix}$$

⊙ Now we perform the computation of the Ricci scalar $\hat{R}$.

▲ First we compute the anholonomy coefficients $\hat{\Omega}$:

$$\hat{\Omega}_{[MN]P} = \left(\partial_M \hat{e}_N{}^A - \partial_N \hat{e}_M{}^A\right) \hat{e}_{PA}$$

• $\hat{\Omega}_{[\mu\nu]\rho} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\, \hat{e}_{\rho A}$

$$= (\partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a)\, \hat{e}_{\rho a} + (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\, \hat{e}_{\rho \underline{z}}$$

$$= \left[\partial_\mu (e^{\alpha\phi} e_\nu{}^a) - \partial_\nu (e^{\alpha\phi} e_\mu{}^a)\right] (e^{\alpha\phi} e_{\rho a})$$

$$+ \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right] (e^{\beta\phi} A_\rho)$$

$$= e^{2\alpha\phi} \left[(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a)\, e_{\rho a} + \alpha (\partial_\mu \phi\, e_\nu{}^a - \partial_\nu \phi\, e_\mu{}^a)\, e_{\rho a} \right]$$

$$+ e^{2\beta\phi} \left[F_{\mu\nu} A_\rho + \beta (\partial_\mu \phi\, A_\nu - \partial_\nu \phi\, A_\mu) A_\rho \right]$$

$$= e^{2\alpha\phi} \left[\Omega_{[\mu\nu]\rho} + 2\alpha\, \partial_{[\mu} \phi\, e_{\nu]}{}^a e_{\rho a} \right]$$

$$+ e^{2\beta\phi} \left[F_{\mu\nu} A_\rho + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} A_\rho \right]$$

• $\hat{\Omega}_{[\mu\nu]z} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\, \hat{e}_{zA} = (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\, \hat{e}_{z\underline{z}}$

$$= \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu) \right] e^{\beta\phi}$$

$$= e^{2\beta\phi} \left[F_{\mu\nu} + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} \right]$$

• $\hat{\Omega}_{[\mu z]\rho} = \partial_\mu \hat{e}_z{}^A\, \hat{e}_{\rho A} = \partial_\mu \hat{e}_z{}^{\underline{z}}\, \hat{e}_{\rho \underline{z}} = \partial_\mu (e^{\beta\phi})\, (e^{\beta\phi} A_\rho)$

$$= e^{2\beta\phi}\, \beta\, \partial_\mu \phi\, A_\rho$$

- $$\hat{\Omega}_{[\mu z]z} = \partial_\mu \hat{e}_z{}^A \, \hat{e}_{zA} = \partial_\mu \hat{e}_z{}^{\underline{z}} \, \hat{e}_{z\underline{z}} = \partial_\mu \left(e^{\beta\phi}\right) e^{\beta\phi}$$
$$= e^{2\beta\phi} \, \beta \, \partial_\mu \phi$$

- $$\hat{\Omega}_{[z\nu]\rho} = -\partial_\nu \hat{e}_z{}^A \, \hat{e}_{\rho A} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \, \hat{e}_{\rho\underline{z}} = -\partial_\nu \left(e^{\beta\phi}\right)\left(e^{\beta\phi} A_\rho\right)$$
$$= -e^{2\beta\phi} \, \beta \, \partial_\nu \phi \, A_\rho$$

- $$\hat{\Omega}_{[z\nu]z} = -\partial_\nu \hat{e}_z{}^A \, \hat{e}_{zA} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \, \hat{e}_{z\underline{z}} = -\partial_\nu \left(e^{\beta\phi}\right)\left(e^{\beta\phi}\right)$$
$$= -e^{2\beta\phi} \, \beta \, \partial_\nu \phi$$

- $$\hat{\Omega}_{[zz]\rho} = \hat{\Omega}_{[zz]z} = 0$$

Using $\hat{\Omega}$ we compute the spin connection with <u>all indices curved</u>

$$\hat{\omega}_{M[NP]}(\hat{e}) = \frac{1}{2}\left(\hat{\Omega}_{[MN]P} - \hat{\Omega}_{[NP]M} + \hat{\Omega}_{[PM]N}\right)$$
$$= \hat{\omega}_M{}^{BC}(\hat{e}) \, \hat{e}_{NB} \, \hat{e}_{PC}$$

- $$\hat{\omega}_{\mu[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]\rho} - \hat{\Omega}_{[\nu\rho]\mu} + \hat{\Omega}_{[\rho\mu]\nu}\right)$$

$$= \frac{1}{2}\Big[e^{2\alpha\phi}\Big(2\,\omega_{\mu[\nu\rho]} + 2\alpha\left(\partial_{[\mu}\phi \, e_{\nu]}{}^a e_{\rho a} - \partial_{[\nu}\phi \, e_{\rho]}{}^a e_{\mu a} + \partial_{[\rho}\phi \, e_{\mu]}{}^a e_{\nu a}\right)\Big)$$
$$+ e^{2\beta\phi}\Big(F_{\mu\nu} A_\rho - F_{\nu\rho} A_\mu + F_{\rho\mu} A_\nu + 2\beta\big(\partial_{[\mu}\phi \, A_{\nu]} A_\rho - \partial_{[\nu}\phi \, A_{\rho]} A_\mu$$
$$+ \partial_{[\rho}\phi \, A_{\mu]} A_\nu\big)\Big)\Big]$$

- $$\hat{\omega}_{z[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]\rho} - \hat{\Omega}_{[\nu\rho]z} + \hat{\Omega}_{[\rho z]\nu}\right)$$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi\, A_\rho - (F_{\nu\rho} + 2\beta\,\partial_{[\nu}\phi\, A_{\rho]}) + \beta\,\partial_\rho\phi\, A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right)$$

- $$\hat{\omega}_{\mu[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]z} - \hat{\Omega}_{[\nu z]\mu} + \hat{\Omega}_{[z\mu]\nu}\right)$$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left((F_{\mu\nu} + 2\beta\,\partial_{[\mu}\phi\, A_{\nu]}) - \beta\,\partial_\nu\phi\, A_\mu - \beta\,\partial_\mu\phi\, A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi\, A_\mu\right)$$

- $$\hat{\omega}_{z[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]z} - \hat{\Omega}_{[\nu z]z} + \overbrace{\hat{\Omega}_{[zz]\nu}}^{0}\right)$$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi - \beta\,\partial_\nu\phi\right)\right] = -e^{2\beta\phi}\beta\,\partial_\nu\phi$$

- $$\hat{\omega}_{\mu[z\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu z]\rho} - \hat{\Omega}_{[z\rho]\mu} + \hat{\Omega}_{[\rho\mu]z}\right)$$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(\beta\,\partial_\mu\phi\, A_\rho + \beta\,\partial_\rho\phi\, A_\mu + (F_{\rho\mu} + 2\beta\,\partial_{[\rho}\phi\, A_{\mu]})\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\rho\mu} + 2\beta\,\partial_\rho\phi\, A_\mu\right)$$

- $$\hat{\omega}_{z[z\rho]} = \frac{1}{2}\left(\underbrace{\hat{\Omega}_{[zz]\rho}}_{0} - \hat{\Omega}_{[z\rho]z} + \hat{\Omega}_{[\rho z]z}\right)$$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(+\beta\,\partial_\rho\phi + \beta\,\partial_\rho\phi\right)\right] = e^{2\beta\phi}\beta\,\partial_\rho\phi$$

- $$\hat{\omega}_{\mu[zz]} = \hat{\omega}_{z[zz]} = 0$$

▲ Using the above results we compute the standard spin connection

$$\hat{\omega}_M{}^{BC} = \hat{\omega}_{M[NP]}\,\hat{e}^{BN}\,\hat{e}^{CP}$$

- $$\hat\omega_\mu{}^{bc} = \hat\omega_{\mu[NP]}\,\hat e^{bN}\,\hat e^{cP}$$

$$= \hat\omega_{\mu[\nu\rho]}\hat e^{b\nu}\hat e^{c\rho} + \hat\omega_{\mu[\nu z]}\hat e^{b\nu}\hat e^{cz} + \hat\omega_{\mu[z\rho]}\hat e^{bz}\hat e^{c\rho} + \overbrace{\hat\omega_{\mu[zz]}}^{0}\hat e^{bz}\hat e^{cz}$$

$$= \hat\omega_{\mu[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat\omega_{\mu[\nu z]}\, e^{-2\alpha\phi} e^{b\nu} A^c - \hat\omega_{\mu[z\rho]}\, e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2}\Big[2\,\omega_\mu{}^{[bc]} + 2\alpha\big(\partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a} e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi\, e_{\rho]}{}^a e_{\mu a} e^{b\nu} e^{c\rho}$$
$$+ \partial_{[\rho}\phi\, e_{\mu]}{}^a e_{\nu a} e^{b\nu} e^{c\rho}\big) + e^{2(\beta-\alpha)\phi}\big(F_{\mu\nu} A_\rho e^{b\nu} e^{c\rho} -$$
$$- F_{\nu\rho} A_\mu e^{b\nu} e^{c\rho} + F_{\rho\mu} A_\nu e^{b\nu} e^{c\rho}\big) +$$
$$+ 2\beta e^{2(\beta-\alpha)\phi}\big(\partial_{[\mu}\phi A_{\nu]} A_\rho e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi A_{\rho]} A_\mu e^{b\nu} e^{c\rho}$$
$$+ \partial_{[\rho}\phi A_{\mu]} A_\nu e^{b\nu} e^{c\rho}\big)\Big]$$
$$- \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\mu\nu} - 2\beta\partial_\nu\phi A_\mu) e^{b\nu} A^c\Big] - \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\rho\mu} + 2\beta\partial_\rho\phi A_\mu) A^b e^{c\rho}\Big]$$

$$= (*)$$

NOTE 1: $$2\alpha\,\frac{1}{2}\big(\partial_\mu\phi\, e_\nu{}^a e_{\rho a} e^{b\nu} e^{c\rho} - \partial_\nu\phi\, e_\mu{}^a e_{\rho a} e^{b\nu} e^{c\rho}$$
$$- \partial_\nu\phi\, e_\rho{}^a e_{\mu a} e^{b\nu} e^{c\rho} + \partial_\rho\phi\, e_\nu{}^a e_{\mu a} e^{b\nu} e^{c\rho}$$
$$+ \partial_\rho\phi\, e_\mu{}^a e_{\nu a} e^{b\nu} e^{c\rho} - \partial_\mu\phi\, e_\rho{}^a e_{\nu a} e^{b\nu} e^{c\rho}\big)$$

$$= \alpha\big(\underline{\partial_\mu\phi\,\eta^{bc}} - \partial_\nu\phi\, e^{b\nu} e_\mu{}^c - \partial_\nu\phi\, e_\mu{}^c e^{b\nu} + \partial_\rho\phi\, e_\mu{}^b e^{c\rho}$$
$$+ \partial_\rho\phi\, e_\mu{}^b e^{c\rho} - \underline{\partial_\mu\phi\,\eta^{bc}}\big)$$

$$= \alpha\big(2\,\partial_\rho\phi\, e_\mu{}^b e^{c\rho} - 2\,\partial_\nu\phi\, e_\mu{}^c e^{b\nu}\big) = \quad [\partial^a \equiv e^{a\rho}\partial_\rho]$$

$$= 2\alpha\big(e_\mu{}^b\,\partial^c\phi - e_\mu{}^c\,\partial^b\phi\big) = 4\alpha\, e_\mu{}^{[b}\,\partial^{c]}\phi$$
$$= 4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]}$$

NOTE 2: $e^{2(\beta-\alpha)\phi}\,(F_{\mu\nu} A_\rho e^{b\nu} e^{c\rho} - F_{\nu\rho} A_\mu e^{b\nu} e^{c\rho} + F_{\rho\mu} A_\nu e^{b\nu} e^{c\rho})$

$$= e^{2(\beta-\alpha)\phi}\,(F_\mu{}^b A^c - F^{bc} A_\mu + F^c{}_\mu A^b)$$

$$= e^{2(\beta-\alpha)\phi}\,(2 F_\mu{}^{[b} A^{c]} - F^{bc} A_\mu)$$

NOTE 3: $2\beta e^{2(\beta-\alpha)\phi}\,\frac{1}{2}\Big(\partial_\mu\phi A_\nu A_\rho e^{b\nu} e^{c\rho} - \partial_\nu\phi A_\mu A_\rho e^{b\nu} e^{c\rho} - \partial_\nu\phi A_\rho A_\mu e^{b\nu} e^{c\rho} + \partial_\rho\phi A_\nu A_\mu e^{b\nu} e^{c\rho} + \partial_\rho\phi A_\mu A_\nu e^{b\nu} e^{c\rho} - \partial_\mu\phi A_\rho A_\nu e^{b\nu} e^{c\rho}\Big)$

$$= \beta e^{2(\beta-\alpha)\phi}\,\big(\partial_\mu\phi A^b A^c - \partial^b\phi A_\mu A^c - \partial^b\phi A^c A_\mu + \partial^c\phi A^b A_\mu + \partial^c\phi A_\mu A^b - \partial_\mu\phi A^c A^b\big)$$

$$= \beta e^{2(\beta-\alpha)\phi}\,(-2 A_\mu \partial^b\phi A^c + 2 A_\mu \partial^c\phi A^b)$$

$$= -4\beta e^{2(\beta-\alpha)\phi} A_\mu \partial^{[b}\phi A^{c]}$$

$$(*) = \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi + e^{2(\beta-\alpha)\phi}(2F_\mu{}^{[b}A^{c]} - F^{bc}A_\mu) - 4\beta e^{2(\beta-\alpha)\phi} A_\mu \partial^{[b}\phi A^{c]}\Big]$$
$$- \frac{1}{2} e^{2(\beta-\alpha)\phi}\Big[\underbrace{F_\mu{}^b A^c - F_\mu{}^c A^b}_{2 F_\mu{}^{[b}A^{c]}} - 2\beta\underbrace{(\partial^b\phi A^c A_\mu - \partial^c\phi A^b)}_{2\partial^{[b}\phi A^{c]}} A_\mu\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi - e^{2(\beta-\alpha)\phi} F^{bc} A_\mu + e^{2(\beta-\alpha)\phi}\big(2F_\mu{}^{[b}A^{c]} - 4\beta A_\mu \partial^{[b}\phi A^{c]} - 2F_\mu{}^{[b}A^{c]} + 4\beta\partial^{[b}\phi A^{c]}\big)\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} + 4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]} - e^{2(\beta-\alpha)\phi} F^{bc} A_\mu\Big] =$$

$$= \omega_\mu{}^{[bc]} + \alpha\left(\partial^c\phi\, e_\mu{}^b - \partial^b\phi\, e_\mu{}^c\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} A_\mu .$$

- $\hat\omega_z{}^{bc} = \hat\omega_{z[NP]}\,\hat e^{bN}\,\hat e^{cP}$

$$= \hat\omega_{z[\nu\rho]}\hat e^{b\nu}\hat e^{c\rho} + \hat\omega_{z[\nu z]}\hat e^{b\nu}\hat e^{cz} + \hat\omega_{z[z\rho]}\hat e^{bz}\hat e^{c\rho} + \overbrace{\hat\omega_{z[zz]}}^{0}\hat e^{bz}\hat e^{cz}$$

$$= \hat\omega_{z[\nu\rho]} e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat\omega_{z[\nu z]} e^{-2\alpha\phi} e^{b\nu} A^c - \hat\omega_{z[z\rho]} e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2} e^{2(\beta-\alpha)\phi}\left[-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right] e^{b\nu} e^{c\rho} + e^{2(\beta-\alpha)\phi}\beta\,\partial_\nu\phi\, e^{b\nu} A^c$$

$$\quad - e^{2(\beta-\alpha)\phi}\beta\,\partial_\rho\phi\, A^b e^{c\rho} =$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} + e^{2(\beta-\alpha)\phi}\beta\left[-\partial^b\phi A^c + \partial^c\phi A^b + \partial^b\phi A^c - \partial^c\phi A^b\right]$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc}.$$

Therefore, using compact notation, we find that

$$\boxed{\hat\omega^{bc} = \omega^{[bc]} + \alpha e^{-\alpha\phi}\left(\partial^c\phi\,\hat e^b - \partial^b\phi\,\hat e^c\right) - \frac{1}{2} F^{bc} e^{(\beta-2\alpha)\phi}\hat e^{\underline z}}$$

- $\hat\omega_\mu{}^{b\underline z} = \hat\omega_{\mu[NP]}\,\hat e^{bN}\,\hat e^{\underline z P}$

$$= \hat\omega_{\mu[\nu\rho]}\hat e^{b\nu}\underbrace{\hat e^{\underline z\rho}}_{0} + \hat\omega_{\mu[\nu z]}\hat e^{b\nu}\hat e^{\underline z z} + \hat\omega_{\mu[z\rho]}\hat e^{bz}\overbrace{\hat e^{\underline z\rho}}^{0} + \overbrace{\hat\omega_{\mu[zz]}}^{0}\hat e^{bz}\hat e^{\underline z z}$$

$$= \hat\omega_{\mu[\nu z]} e^{-(\alpha+\beta)\phi} e^{b\nu} = \frac{1}{2} e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi A_\mu\right) e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= \frac{1}{2} e^{(\beta-\alpha)\phi}\left(F_\mu{}^b - 2\beta\,\partial^b\phi\, A_\mu\right) \quad \rightarrow F^b{}_c\, e_\mu{}^c$$

$$= -\beta e^{(\beta-\alpha)\phi}\partial^b\phi A_\mu - \frac{1}{2} e^{(\beta-\alpha)\phi} F^b{}_\mu$$

$$= -e^{(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi A_\mu + \frac{1}{2}F^b{}_\mu\right] = -\hat\omega_\mu{}^{\underline z b}$$

- $\hat{\omega}_z{}^{b\underline{z}} = \hat{\omega}_{z[NP]}\,\hat{e}^{bN}\,\hat{e}^{\underline{z}P}$

$$= \hat{\omega}_{z[\nu\rho]}\hat{e}^{b\nu}\underbrace{\hat{e}^{\underline{z}\rho}}_{0} + \hat{\omega}_{z[\nu z]}\hat{e}^{b\nu}\hat{e}^{\underline{z}z} + \hat{\omega}_{z[z\rho]}\overbrace{\hat{e}^{bz}}^{0}\hat{e}^{\underline{z}\rho} + \hat{\omega}_{z[zz]}\overbrace{\hat{e}^{bz}\hat{e}^{\underline{z}z}}^{0}$$

$$= \hat{\omega}_{z[\nu z]}\, e^{-(\alpha+\beta)\phi}\, e^{b\nu} = -e^{2\beta\phi}\beta\partial_\nu\phi\, e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\,\partial^b\phi \;=\; -\hat{\omega}_z{}^{\underline{z}b}$$

Using again compact notation we find:

$$\boxed{\hat{\omega}^{b\underline{z}} = -\omega^{\underline{z}b} = -\beta e^{-\alpha\phi}\partial^b\phi\,\hat{e}^z - \frac{1}{2}{}^{(\beta-2\alpha)\phi} F^b{}_c\,\hat{e}^c}$$

- $\hat{\omega}_\mu{}^{\underline{z}\underline{z}} = \hat{\omega}_z{}^{\underline{z}\underline{z}} = 0$

▲ Once we have the spin connection we compute the Riemann tensor

$$\hat{R}_{MN}{}^{BC} = \partial_M\hat{\omega}_N{}^{BC} - \partial_N\hat{\omega}_M{}^{DC} + \hat{\omega}_M{}^B{}_D\,\hat{\omega}_N{}^{DC} - \hat{\omega}_N{}^B{}_D\,\hat{\omega}_M{}^{DC}$$

- $\hat{R}_{\mu\nu}{}^{bc} = \partial_\mu\hat{\omega}_\nu{}^{bc} + \hat{\omega}_\mu{}^b{}_D\,\hat{\omega}_\nu{}^{Dc} - \partial_\nu\hat{\omega}_\mu{}^{bc} - \hat{\omega}_\nu{}^b{}_D\,\hat{\omega}_\mu{}^{Dc}$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}\partial_\mu\left(e^{2(\beta-\alpha)\phi}F^{bc}A_\nu\right)$$
$$+ \hat{\omega}_\mu{}^b{}_d\,\hat{\omega}_\nu{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\,\hat{\omega}_\nu{}^{\underline{z}c} - (\mu\leftrightarrow\nu)$$
$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[2(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu$$
$$+ \partial_\mu F^{bc}A_\nu + F^{bc}\partial_\mu A_\nu\Big]$$
$$+ \Big[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d A_\mu\Big]$$
$$\Big[\omega_\nu{}^{dc} + \alpha(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\nu\Big]$$

$$- e^{2(\beta-\alpha)\phi}\Big[\beta\,\partial^b\phi\, A_\mu + \tfrac{1}{2}F^b{}_\mu\Big]\Big[\beta\,\partial^c\phi\, A_\nu + \tfrac12 F^c{}_\nu\Big] - (\mu\leftrightarrow\nu)$$

$$= \overbrace{\partial_\mu \omega_\nu{}^{bc} + \omega_\mu{}^b{}_d\, \omega_\nu{}^{dc}}^{R_{\mu\nu}{}^{bc}} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c)$$

$$- e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu + \tfrac12\partial_\mu F^{bc}A_\nu + \tfrac12 F^{bc}\partial_\mu A_\nu$$

$$+\beta^2\, \underline{\partial^b\phi\,\partial^c\phi\, A_\mu A_\nu} + \tfrac12\beta\,\partial^b\phi\, A_\mu F^c{}_\nu + \tfrac12\beta\,\partial^c\phi\, A_\nu F^b{}_\mu + \tfrac14 F^b{}_\mu F^c{}_\nu\Big]$$

$$+\alpha\,\omega_\mu{}^b{}_d(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c) - \tfrac12 e^{2(\beta-\alpha)\phi}\,\omega_\mu{}^b{}_d F^{dc}A_\nu$$

$$+\alpha\,\omega_\nu{}^{dc}(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \tfrac12\alpha\, e^{2(\beta-\alpha)\phi}(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d})F^{dc}A_\nu$$

$$+\alpha^2(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - \underline{\partial^b\phi\,\partial^c\phi\, g_{\mu\nu}} + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c)$$

$$-\tfrac12 e^{2(\beta-\alpha)\phi}\,\omega_\nu{}^{dc}F^b{}_d A_\mu + \underline{\tfrac14 e^{4(\beta-\alpha)\phi}F^b{}_d F^{dc}A_\mu A_\nu}$$

$$-\tfrac12\alpha\, e^{2(\beta-\alpha)\phi}(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c)F^b{}_d A_\mu - \underline{(\mu\leftrightarrow\nu)}$$

NOTE: Underlined terms vanish because they are $\mu\leftrightarrow\nu$ symmetric

$$= \tfrac12 R_{\mu\nu}{}^{bc} + \alpha\big(\partial_\mu\partial^c\phi\, e_\nu{}^b + \partial^c\phi\,\partial_\mu e_\nu{}^b - \partial_\mu\partial^b\phi\, e_\nu{}^c - \partial^b\phi\,\partial_\mu e_\nu{}^c\big)$$

$$- e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu + \tfrac12\beta\,\partial^b\phi\, F^c{}_\nu A_\mu + \tfrac12\beta\,\partial^c\phi\, F^b{}_\mu A_\nu$$

$$+\tfrac12\alpha\,\partial_d\phi\, F^{dc}A_\nu e_\mu{}^b - \tfrac12\alpha\,\partial^b\phi\, F_\mu{}^c A_\nu$$

$$+\tfrac12\alpha\,\partial^c\phi\, F^b{}_\nu A_\mu - \tfrac12\alpha\,\partial^d\phi\, F^b{}_d A_\mu e_\nu{}^c$$

$$+\tfrac12\partial_\mu F^{bc}A_\nu + \tfrac12 F^{bc}\partial_\mu A_\nu + \tfrac14 F^b{}_\mu F^c{}_\nu$$

$$+\tfrac12\omega_\mu{}^b{}_d F^{dc}A_\nu + \tfrac12\omega_\nu{}^{dc}F^b{}_d A_\mu\Big]$$

$$+\alpha\,\omega_\mu{}^b{}_d(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c) + \alpha\,\omega_\nu{}^{dc}(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d})$$

$$+\alpha^2(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - \underline{(\mu\leftrightarrow\nu)}$$

$$
\begin{aligned}
\bullet\ \hat{R}_{\mu\nu}{}^{b\underline{z}} &= \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{D}\, \hat{\omega}_\nu{}^{D\underline{z}} - (\mu \leftrightarrow \nu) \\
&= \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{c}\, \hat{\omega}_\nu{}^{c\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{\underline{z}}\, \overbrace{\hat{\omega}_\nu{}^{\underline{z}\underline{z}}}^{0} - (\mu \leftrightarrow \nu) \\
&= -\beta\, e^{(\beta-\alpha)\phi} \left[(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \partial_\mu \partial^b\phi\, A_\nu + \partial^b\phi\, \partial_\mu A_\nu \right] \\
&\quad - \tfrac{1}{2} e^{(\beta-\alpha)\phi} \left[(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \partial_\mu F^b{}_\nu \right] \\
&\quad - \left[\omega_\mu{}^{b}{}_{c} + \alpha \left(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c} \right) - \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu \right] \\
&\qquad e^{(\beta-\alpha)\phi} \left[\beta\, \partial^c\phi\, A_\nu + \tfrac{1}{2} F^c{}_\nu \right] - (\mu \leftrightarrow \nu) \\
&= -e^{(\beta-\alpha)\phi} \Big[\beta(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \beta\, \partial_\mu \partial^b\phi\, A_\nu + \beta\, \partial^b\phi\, \partial_\mu A_\nu \\
&\qquad + \tfrac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \tfrac{1}{2}\partial_\mu F^b{}_\nu + \beta\, \omega_\mu{}^b{}_c\, \partial^c\phi\, A_\nu + \tfrac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu \\
&\qquad + \alpha\beta\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b A_\nu + \tfrac{1}{2}\alpha\, \partial_c\phi\, e_\mu{}^b F^c{}_\nu \\
&\qquad - \alpha\beta\, \partial_\mu\phi\, \partial^b\phi\, A_\nu - \tfrac{1}{2}\alpha\, \partial^b\phi\, F_{\mu\nu} \Big] \\
&\qquad + e^{3(\beta-\alpha)\phi} \left[\underline{\tfrac{1}{2}\beta\, F^b{}_c\, \partial^c\phi\, A_\mu A_\nu} + \tfrac{1}{4} F^b{}_c F^c{}_\nu A_\mu \right] - \underline{(\mu \leftrightarrow \nu)} \\
&= -e^{(\beta-\alpha)\phi} \Big[\beta^2 \partial_\mu\phi\, \partial^b\phi\, A_\nu - 2\alpha\beta\, \partial_\mu\phi\, \partial^b\phi\, A_\nu + \beta\, \partial_\mu \partial^b\phi\, A_\nu \\
&\qquad + \alpha\beta\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b A_\nu + \tfrac{1}{2}\alpha\, \partial_c\phi\, F^c{}_\nu\, e_\mu{}^b + \tfrac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu \\
&\qquad + \beta\, \partial^b\phi\, \partial_\mu A_\nu - \tfrac{1}{2}\alpha\, \partial^b\phi\, F_{\mu\nu} + \tfrac{1}{2}\partial_\mu F^b{}_\nu \\
&\qquad + \beta\, \omega_\mu{}^b{}_c\, \partial^c\phi\, A_\nu + \tfrac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu \Big] \\
&\qquad + e^{3(\beta-\alpha)\phi}\, \tfrac{1}{4} F^b{}_c F^c{}_\nu A_\mu - (\mu \leftrightarrow \nu) = -\hat{R}_{\mu\nu}{}^{\underline{z}b}
\end{aligned}
$$

$$\bullet\ \hat{R}_{\mu z}{}^{bc} = \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{Dc} - (\mu \leftrightarrow z)$$

$$= \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_d\, \hat{\omega}_z{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \hat{\omega}_z{}^{\underline{z}c}$$

$$- \underbrace{\partial_z \hat{\omega}_\mu{}^{bc}}_{0} - \hat{\omega}_z{}^b{}_d\, \hat{\omega}_\mu{}^{dc} - \hat{\omega}_z{}^b{}_{\underline{z}}\, \hat{\omega}_\mu{}^{\underline{z}c}$$

$$= -\tfrac{1}{2} e^{2(\beta-\alpha)\phi}\left[2(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \partial_\mu F^{bc}\right]$$

$$-\left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \tfrac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d A_\mu\right]\tfrac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}$$

$$- e^{(\beta-\alpha)\phi}\left[\beta\partial^b\phi A_\mu + \tfrac{1}{2}F^b{}_\mu\right]\beta e^{(\beta-\alpha)\phi}\partial^c\phi$$

$$+ \tfrac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d\left[\omega_\mu{}^{dc} + \alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c) - \tfrac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\mu\right]$$

$$+ \beta e^{(\beta-\alpha)\phi}\partial^b\phi\; e^{(\beta-\alpha)\phi}\left[\beta\partial^c\phi A_\mu + \tfrac{1}{2}F^c{}_\mu\right]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi F^{bc} + \tfrac{1}{2}\partial_\mu F^{bc} + \tfrac{1}{2}\omega_\mu{}^b{}_d F^{dc}$$

$$+ \tfrac{1}{2}\alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d})F^{dc} + \beta^2\partial^b\phi\partial^c\phi A_\mu + \tfrac{1}{2}\beta F^b{}_\mu\partial^c\phi$$

$$- \tfrac{1}{2}\omega_\mu{}^{dc}F^b{}_d - \tfrac{1}{2}\alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c)F^b{}_d - \beta^2\partial^b\phi\partial^c\phi A_\mu$$

$$- \tfrac{1}{2}\beta\partial^b\phi F^c{}_\mu\Big]$$

$$+ e^{4(\beta-\alpha)\phi}\left[\tfrac{1}{4}F^b{}_d F^{dc}A_\mu - \tfrac{1}{4}F^b{}_d F^{dc}A_\mu\right]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi F^{bc} - \tfrac{1}{2}\alpha\partial^d\phi F^c{}_d e_\mu{}^b + \tfrac{1}{2}\alpha\partial^b\phi F^c{}_\mu$$

$$- \tfrac{1}{2}\alpha\partial^c\phi F^b{}_\mu + \tfrac{1}{2}\alpha\partial^d\phi F^b{}_d e_\mu{}^c + \tfrac{1}{2}\beta\partial^c\phi F^b{}_\mu - \tfrac{1}{2}\beta\partial^b\phi F^c{}_\mu$$

$$- \tfrac{1}{2}\omega_\mu{}^b{}_d F^{cd} + \tfrac{1}{2}\omega_\mu{}^c{}_d F^{bd} + \tfrac{1}{2}\partial_\mu F^{bc}\Big]$$

$$
\begin{aligned}
&= - e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\,\partial_\mu\phi\, F^{bc} + \alpha\,\partial^{[b}\phi\, F^{c]}{}_\mu + \alpha\,\partial^d\phi\, F^{[b}{}_d\, e_\mu{}^{c]} \\
&\qquad\qquad + \beta\, F^{[b}{}_\mu\, \partial^{c]}\phi - \omega_\mu{}^{[b}{}_d\, F^{c]d} + \tfrac{1}{2}\,\partial_\mu F^{bc} \Big] \\
&= - \hat{R}_{z\mu}{}^{bc}
\end{aligned}
$$

- $\hat{R}_{\mu z}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{D\underline{z}} \underset{0}{-} (\mu \leftrightarrow z)$

$$
\begin{aligned}
&= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \overbrace{\hat{\omega}_z{}^{\underline{z}\underline{z}}}^{0} \\
&\quad - \underbrace{\partial_z \hat{\omega}_\mu{}^{b\underline{z}}}_{0} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}} - \omega_z{}^b{}_{\underline{z}}\, \underbrace{\hat{\omega}_\mu{}^{\underline{z}\underline{z}}}_{0} \\
&= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}} \\
&= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\,\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi \Big] \\
&\quad - \Big[\omega_\mu{}^b{}_c + \alpha\,(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}) - \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu \Big]\, \beta\, e^{(\beta-\alpha)\phi}\, \partial^c\phi \\
&\quad - \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c \cdot e^{(\beta-\alpha)\phi}\Big[\beta\,\partial^c\phi\, A_\mu + \tfrac{1}{2} F^c{}_\mu \Big] \\
&= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\,\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \omega_\mu{}^b{}_c\,\partial^c\phi \\
&\qquad\qquad + \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b - \alpha\,\partial^b\phi\,\partial_\mu\phi \Big] \\
&\quad + e^{3(\beta-\alpha)\phi}\Big[\underline{\tfrac{1}{2}\beta\, F^b{}_c\,\partial^c\phi\, A_\mu} - \underline{\tfrac{1}{2}\beta\, F^b{}_c\,\partial^c\phi\, A_\mu} \\
&\qquad\qquad - \tfrac{1}{4} F^b{}_c\, F^c{}_\mu \Big] \\
&= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-2\alpha)\,\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b + \omega_\mu{}^b{}_c\,\partial^c\phi \Big] \\
&\quad - \tfrac{1}{4} e^{3(\beta-\alpha)\phi}\, F^b{}_c\, F^c{}_\mu = - \hat{R}_{\mu z}{}^{\underline{z}b} = - \hat{R}_{z\mu}{}^{b\underline{z}} = \hat{R}_{z\mu}{}^{\underline{z}b}
\end{aligned}
$$

▲ With the Riemann tensor we compute now the curved / flat Ricci tensor

$$\hat{R}_{Nc} = \hat{R}_{MN}{}^{B}{}_{c}\,\hat{e}_{B}{}^{M}$$

• $$\hat{R}_{\nu c} = \hat{R}_{M\nu}{}^{B}{}_{c}\,\hat{e}_{B}{}^{M}$$

$$= \hat{R}_{\mu\nu}{}^{b}{}_{c}\,\hat{e}_{b}{}^{\mu} + \hat{R}_{\mu\nu}{}^{\underline{z}}{}_{c}\,\overbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0} + \hat{R}_{z\nu}{}^{b}{}_{c}\,e_{b}{}^{z} + \hat{R}_{z\nu}{}^{\underline{z}}{}_{c}\,\hat{e}_{\underline{z}}{}^{z}$$

$$= e^{-\alpha\phi} e_{b}{}^{\mu}\,\hat{R}_{\mu\nu}{}^{b}{}_{c} - e^{-\alpha\phi} A_{b}\,\hat{R}_{z\nu}{}^{b}{}_{c} + e^{-\beta\phi} R_{z\nu}{}^{\underline{z}}{}_{c}$$

$$= e^{-\alpha\phi} R_{\nu c} + e^{-\alpha\phi}\alpha\big(\partial_b\partial_c\phi\, e_\nu{}^{b} - \partial_\nu\partial_c\phi\, D + \partial_c\phi\,\partial_b e_\nu{}^{b} - \partial_c\phi\,\partial_\nu e_\mu{}^{b} e_b{}^{\mu}$$
$$- \partial^2\phi\, e_{\nu c} + \partial_\nu\partial^b\phi\,\eta_{bc} - \partial^b\phi\,\partial_b e_{\nu c}$$
$$+ \partial^b\phi\,\partial_\nu e_{\mu c}\, e_b{}^{\mu}\big)$$

$$- e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\big(\partial_b\phi F^{b}{}_{c}A_\nu - \partial_\nu\phi F^{b}{}_{c}A_b\big)$$
$$+ \tfrac{1}{2}\beta\big(\partial^b\phi F_{c\nu}A_b + \partial_b\phi F^{b}{}_{c}A_\nu\big)$$
$$+ \tfrac{1}{2}\beta\big(\cancelto{0}{\partial_c\phi F^{b}{}_{b}A_\nu} - \partial_c\phi F^{b}{}_{\nu}A_b\big)$$
$$+ \tfrac{1}{2}\alpha\,\partial_b\phi F^{b}{}_{c}\big(A_\nu \underbrace{\delta_d{}^{d}}_{D} - \underbrace{A_d e_\nu{}^{d}}_{A_\nu}\big)$$
$$- \tfrac{1}{2}\alpha\,\partial_b\phi\big(F^{b}{}_{c}A_\nu - F_{\nu c}A^{b}\big)$$
$$+ \tfrac{1}{2}\alpha\,\partial_c\phi\big(F^{b}{}_{\nu}A_b - \cancelto{0}{F^{b}{}_{b}A_\nu}\big)$$
$$+ \tfrac{1}{2}\alpha\,\partial_b\phi F^{bd}\big(A_d e_{\nu c} - A_\nu\eta_{dc}\big)$$
$$+ \tfrac{1}{2}\big(\partial_b F^{b}{}_{c}A_\nu - \partial_\nu F^{b}{}_{c}A_b\big) + \tfrac{1}{2}F^{b}{}_{c}F_{b\nu}$$
$$+ \tfrac{1}{4}\big(\cancelto{0}{F^{b}{}_{b}F_{c\nu}} - F^{b}{}_{\nu}F_{cb}\big)$$
$$+ \tfrac{1}{2}F^{d}{}_{c}\big(\omega_b{}^{b}{}_{d}A_\nu - \omega_\nu{}^{b}{}_{d}A_b\big)$$
$$+ \tfrac{1}{2}F^{b}{}_{d}\big(\omega_\nu{}^{d}{}_{c}A_b - \omega_b{}^{d}{}_{c}A_\nu\big)\Big]$$

SO(1,D-1) generators are antisymmetric

$$+ e^{-\alpha\phi}\Big[\alpha\,\omega_b{}^b{}_d(\partial_c\phi\, e_\nu{}^d - \partial^d\phi\, e_{\nu c}) - \alpha\,\omega_\nu{}^b{}_d(\underbrace{\cancel{\partial_c\phi\,\delta_b^d}}_{0} - \partial^d\phi\,\eta_{bc})$$

$$+\alpha\,\omega_\nu{}^d{}_c(\partial_d\phi\,\overbrace{\delta_b^b}^{D} - \partial^b\phi\,\eta_{bd}) - \alpha\,\omega_b{}^d{}_c(\partial_d\phi\, e_\nu{}^b - \partial^b\phi\, e_{\nu d})$$

$$+\alpha^2\big(\partial_\nu\phi\,\partial_c\phi\,\overbrace{\delta_b^b}^{D} - \partial_b\phi\,\partial_c\phi\, e_\nu{}^b + \partial^b\phi\,\partial_b\phi\, e_{\nu c} - \partial^b\phi\,\partial_\nu\phi\,\eta_{bc}$$

$$-\partial_d\phi\,\partial^d\phi\,\underbrace{\delta_b^b}_{D}\, e_{\nu c} + \partial_d\phi\,\partial^d\phi\, e_\nu{}^b\,\eta_{bc}\big)\Big]$$

$$- A_b\, e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\,\partial_\nu\phi\, F^b{}_c - \tfrac12\alpha\,\partial^d\phi\, F_{cd}\, e_\nu{}^b + \tfrac12\alpha\,\partial^b\phi\, F_{c\nu}$$

$$-\tfrac12\alpha\,\partial_c\phi\, F^b{}_\nu + \tfrac12\alpha\,\partial^d\phi\, F^b{}_d\, e_{\nu c} + \tfrac12\beta\,\partial_c\phi\, F^b{}_\nu$$

$$-\tfrac12\beta\,\partial^b\phi\, F_{c\nu} - \tfrac12\omega_\nu{}^b{}_d\, F_c{}^d + \tfrac12\omega_{\nu cd}\, F^{bd}$$

$$+\tfrac12\partial_\nu F^b{}_c\Big]$$

$$-\beta\, e^{-\alpha\phi}\Big[(\beta-2\alpha)\,\partial_\nu\phi\,\partial_c\phi + \partial_\nu\partial_c\phi + \alpha\,\partial_d\phi\,\partial^d\phi\, e_{\nu c} + \omega_{\nu cd}\,\partial^d\phi\Big]$$

$$-\tfrac14 e^{(2\beta-3\alpha)\phi}\, F_{cd}\, F^d{}_\nu$$

$$= e^{-\alpha\phi}\Big[R_{\nu c} + \alpha\big(\partial_b\partial_c\phi\, e_\nu{}^b - \partial^2\phi\, e_{\nu c} - (D-1)\,\partial_\nu\partial_c\phi$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b - \partial^b\phi\,\partial_b e_{\nu c} - \partial_c\phi\,\partial_\nu e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_\nu e_{\mu c}\, e_b{}^\mu$$

$$+\omega_b{}^b{}_\nu\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\, e_{\nu c} + (3-D)\,\omega_{\nu c}{}^d\,\partial_d\phi$$

$$+\omega^d{}_{\nu c}\,\partial_d\phi\big) + \alpha^2(D-2)\big(\partial_\nu\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\, e_{\nu c}\big)$$

$$-\beta^2\,\partial_\nu\phi\,\partial_c\phi + \alpha\beta\big(2\,\partial_\nu\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\, e_{\nu c}\big)$$

$$-\beta\big(\partial_\nu\partial_c\phi + \omega_{\nu c}{}^d\,\partial_d\phi\big)\Big]$$

$$-e^{(2\beta-3\alpha)\phi}\Big[\alpha\,\frac{D-4}{2}\,\partial_b\phi\, F^b{}_c A_\nu + \beta\,\frac32\,\partial_b\phi\, F^b{}_c A_\nu + \tfrac12\partial_b F^b{}_c A_\nu + \tfrac12 F_{b\nu} F^b{}_c$$

$$+\tfrac12\omega_b{}^b{}_d\, F^d{}_c A_\nu - \tfrac12\omega_b{}^d{}_c\, F^b{}_d A_\nu\Big]$$

- $$\hat{R}_{zc} = \hat{R}_{Mz}{}^{B}{}_{c}\,\hat{e}_{B}{}^{M}$$

$$= \hat{R}_{\mu z}{}^{b}{}_{c}\,\hat{e}_{b}{}^{\mu} + \hat{R}_{\mu z}{}^{z}{}_{c}\,\overbrace{\hat{e}_{z}{}^{\mu}}^{0} + \overbrace{\hat{R}_{zz}{}^{b}{}_{c}}^{0}\,\hat{e}_{b}{}^{z} + \overbrace{\hat{R}_{zz}{}^{z}{}_{c}}^{0}\,\hat{e}_{z}{}^{z}$$

$$= e^{-\alpha\phi}\,e_{b}{}^{\mu}\,\hat{R}_{\mu z}{}^{b}{}_{c}$$

$$= -e^{(2\beta-3\alpha)\phi}\Big[\alpha\Big(-\partial_b\phi\,F^{b}{}_{c} + \frac{D}{2}\partial^{d}\phi\,F_{dc} - \partial^{b}\phi\,F_{bc}\Big) + \frac{1}{2}\partial_b F^{b}{}_{c}$$

$$+ \frac{3}{2}\partial_b\phi\,F^{b}{}_{c} + \frac{1}{2}\omega_{b}{}^{b}{}_{d}\,F^{d}{}_{c} - \frac{1}{2}\omega_{bcd}\,F^{db}\Big]$$

- $$\hat{R}_{\nu z} = \hat{R}_{M\nu}{}^{B}{}_{z}\,\hat{e}_{B}{}^{M}$$

$$= \hat{R}_{\mu\nu}{}^{b}{}_{z}\,\hat{e}_{b}{}^{\mu} + \hat{R}_{\mu\nu}{}^{z}{}_{z}\,\overbrace{\hat{e}_{z}{}^{\mu}}^{0} + \hat{R}_{z\nu}{}^{b}{}_{z}\,\hat{e}_{b}{}^{z} + \overbrace{\hat{R}_{z\nu}{}^{z}{}_{z}}^{0}\,\hat{e}_{z}{}^{z}$$

$$= e^{-\alpha\phi}\,e_{b}{}^{\mu}\,\hat{R}_{\mu\nu}{}^{b}{}_{z} - e^{-\alpha\phi}A_b\,\hat{R}_{z\nu}{}^{b}{}_{z}$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\big(\partial_b\phi\,\partial^b\phi\,A_\nu - \partial_\nu\phi\,\partial^b\phi\,A_b\big)$$

$$+ \alpha\beta\big(-2\partial_b\phi\,\partial^b\phi\,A_\nu + 2\partial_\nu\phi\,\partial^b\phi\,A_b + (D-1)\partial_c\phi\,\partial^c\phi\,A_\nu\big)$$

$$+ \alpha\Big(\frac{1}{2}\partial_c\phi\,(D-1)F^{c}{}_{\nu} - \frac{1}{2}\partial_b\phi\,F^{b}{}_{\nu} - \partial^b\phi\,F_{b\nu}\Big)$$

$$+ \beta\Big(\partial^2\phi\,A_\nu - \partial_\nu\partial^b\phi\,A_b + \frac{1}{2}\partial_b\phi\,F^{b}{}_{\nu} + \partial^b\phi\,F_{b\nu}$$

$$+ \omega_{b}{}^{b}{}_{c}\,\partial^c\phi\,A_\nu - \omega_{\nu}{}^{b}{}_{c}\,\partial^c\phi\,A_b\Big) + \frac{1}{2}\partial_b F^{b}{}_{\nu}$$

$$-\frac{1}{2}\partial_\nu F^{b}{}_{\mu}\,e_{b}{}^{\mu} + \frac{1}{2}\omega_{b}{}^{b}{}_{c}\,F^{c}{}_{\nu} - \frac{1}{2}\omega_{\nu}{}^{b}{}_{c}\,F^{c}{}_{b}\Big]$$

$$+ e^{(3\beta-4\alpha)\phi}\Big[\frac{1}{4}F^{b}{}_{c}F^{c}{}_{\nu}A_b - \frac{1}{4}F^{b}{}_{c}F^{c}{}_{b}A_\nu\Big]$$

$$-\beta e^{(\beta-2\alpha)\phi}A_b\Big[\alpha\big(-2\partial_\nu\phi\,\partial^b\phi + \partial_c\phi\,\partial^c\phi\,e_{\nu}{}^{b}\big) + \beta\,\partial_\nu\phi\,\partial^b\phi$$

$$+ \partial_\nu\partial^b\phi + \omega_{\nu}{}^{b}{}_{c}\,\partial^c\phi\Big] - \frac{1}{4}e^{(3\beta-4\alpha)\phi}A_b\,F^{b}{}_{c}F^{c}{}_{\nu}$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2 \partial_b\phi\,\partial^b\phi\, A_\nu + \alpha\beta(D-2)\partial_b\phi\,\partial^b\phi\, A_\nu$$

$$+ \alpha\,\frac{D-4}{2}\partial_b\phi\, F^b{}_\nu + \beta\Big(\partial^2\phi\, A_\nu + \frac{3}{2}\partial_b\phi\, F^b{}_\nu + \omega_b{}^b{}_c\,\partial^c\phi\, A_\nu\Big)$$

$$+ \frac{1}{2}\partial_b F^b{}_\nu - \frac{1}{2}\partial_\nu F^b{}_\mu\, e_b{}^\mu + \frac{1}{2}\omega_b{}^b{}_c F^c{}_\nu - \frac{1}{2}\omega_\nu{}^b{}_c F^c{}_b\Big]$$

$$- \frac{1}{4} e^{(3\beta-4\alpha)\phi} F^b{}_c F^c{}_b A_\nu$$

- $$\hat{R}_{z\underline{z}} = \hat{R}_{Mz}{}^B{}_{\underline{z}}\,\hat{e}_B{}^M$$

$$= \hat{R}_{\mu z}{}^b{}_{\underline{z}}\,\hat{e}_b{}^\mu + \underbrace{\hat{R}_{\mu z}{}^{\underline{z}}{}_{\underline{z}}}_{0}\,\hat{e}_{\underline{z}}{}^\mu + \underbrace{\hat{R}_{zz}{}^b{}_{\underline{z}}\,\hat{e}_b{}^z}_{0} + \underbrace{\hat{R}_{zz}{}^{\underline{z}}{}_{\underline{z}}}_{0}\, e_{\underline{z}}{}^z$$

$$= e^{-\alpha\phi} e_b{}^\mu\,\hat{R}_{\mu z}{}^b{}_{\underline{z}}$$

$$= -\beta\, e^{(\beta-2\alpha)\phi}\Big[(\beta-2\alpha)\partial_b\phi\,\partial^b\phi + \partial^2\phi + \alpha D\,\partial_c\phi\,\partial^c\phi + \omega_b{}^b{}_c\,\partial^c\phi\Big]$$

$$- \frac{1}{4} e^{(3\beta-4\alpha)\phi} F^b{}_c F^c{}_b$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\partial_b\phi\,\partial^b\phi + \alpha\beta(D-2)\partial_b\phi\,\partial^b\phi + \beta(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi)\Big]$$

$$- \frac{1}{4} e^{(3\beta-4\alpha)\phi} F^b{}_c F^c{}_b$$

Now we compute the vielbein components of the Ricci tensor

$$\hat{R}_{AC} = \hat{e}_A{}^N\,\hat{R}_{NC}$$

- $$\hat{R}_{ac} = \hat{e}_a{}^N\,\hat{R}_{Nc} = \hat{e}_a{}^\nu\,\hat{R}_{\nu c} + \hat{e}_a{}^z\,\hat{R}_{zc}$$

$$= e^{-\alpha\phi} e_a{}^\nu\,\hat{R}_{\nu c} - e^{-\alpha\phi} A_a\,\hat{R}_{zc}$$

$$= e^{-2\alpha\phi}\Big[R_{ac} + \alpha\Big(\partial_a \partial_c \phi - \partial^2 \phi\, \eta_{ac} - (D-1)\, \partial_a \partial_c \phi$$

$$+ \partial_c \phi\, \partial_b e_\nu{}^b e_a{}^\nu - \partial^b \phi\, \partial_b e_{\nu c} e_a{}^\nu - \partial_c \phi\, \partial_a e_\mu{}^b e_b{}^\mu + \partial^b \phi\, \partial_a e_{\mu c} e_b{}^\mu$$

$$+ \omega_b{}^b{}_a \partial_c \phi - \omega_b{}^{bd} \partial_d \phi\, \eta_{ca} + (3-D)\, \omega_{ac}{}^d \partial_d \phi + \omega^d{}_{ac} \partial_d \phi \Big)$$

$$+ \alpha^2 (D-2) \Big(\partial_a \phi\, \partial_c \phi - \partial^d \phi\, \partial_d \phi\, \eta_{ac} \Big) - \beta^2 \partial_a \phi\, \partial_c \phi$$

$$+ \alpha\beta \Big(2\, \partial_a \phi\, \partial_c \phi - \partial_d \phi\, \partial^d \phi\, \eta_{ac} \Big) - \beta \Big(\partial_a \partial_c \phi + \omega_{ac}{}^d \partial_d \phi \Big) \Big]$$

$$- e^{(2\beta - 4\alpha)\phi} \Big[\alpha\, \frac{D-4}{2}\, \partial_b \phi\, F^b{}_c A_a + \beta\, \frac{3}{2}\, \partial_b \phi\, F^b{}_c A_a + \frac{1}{2} \partial_b F^b{}_c A_a$$

$$+ \frac{1}{2} F_{ba} F^b{}_c + \frac{1}{2} \omega_b{}^b{}_d F^d{}_c A_a - \frac{1}{2} \omega_b{}^d{}_c F^b{}_d A_a \Big]$$

$$+ e^{(2\beta - 4\alpha)\phi} \Big[\alpha \Big(-\partial_b \phi\, F^b{}_c A_a + \frac{D}{2}\, \partial^d \phi\, F_{dc} A_a - \partial^b \phi\, F_{bc} A_a \Big)$$

$$+ \frac{1}{2} \partial_b F^b{}_c A_a + \beta\, \frac{3}{2}\, \partial_b \phi\, F^b{}_c A_a$$

$$+ \frac{1}{2} \omega_b{}^b{}_d F^d{}_c A_a - \frac{1}{2} \omega_{bcd} F^{db} A_a \Big]$$

$$= e^{-2\alpha\phi}\Big[R_{ac} + \alpha\Big(-(D-2)\, \partial_a \partial_c \phi - \partial^2 \phi\, \eta_{ac}$$

$$+ \partial_c \phi\, \partial_b e_\nu{}^b e_a{}^\nu - \partial^b \phi\, \partial_b e_{\nu c} e_a{}^\nu - \partial_c \phi\, \partial_a e_\mu{}^b e_b{}^\mu + \partial^b \phi\, \partial_a e_{\mu c} e_b{}^\mu$$

$$+ \omega_b{}^b{}_a \partial_c \phi - \omega_b{}^{bd} \partial_d \phi\, \eta_{ca} + (3-D)\, \omega_{ac}{}^d \partial_d \phi + \omega^d{}_{ac} \partial_d \phi \Big)$$

$$+ \alpha^2 (D-2) \Big(\partial_a \phi\, \partial_c \phi - \partial^d \phi\, \partial_d \phi\, \eta_{ac} \Big) - \beta^2 \partial_a \phi\, \partial_c \phi$$

$$+ \alpha\beta \Big(2\, \partial_a \phi\, \partial_c \phi - \partial_d \phi\, \partial^d \phi\, \eta_{ac} \Big) - \beta \Big(\partial_a \partial_c \phi + \omega_{ac}{}^d \partial_d \phi \Big) \Big]$$

$$- e^{(2\beta - 4\alpha)\phi} \Big[\partial_b \phi\, F^b{}_c A_a\, \frac{(D-4)\alpha + 3\beta}{2} + \frac{1}{2} \partial_b F^b{}_c A_a + \frac{1}{2} F_{ba} F^b{}_c$$

$$+ \frac{1}{2} \omega_b{}^b{}_d F^d{}_c A_a - \frac{1}{2} \omega_b{}^d{}_c F^b{}_d A_a$$

$$- \alpha\, \frac{D-4}{2}\, \partial_b \phi\, F^b{}_c A_a - \frac{1}{2} \partial_b F^b{}_c A_a$$

$$-\beta\,\frac{3}{2}\,\partial_b\phi\, F^b{}_c A_a$$

$$-\frac{1}{2}\,\omega_b{}^b{}_d\, F^d{}_c A_a + \frac{1}{2}\,\omega_b{}^d{}_c\, F^b{}_d A_a \Big]$$

$$= e^{-2\alpha\phi}\Big[R_{ac} + \alpha\Big(-(D-2)\,\partial_a\partial_c\phi \;\underbrace{- \partial^2\phi\,\eta_{ac}}_{} \to \Box\phi$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu$$

$$+\omega_b{}^b{}_a\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\,\eta_{ac} \underbrace{-(D-3)\,\omega_{ac}{}^d\,\partial_d\phi}_{-(D-2)+1} + \omega^d{}_{ac}\,\partial_d\phi\Big)$$

$$+\alpha^2(D-2)\left(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\right) - \beta^2\,\partial_a\phi\,\partial_c\phi$$

$$+\alpha\beta\left(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\right) - \beta\Big(\underbrace{\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi}_{\nabla_a\nabla_c\phi}\Big)\Big]$$

$$-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b F_{cb} = (*)$$

NOTE 4: We will see later that one must set $\beta = -(D-2)\alpha$

$$(*) = -\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b F_{cb} + e^{-2\alpha\phi}\Big[R_{ac} + \overbrace{(D-2)\alpha}^{-\beta}\,\nabla_a\nabla_c\phi$$

$$+\partial_a\phi\,\partial_c\phi\,\underbrace{\left(\alpha^2(D-2) - \beta^2 + 2\alpha\beta\right)}_{\alpha\beta-\beta^2 = -(D-2)(D-1)\alpha^2} - \partial^b\phi\,\partial_b\phi\,\eta_{ac}\,\underbrace{\left(\alpha^2(D-2) + \alpha\beta\right)}_{0}$$

$$+\alpha\Big(-\Box\phi\,\eta_{ac} - (D-2)\,\nabla_a\nabla_c\phi + \omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi$$

$$+\omega_b{}^b{}_a\,\partial_c\phi + \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial_d\phi\,\partial^d e_{\nu c}\, e_a{}^\nu$$

$$-\partial_c\phi\,\partial_a e_\nu{}^b e_b{}^\nu + \partial_d\phi\,\partial_a e_{\nu c}\, e^{d\nu} \Big)\Big]$$

NOTE 5: $\alpha^2 = \frac{1}{2(D-2)(D-1)}$ [We will see later]

$$= -\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b F_{cb} + e^{-2\alpha\phi}\Big[R_{ac} - \frac{1}{2}\,\partial_a\phi\,\partial_c\phi - \alpha\,\Box\phi\,\eta_{ac}$$

$$+\partial_d\phi\,\underbrace{\left(\omega_{ac}{}^d + \omega^d{}_{ac} - \partial^d e_{\nu c}\, e_a{}^\nu + \partial_a e_{\nu c}\, e^{d\nu}\right)}_{0} + \partial_c\phi\,\underbrace{\left(\omega_b{}^b{}_a + \partial_b e_\nu{}^b e_a{}^\nu - \partial_a e_\nu{}^b e_b{}^\nu\right)}_{0} = (*)$$

Remark 1

$$\omega_b{}^{ba} = e_b{}^\mu \omega_\mu{}^{ba} = -e_b{}^\mu \omega_\mu{}^{ab}(e)$$

$$= -e_b{}^\mu \tfrac{1}{2}\Big[e^{\nu a}\partial_\mu e_\nu{}^b - e^{\nu b}\partial_\mu e_\nu{}^a - e^{\nu a}\partial_\nu e_\mu{}^b + e^{\nu b}\partial_\nu e_\mu{}^a - e^{\nu a}e^{b\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c + e^{\nu b}e^{a\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c\Big]$$

$$= -\tfrac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial_b e_\nu{}^a e^{\nu b} - \partial^a e_\mu{}^b e_b{}^\mu + \partial^b e_\mu{}^a e_b{}^\mu - e^{\nu a}e^{b\sigma}\partial_\nu e_{\sigma b} + e^{\nu b}e^{a\sigma}\partial_\nu e_{\sigma b}\Big]$$

$$= -\tfrac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial^\nu e_\nu{}^a - \partial^a e_\mu{}^b e_b{}^\mu + \partial^\mu e_\mu{}^a - \partial^a e_{\nu b} e^{b\nu} + \partial^b e_{\nu b} e^{a\nu}\Big]$$

$$= -\tfrac{1}{2}\Big[2\partial_b e_\nu{}^b e^{\nu a} - \partial^a e_{\nu b} e^{b\nu}\Big] = \partial^a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e^{\nu a}$$

$$\Rightarrow \omega_b{}^b{}_a = \partial_a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e_a{}^\nu$$

Remark 2 [We can compute faster by using anholonomy coefficients]

$$\omega_{acd} + \omega_{dac} = \tfrac{1}{2}\Big[\Omega_{[ac]d} - \Omega_{[cd]a} + \Omega_{[da]c} + \Omega_{[da]c} - \Omega_{[ac]d} + \Omega_{[cd]a}\Big]$$

$$= \Omega_{[da]c}$$

see note below

$$\Rightarrow \omega_{ac}{}^d + \omega^d{}_{ac} = \Omega_{[ba]c}\,\eta^{bd} = -(\partial_b e_a{}^\rho - \partial_a e_b{}^\rho)e_{\rho c}\,\eta^{bd}$$

$$= -\partial^d e_a{}^\nu e_{\nu c} + \partial_a e^{d\nu} e_{\nu c}$$

$$= \partial^d e_{\nu c} e_a{}^\nu - \partial_a e_{\nu c} e^{d\nu}$$

NOTE: $\Omega_{[\mu\nu]\rho} = (\partial_\mu e_\nu{}^d - \partial_\nu e_\mu{}^d)e_{d\rho}$

$$\Omega_{[ab]c} = e_a{}^\mu e_b{}^\nu e_c{}^\rho\,\Omega_{[\mu\nu]\rho} = (\partial_a e_\nu{}^d e_b{}^\nu - \partial_b e_\mu{}^d e_a{}^\mu)\eta_{cd}$$

Important $\leftarrow$ $= -(\partial_a e_b{}^\rho - \partial_b e_a{}^\rho)e_{\rho c}$

$$(*) = e^{-2\alpha\phi}\left[R_{ac} - \frac{1}{2}\partial_a\phi\partial_c\phi - \alpha\Box\phi\,\eta_{ac}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^b F_{cb}$$

- $$\hat{R}_{\underline{z}\underline{z}} = \hat{e}_{\underline{z}}{}^N \hat{R}_{N\underline{z}} = \overbrace{\hat{e}_{\underline{z}}{}^0}^{0} \hat{R}_{0\underline{z}} + \hat{e}_{\underline{z}}{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\beta\phi}\hat{R}_{z\underline{z}}$$

$$= -e^{-2\alpha\phi}\left[\partial_b\phi\partial^b\phi\left(\beta^2 + (D-2)\alpha\beta\right) + \beta\overbrace{\left(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\right)}^{\eta^{ab}\nabla_a\nabla_b\phi = \Box\phi}\right]$$

$$+ \frac{1}{4}e^{(2\beta-4\alpha)\phi}F_c{}^b F^c{}_b$$

$$= e^{-2\alpha\phi}\left[-\underbrace{(\beta^2 + (D-2)\alpha\beta)}_{0 \text{ (see note 4)}}\partial_b\phi\partial^b\phi - \beta\Box\phi\right] + \frac{1}{4}e^{(2\beta-4\alpha)\phi}F^2$$

▲ Using these two vielbein components of the Ricci tensor it is finally straightforward to compute the Ricci scalar

$$\hat{R} = \hat{R}_{AB}\,\eta^{AB}$$

We find:

$$\hat{R} = \hat{R}_{ac}\eta^{ac} + \hat{R}_{\underline{z}\underline{z}} = e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \overbrace{(D\alpha+\beta)}^{D\alpha-(D-2)\alpha = 2\alpha}\Box\phi\right]$$

$$-\frac{1}{2}e^{-2D\alpha\phi}\left(F^2 - \frac{1}{2}F^2\right) =$$

$$= e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \underbrace{2\alpha\Box\phi}_{\text{boundary term}} - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]$$

▶ The full (D+1)-dimensional action then reduces to

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2} \int d^{D+1}x \; \hat{e} \; \hat{R}$$

$$= \frac{1}{2\kappa_{D+1}^2} \int_0^{2\pi L} dz \int d^D x \; e^{(\alpha D+\beta)\phi} e \; \hat{R}$$

$$= \frac{1}{2 \underbrace{\frac{\kappa_{D+1}^2}{2\pi L}}_{\kappa_D^2 = \frac{\kappa_{D+1}^2}{2\pi L}}} \int d^D x \; \underbrace{e^{[(D-2)\alpha+\beta]\phi}}_{\substack{\text{Canonical E-H if} \\ \beta = -(D-2)\alpha \\ \text{(see note 4)}}} e \left[R - \underbrace{\tfrac{1}{2}(\partial\phi)^2}_{\substack{\text{Proper normalisation if} \\ \alpha^2 = \frac{1}{2(D-2)(D-1)}}} - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right]$$

(see note 5)

$$= \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right] = S_D$$

Therefore we recover an Einstein - Maxwell - Dilaton theory !!

$$\boxed{S_{D+1} = \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right]}$$

with $\kappa_D^2 = \dfrac{\kappa_{D+1}^2}{2\pi L}$

<u>Example</u> : If $D=4 \Rightarrow e^{-2(D-1)\alpha\phi} = e^{-\sqrt{3}\phi}$

Exercise: Compute the $\hat{R}_{b\underline{z}}$ component of the Ricci tensor

$$\hat{R}_{b\underline{z}} = \hat{e}_b{}^N \hat{R}_{N\underline{z}} = \hat{e}_b{}^\nu \hat{R}_{\nu\underline{z}} + \hat{e}_b{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\alpha\phi} e_b{}^\nu \hat{R}_{\nu\underline{z}} - e^{-\alpha\phi} A_b \hat{R}_{z\underline{z}}$$

$$= -e^{(\beta-3\alpha)\phi}\Big[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \alpha\frac{D-4}{2}\partial_c\phi F^c{}_b$$

$$+\beta\left(\partial^2\phi A_b + \frac{3}{2}\partial_c\phi F^c{}_b + \omega_c{}^c{}_d\partial^d\phi A_b\right) + \frac{1}{2}\partial_c F^c{}_\nu e_b{}^\nu$$

$$-\frac{1}{2}\partial_b F^c{}_\nu e_c{}^\nu + \frac{1}{2}\omega_c{}^c{}_d F^d{}_b - \frac{1}{2}\omega_b{}^c{}_d F^d{}_c\Big]$$

$$+\frac{1}{4}e^{(3\beta-5\alpha)\phi}F^2 A_b$$

$$+e^{(\beta-3\alpha)\phi}\left[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \beta\Box\phi A_b\right]$$

$$-\frac{1}{4}e^{(3\beta-5\alpha)\phi}F^2 A_b$$

$$= -\frac{1}{2}\underbrace{e^{(\beta-3\alpha)\phi}}_{e^{-(D+1)\alpha}}\Big[-\big(\overbrace{(D-4)\alpha+3\beta}^{-2(D-1)\alpha}\big)\partial_c\phi F_b{}^c$$

$$+\partial_c F^c{}_\nu e_b{}^\nu - \partial_b F^c{}_\nu e_c{}^\nu$$

$$+\omega_c{}^c{}_d F^d{}_b + \omega_b{}^c{}_d F_c{}^d\Big]$$

NOTE: $-(D+1)\alpha = (D-3)\alpha - 2(D-1)$

$$= \frac{1}{2}e^{(D-3)\alpha\phi}\times e^{-2(D-1)\alpha\phi}\Big[-2(D-1)\alpha\,\partial_c\phi F_b{}^c$$

$$\underbrace{-\partial_c F^c{}_\nu e_b{}^\nu + \partial_b F^c{}_\nu e_c{}^\nu}_{\partial_c F_b{}^c - F_\nu{}^c\partial_c e_b{}^\nu + F_\nu{}^c\partial_b e_c{}^\nu} + \omega_c{}^c{}_d F_b{}^d + \omega_b{}^c{}_d F^d{}_c\Big]$$

NOTE: $\nabla_c F_b{}^c = \partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi} \Big[-2(D-1)\alpha\, \overbrace{\partial_c \phi}^{\nabla_c \phi}\, F_b{}^c$$

$$+ \underbrace{\partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c}_{\nabla_c F_b{}^c} + \omega_{cdb} F^{dc} - F_0{}^c \partial_c e_b{}^0 + F_0{}^c \partial_b e_c{}^0$$

$$+ \omega_{bcd} F^{dc} \Big] = (*)$$

Remark 3

$$\omega_{cdb} + \omega_{bcd} = \frac{1}{2} \Big[\underline{\Omega_{[cd]b}} - \underline{\Omega_{[db]c}} + \Omega_{[cb]d}$$

$$+ \Omega_{[cb]d} - \underline{\Omega_{[cd]b}} + \underline{\Omega_{[db]c}} \Big]$$

$$= \Omega_{[cb]d} = -(\partial_b e_c{}^0 - \partial_c e_b{}^0)\, e_{0d}$$

$$\Rightarrow (\omega_{cdb} + \omega_{bcd}) F^{dc} = -\partial_b e_c{}^0 e_{0d} F^{dc} + \partial_c e_b{}^0 e_{0d} F^{dc}$$

$$= F_0{}^c \partial_c e_b{}^0 - F_0{}^c \partial_b e_c{}^0$$

$$(*) = \frac{1}{2} e^{(D-3)\alpha\phi} \nabla_c \Big[e^{-2(D-1)\alpha\phi} F_b{}^c \Big] = \hat{R}_{\underline{z}b}$$

IX. (D+1)-dimensional vs D-dimensional EOMs and symmetries

In this section we discuss the equations of motion (EOMs) that result from S_{D+1} and S_D.

i) (D+1)-dimensional EOMs

$$S_{D+1} = \frac{1}{2K^2_{D+1}} \int d\hat{x}^{D+1}\, \hat{e}\, \hat{R} \implies \hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2} \hat{g}_{MN} \hat{R} = 0$$

NOTE: $\hat{g}^{MN} \hat{G}_{MN} = \hat{R} - \frac{1}{2}(D+1)\hat{R} = \left(1 - \frac{1}{2}(D+1)\right)\hat{R} = 0$

$$\implies \underbrace{\hat{R} = 0}_{D \neq 1} \implies \hat{R}_{MN} = 0$$

$$\implies \hat{R}_{AB} = 0$$

$$\hat{R}_{AB} = 0 \quad \begin{cases} \hat{R}_{ab} = e^{-2\alpha\phi}\left[R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\,\Box\phi\,\eta_{ab}\right] - \frac{1}{2} e^{-2D\alpha\phi} F_a{}^c F_{bc} = 0 \\[2ex] \hat{R}_{a\underline{z}} = \frac{1}{2} e^{(D-3)\alpha\phi} \nabla_c \left[e^{-2(D-1)\alpha\phi} F_b{}^c \right] = 0 \\[2ex] \hat{R}_{\underline{z}\underline{z}} = (D-2)\alpha\, e^{-2\alpha\phi}\, \Box\phi + \frac{1}{4} e^{-2D\alpha\phi} F^2 = 0 \end{cases}$$

- It is important to notice that

$$\phi = 0 \implies \hat{R}_{\underline{z}\underline{z}} = \frac{1}{4}F^2 = 0 \implies \underline{F=0}$$

↳ Trivial Maxwell !!

ii) D-dimensional EOMs

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx \; e \left[R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4} e^{-2(D-1)\alpha\phi} F_{\mu\nu}F^{\mu\nu} \right]$$

The EOMs that follow from the above action are:

- $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) + \frac{1}{2} e^{-2(D-1)\alpha\phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^2 g_{\mu\nu}\right)$$

- $$\nabla^\mu\left(e^{-2(D-1)\alpha\phi} F_{\mu\nu}\right) = 0$$

- $$\Box\phi = -\frac{1}{2}(D-1)\alpha\, e^{-2(D-1)\alpha\phi} F^2$$

▲ It is important to notice again that

$$\phi = 0 \implies F^2 = 0 \implies \underline{F=0}$$

↳ Trivial Maxwell !!

IMPORTANT: Having set $\phi = 0$ in the Ansatz for the (D+1)-dimensional metric would have been inconsistent !! [common mistake]

[Einstein - Maxwell - DILATON]

iii) (D+1)-dimensional symmetries

The symmetry group is (D+1)-dimensional general coordinate transformations. At the infinitesimal level we have

$$\delta_{\hat\xi}\, \hat g_{MN} = \hat\xi^P \partial_P \hat g_{MN} + \hat g_{PN} \partial_M \hat\xi^P + \hat g_{MP} \partial_N \hat\xi^P$$

with

$$\hat\xi^M(x,z) = \left(\hat\xi^\mu(x,z)\ ,\ \hat\xi^z(x,z) \right)$$

- However, in order to preserve the KK Ansatz of the (D+1)-dimensional metric, there are the restrictions:

Diffeom : $\hat\xi^\mu = \xi^\mu(x)$, $\hat\xi^z = \lambda(x) + \underbrace{c\, z}_{\text{linear dependence on } S^1}$

- On the other hand, the (D+1)-dimensional EOMs have a scaling symmetry [not the E-H action!!] of the form:

$$\left.\begin{array}{l} \hat g_{MN} \to a^2\, \hat g_{MN} \\ \hat e \to a^{(D+1)}\, \hat e \\ \hat R \to a^{-2}\, \hat R \end{array}\right\} \quad \hat e\, \hat R \to \underbrace{a^{(D-1)}}_{a\in\mathbb{R}}\, \hat e\, \hat R \implies \delta_a\, \hat g_{MN} = 2\, a\, \hat g_{MN}$$

infinitesim.

iv) D-dimensional symmetries

Starting from (D+1)-dimensional diffeomorphisms we will obtain D-dimensional diff + U(1) gauge symmetry + Global symmetries.

Ex : Using
$$\left\{\begin{aligned} \hat{g}_{\mu\nu} &= e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu \\ \hat{g}_{\mu z} = \hat{g}_{z\mu} &= e^{2\beta\phi} A_\mu \\ \hat{g}_{zz} &= e^{2\beta\phi} \end{aligned}\right\} \quad \text{with } \beta = -(D-2)\alpha$$

show that $\delta \hat{g}_{MN} = (\delta_\xi + \delta_a)\, \hat{g}_{MN}$ gives rise to :

$$\delta\phi = \xi^\rho \partial_\rho \phi - \frac{1}{(D-2)\alpha}(c+a)$$

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda - c A_\mu$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \frac{2}{(D-2)} \left[c + a(D-1) \right] g_{\mu\nu}$$

- Setting $a = -\dfrac{c}{(D-1)}$ one finds :

$$\delta\phi = \underbrace{\delta_\xi \phi} - \overbrace{\frac{c}{(D-1)\alpha}}^{\text{shift} \leftrightarrow \text{Non-linear action}}$$

$$\delta A_\mu = \underbrace{\delta_\xi A_\mu} + \underbrace{\partial_\mu \lambda} \overbrace{- c A_\mu}^{\text{scaling} \leftrightarrow \text{linear action}}$$

$$\delta g_{\mu\nu} = \underbrace{\delta_\xi g_{\mu\nu}}$$

→ Global symmetry $\equiv \mathbb{R}$ (real parameter)

→ $U(1)$ gauge symmetry

→ D-dimensional diffeomorphisms

- Setting $a = -c$ one finds : $\underbrace{n\text{-legs} \Rightarrow nc}$

$$\delta\phi = \delta_\xi \phi \qquad (0\text{-legs})$$

$$\delta A_\mu = \delta_\xi A_\mu + \partial_\mu \lambda - \underbrace{c A_\mu} \qquad (1\text{-leg})$$

$$\delta g_{\mu\nu} = \delta_\xi g_{\mu\nu} - \underbrace{2c\, g_{\mu\nu}} \qquad (2\text{-legs})$$

$\longrightarrow$ Real scaling $\mathbb{R}$ symmetry of the D-dimensional EOMs known as "trombone" scaling symmetry.

Important : There are two inequivalent $\mathbb{R}$ global symmetries. One is an actual symmetry of the D-dimens action whereas the other is only of the EOMs.

Important : In modern language, the global symmetries of the action upon dimensional reduction are referred to as "dualities". In this case the duality group is just $G_{global} = \mathbb{R}$ symmetry and affects scalar and vector fields in the reduced theory.

X. Kaluza-Klein reduction of Maxwell and scalar on S^1

In this section we look at other reductions on S^1. The starting point is a $(D+1)$-dimensional Maxwell field $\hat{B}_M$ with field strength $\hat{F}_{MN} = \partial_M \hat{B}_N - \partial_N \hat{B}_M$.

- The K-K Ansatz for $\hat{B}_M$ reads:

$$\boxed{\hat{B}_M = \left(\hat{B}_\mu , \hat{B}_z \right) = \left(B_\mu(x) , \chi(x) \right)}$$

which yields a field strength of the form:

$$\hat{F}_{MN} = \begin{bmatrix} \hat{F}_{\mu\nu} & \hat{F}_{\mu z} \\ \hat{F}_{z\nu} & 0 \end{bmatrix} = \begin{bmatrix} F_{\mu\nu} & \partial_\mu \chi \\ -\partial_\nu \chi & 0 \end{bmatrix}$$

with:

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$F_{\mu z} = \partial_\mu \chi$$

$$F_{z\nu} = -\partial_\nu \chi$$

- The Maxwell's action in $(D+1)$-dimensions then reduces to:

$$S_{\hat{B}} = -\frac{1}{4}\int d^{D+1}x \underbrace{\sqrt{|\hat{g}|}}_{\hat{e}} \hat{F}_{MN} \hat{F}^{MN} = -\frac{1}{4}\int d^{D+1}x \; \hat{e} \, \hat{F}_{AB} \hat{F}^{AB} = (*)$$

NOTE 1 : $\hat{F}_{AB} = \hat{e}_A{}^M \hat{e}_B{}^N \hat{F}_{MN}$

- $$\hat{F}_{ab} = \hat{e}_a{}^M \hat{e}_b{}^N \hat{F}_{MN}$$
$$= \hat{e}_a{}^\mu \hat{e}_b{}^\nu \hat{F}_{\mu\nu} + \hat{e}_a{}^z \hat{e}_b{}^\nu \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_b{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_b{}^z \underbrace{\hat{F}_{zz}}_{0}$$
$$= e^{-2\alpha\phi} F_{ab} + e^{-2\alpha\phi} A_a \partial_b \chi - e^{-2\alpha\phi} A_b \partial_a \chi$$
$$= e^{-2\alpha\phi} \Big[\underbrace{F_{ab} - (\partial_a \chi A_b - \partial_b \chi A_a)}_{\mathcal{F}_{ab} \equiv F_{ab} - 2\partial_{[a}\chi A_{b]}} \Big] = e^{-2\alpha\phi} \mathcal{F}_{ab}$$

- $$\hat{F}_{a\underline{z}} = \hat{e}_a{}^M \hat{e}_{\underline{z}}{}^N \hat{F}_{MN}$$
$$= \hat{e}_a{}^\mu \underbrace{\hat{e}_{\underline{z}}{}^\nu}_{0} \hat{F}_{\mu\nu} + \hat{e}_a{}^z \underbrace{\hat{e}_{\underline{z}}{}^\nu}_{0} \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_{\underline{z}}{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_{\underline{z}}{}^z \underbrace{\hat{F}_{zz}}_{0}$$
$$= e^{-(\alpha+\beta)\phi} \partial_a \chi = -\hat{F}_{\underline{z}a}$$

- $$\hat{F}_{\underline{z}\underline{z}} = 0$$

NOTE 2 : $\hat{e} = e^{(\alpha D + \beta)\phi}\, e$

$$(*) = -\frac{1}{4} e^{(\alpha D+\beta)\phi} (2\pi L) \int d^D x\, e \Big[\hat{F}_{ab} \hat{F}^{ab} + \underbrace{\hat{F}_{a\underline{z}} \hat{F}^{a\underline{z}} + \hat{F}_{\underline{z}b} \hat{F}^{\underline{z}b}}_{2 \hat{F}_{a\underline{z}} \hat{F}^{a\underline{z}}} \Big]$$
$$= -\frac{1}{4} (2\pi L) e^{(\alpha D+\beta)\phi} \int d^D x\, e \Big[e^{-4\alpha\phi} \mathcal{F}_{ab} \mathcal{F}^{ab} + 2 e^{-2(\alpha+\beta)\phi} \partial_a \chi \partial^a \chi \Big]$$

$$\boxed{S_{\hat{B}} = (2\pi L) \int d^D x\, e \Big[-\frac{1}{4} e^{-2\alpha\phi} \mathcal{F}^2 - \frac{1}{2} e^{2(D-2)\alpha\phi} (\partial\chi)^2 \Big]}$$

- The reduction of the scalar field is straightforward. The KK Ansatz reads:

$$\boxed{\hat{\Phi} = \Phi(x)}$$

So that the action reduces as:

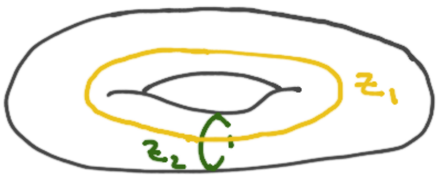
$$\boxed{S_{\hat{\Phi}} = -\frac{1}{2}\int d^{D+1}x \; \hat{e} \; \partial_M \hat{\Phi} \, \partial^M \hat{\Phi} = (2\pi L)\int d^D x \, e\left[-\frac{1}{2}\partial_\mu \Phi \, \partial^\mu \Phi\right]}$$

NOTE 1: $\hat{e} = e^{(D\alpha+\beta)\phi} \, e \overset{\beta=-(D-2)\alpha}{=} e^{2\alpha\phi} e$

NOTE 2: $\partial_A \hat{\Phi} = (\hat{e}_a{}^\mu \partial_\mu \Phi, 0) = e^{-\alpha\phi}(\partial_a \Phi, 0)$

x1. Kaluza-Klein reduction on T^2 and $GL(2)$ duality

In this section we are combining the previous results above and will present the KK reduction of gravity in $(D+2)$ dimensions:



$T^2 \equiv$ 2-torus

coordinates (z_1, z_2)

$$\hat{\hat{g}}_{\mathcal{M}\mathcal{N}} \Rightarrow \hat{g}_{MN} + \hat{A}_{M1} + \hat{\phi}_1 \Rightarrow g_{\mu\nu} + A_{\mu 2} + \phi_2 + A_{\mu 1} + \chi + \phi_1$$

step 1 $\qquad$ step 2

$\mathcal{M} = M, z_1$ $\qquad$ $M = \mu, z_2$

- Reduction along z_1 :

$$S_{D+2} = \frac{1}{2\kappa_{D+2}^2} \int d^D x \, dz_2 \, dz_1 \; \hat{\hat{e}} \; \hat{\hat{R}}$$

$$= \frac{1}{2\kappa_{D+1}^2} \int d^D x \, dz_2 \; \hat{e} \left[\hat{R} - \frac{1}{2} (\partial \hat{\phi}_1)^2 - \frac{1}{4} e^{-2D\alpha_1 \hat{\phi}_1} \hat{F}_1^{\,2} \right] \equiv S_{D+1}$$

with $\kappa_{D+1}^2 = \dfrac{\kappa_{D+2}^2}{2\pi L_1}$ and $\alpha_1^2 = \dfrac{1}{2(D-1)D}$

- Reduction along z_2 :

$$S_D = \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^{\,2} \right.$$

$$- \frac{1}{2} (\partial \phi_1)^2$$

$$\left. + e^{-2D\alpha_1 \phi_1} \left(-\frac{1}{4} e^{-2\alpha_2 \phi_2} \mathcal{F}_1^{\,2} - \frac{1}{2} e^{2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right) \right]$$

$$= \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{2} e^{-2D\alpha_1 \phi_1 + 2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right.$$

$$\left. - \frac{1}{4} e^{-2D\alpha_1 \phi_1 - 2\alpha_2 \phi_2} \mathcal{F}_1^{\,2} - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^{\,2} \right]$$

with $\alpha_2^2 = \dfrac{1}{2(D-2)(D-1)}$ and $\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2 \partial_{[\mu} \chi A_{\nu]2}$

The action S_D can be enlighteningly rewritten as

$$S_D = \frac{1}{2K_D^2}\int d^Dx \, e\left[R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}e^{\vec{c}\vec{\phi}}(\partial\chi)^2 - \frac{1}{4}e^{\vec{c}_1\vec{\phi}}\mathcal{F}_1^2 - \frac{1}{4}e^{\vec{c}_2\vec{\phi}}\mathcal{F}_2^2\right]$$

with

$$\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2\,\partial_{[\mu}\chi A_{\nu]2}$$

$$\mathcal{F}_{\mu\nu 2} = F_{\mu\nu 2}$$

and

$$K_D^2 = \frac{K_{D+1}^2}{2\pi L_2} = \frac{K_{D+2}^2}{(2\pi L_1)(2\pi L_2)} = \frac{K_{D+2}^2}{\mathrm{Vol}(T^2)}$$

$$\vec{c} = \left[-\sqrt{\frac{2D}{D-1}}\,,\ \sqrt{\frac{2(D-2)}{D-1}}\right]$$

$$\vec{c}_1 = \left[-\sqrt{\frac{2D}{D-1}}\,,\ -\sqrt{\frac{2}{(D-1)(D-2)}}\right]$$

$$\vec{c}_2 = \left[\ 0\ ,\ -\sqrt{\frac{2(D-1)}{D-2}}\right]$$

- Let us rotate the two dilatons $\vec{\phi} = (\phi_1, \phi_2)$ to new ones:

$$\phi = -\frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_1 + \frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_2$$

$$\varphi = -\frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_1 - \frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_2$$

so the D-dimensional action becomes

$$S_D = \frac{1}{2K_D^2}\int d^Dx\, e\Big[R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2$$
$$- \frac{1}{4}e^{q\varphi+\phi}\mathcal{F}_1^{\,2} - \frac{1}{4}e^{q\varphi-\phi}\mathcal{F}_2^{\,2}\Big]$$

with $q^2 = \frac{D}{D-2}$ and the $(D+2)$-dimensional metric reads

$$ds_{D+2}^2 = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}\,ds_D^2 + e^{\sqrt{\frac{D-2}{D}}\varphi}\,ds_2^2$$

with

$$ds_2^2 = e^{\phi}(dz_1 + A_{\mu 1}dx^\mu + \chi\, dz_2)^2 + e^{-\phi}(dz_2 + A_{\mu 2}dx^\mu)^2$$
$$\Rightarrow\; ds_2^2\Big|_{\phi=\chi=A_{\mu 1,2}=0} = dz_1^2 + dz_2^2$$

Moduli space: (scalars ≡ "moduli")

- The scalar φ parameterises the volume of volume of T^2 as it appears as a factor in front of ds_2^2.

- The scalar ϕ and χ play different roles. The scalar ϕ parameterises a shape-changing of the torus. It scales the z_1-cycle and the z_2-cycle in opposite manners. The scalar χ varies the angle between the z_1-cycle and the z_2-cycle.

Scalar sector and SL(2) duality

Let's focus on the scalar sector of the above 8D action:

$$\mathcal{L}_{scalar} = -\frac{1}{2}(\partial\varphi)^2 \underbrace{- \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi^2)}_{\mathcal{L}(\phi,\chi)}$$

Global symmetries (or dualities)

i) The scalar φ decouples from the others. It has a global $\mathbb{R}$ shift symmetry

$$\varphi \longrightarrow \varphi + K \qquad \text{with} \quad K \in \mathbb{R}$$

↳ Non-linear action

ii) The symmetry analysis for $\mathcal{L}(\phi,\chi)$ is more interesting. To make the symmetry manifest we define a complex modulus field on T^2 as

$$\tau = \chi + i\, e^{-\phi}$$

in terms of which

$$\mathcal{L}(\phi,\chi) = -\frac{1}{2}\left[(\partial\phi)^2 + e^{2\phi}(\partial\chi)^2\right] = -\frac{\partial_\mu\tau\,\partial^\mu\bar{\tau}}{2\,\mathrm{Im}^2(\tau)}$$

Ex: Show that $\mathcal{L}(\phi,\chi)$ is invariant under the global fractional linear transformation:

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

with $ad - bc = 1$. Show that this transformation acts on (ϕ,χ) as:

$$\left.\begin{aligned} e^{\phi} &\longrightarrow e^{\phi'} = (c\chi+d)^2 e^{\phi} + c^2 e^{-\phi} \\ \chi e^{\phi} &\longrightarrow \chi' e^{\phi'} = (a\chi+b)(c\chi+d)\, e^{\phi} + ac\, e^{-\phi} \end{aligned}\right\} \text{Non-linear SL(2) action}$$

iii) As scalars couple to vectors, these must also transform. Let us write a constant 2×2 matrix Λ of the form

$$\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with} \quad |\Lambda| = ad - bc = 1$$

so that $\Lambda \in SL(2)$. Using this matrix Λ, the transformation

$$\begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \longrightarrow (\Lambda^t)^{-1} \begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \quad \Rightarrow \text{Linear SL(2) action}$$

precisely cancells against the transformation of scalars and leaves the action invariant.

As a result, the global symmetry (duality) of the KK reduction of gravity on T^2 turns out to be:

$$G_{global} = \mathbb{R} \times SL(2) \equiv GL(2)$$

XII. Kaluza-Klein reduction on T^n and $GL(n)$ duality

The discussion above can be generalised to a reduction on a n-dimensional torus T^n.

$$\overset{\hat{\vdots}}{\hat{g}}\Big\{ n \qquad \hat{g}_{\mu\nu} \underset{\text{step 1}}{\Longrightarrow} \cdots \underset{\text{step } n}{\Longrightarrow} g_{\mu\nu} \oplus \underbrace{A_\mu^{1,2,\dots,n}}_{n \text{ vectors}} \oplus \overbrace{\underbrace{\phi_{1,2,\dots,n}}_{\vec{\phi}} \oplus \underbrace{\chi^n{}_m}_{m>n}}$$

$$\# \text{ scalars} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

$$\frac{G}{H} = \frac{GL(n)}{SO(n)}$$

The $(D+n)$-dim metric

$$ds^2_{D+n} = e^{\frac{1}{3}\vec{g}\vec{\phi}}\, ds^2_D + \sum_{i=1}^{n} e^{2\vec{\gamma}_i \vec{\phi}} \left(dz^i + A^i{}_\mu dx^\mu + \chi^i{}_j\, dz^j \right)^2$$

whose structure is preserved by $(D+n)$-dim diffeomorphisms

$$\hat{\xi}^\mu = \xi^\mu(x) \quad , \quad \hat{\xi}^i(x,z) = \underbrace{\lambda^i(x)}_{U(1)^n} + \underbrace{\Lambda^i{}_j}_{GL(n)}\, z^j$$

* 11D reduction on T^n and $\frac{E_{n(n)}}{K[E_{n(n)}]}$ coset

▶ We have seen that when (D+n)-dim gravity is reduced on T^n then the duality group becomes $G_{global} = \mathbb{R} \times SL(n)$

▶ If we start from the 11D supergravity theory and reduce it on T^n then the duality group gets enhanced to the exceptional $G_{global} = E_{n(n)}$

$$S_{11D}^{SUGRA} = \frac{1}{2\kappa_{11D}^2} \int d^{11}x \, \hat{e} \left[\underset{\downarrow \; GL(n) = \mathbb{R} \times SL(n)}{\hat{R}} - \frac{1}{2 \times 4!} \underset{\downarrow \; \text{enhancement to } E_{n(n)}}{\hat{F}_{(4)}^2} \right] + \dots$$

where $\hat{F}_{(4)}^2 = \hat{F}_{MNPQ} \hat{F}^{MNPQ}$ with $\hat{F}_{MNPQ} = \partial_{[M} \hat{A}_{NPQ]}$

▶ Upon reduction on T^n one finds maximal supergravity in D-dimensions with $\frac{G}{H} = \frac{E_{n(n)}}{K[E_{n(n)}]}$ coset spaces :

maximal compact subgroup K

D	n	$G = E_{n(n)}$	K (max comp)
9	2	$GL(2)$	$SO(2)$
8	3	$SL(2) \times SL(3)$	$SO(2) \times SO(3)$
7	4	$SL(5)$	$SO(5)$
6	5	$SO(5,5)$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$
2	9	$E_{9(9)}$	$K[E_{9(9)}]$