I. Postulates and general coordinate transformations

Principle of general relativity: The laws of Nature are invariant under general coordinate transformations (g.c.t) $\Rightarrow$ General covariance

BUT... what determines the laws of gravity?... the local Lorentz symmetry !!

Denoting coordinates as $x^\mu = (\underbrace{x^0}_{ct}, \underbrace{x^1, x^2, x^3}_{\vec{x}})$, a g.c.t $x^\mu \to x'^\mu$ has an action

$$dx'^\mu = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}}_{\text{arbitrary}} dx^\nu \quad \text{and} \quad \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\rho} \quad \text{with} \quad \underbrace{\left|\frac{\partial x'^\mu}{\partial x^\nu}\right|}_{\text{non-singular}} \neq 0$$

- Line element (proper time): $ds^2 = \underbrace{g_{\mu\nu}}_{\text{NOT invariant under g.c.t !!}} dx^\mu dx^\nu \Rightarrow$ Invariant under g.c.t !!

Proof: $ds'^2 = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\epsilon} dx^\lambda dx^\epsilon = g_{\rho\sigma} dx^\rho dx^\sigma = ds^2$

(with $\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\lambda} = \delta^\rho_\lambda$ and $\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\epsilon} = \delta^\sigma_\epsilon$)

Equivalence Principle: Consider a particle moving under the influence of purely gravitational forces. Then, there is a freely falling local coordinate system with basis $\hat{\theta}^a = e_\mu{}^a dx^\mu$ such that

$$g_{\mu\nu}(x) = e_\mu{}^a(x)\, \underbrace{e_\nu{}^b(x)}_{\text{"frame or vielbein"}}\, \eta_{ab}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \eta_{ab}\, \hat{\theta}^a \hat{\theta}^b$$

(η_{ab}: Minkowski (flat) metric for a freely falling observer) $\Rightarrow$ GRAVITY = GEOMETRY !!

* **Vectors and tensors**: Objects in the theory of general relativity transform covariantly under g.c.t.

- Contravariant vectors V^μ
- Covariant vectors V_μ

$$V_\mu = g_{\mu\nu} V^\nu \qquad V^\mu = g^{\mu\nu} V_\nu \quad \Rightarrow \quad g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$

($g^{\mu\nu}$ → inverse metric; $g_{\nu\rho}$ → metric)

Under a g.c.t one has

$$V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu \quad \text{and} \quad V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

so that full-contractions are g.c.t-invariant quantities

$$u'^\mu V'_\mu = \frac{\partial x'^\mu}{\partial x^\nu} u^\nu \frac{\partial x^\rho}{\partial x'^\mu} V_\rho = \delta^\rho_\nu u^\nu V_\rho = u^\rho V_\rho$$

- **Mixed tensors**: General tensors can be constructed transforming multilinearly under g.c.t. They satisfy
  - Linearity: $T^\mu{}_\nu \equiv \alpha \underbrace{R^\mu{}_\nu}_{\text{tensor}} + \beta \underbrace{S^\mu{}_\nu}_{\text{tensor}}$ is a tensor.
  - Direct product: $T_\mu{}_\nu{}^\rho \equiv \underbrace{A_\mu}_{\text{tensor}} \underbrace{B_\nu{}^\rho}_{\text{tensor}}$ is a tensor $[A_\mu \neq \partial_\mu]$ IMPORTANT !!
  - Contraction: $T^{\mu\nu} \equiv \underbrace{T^\mu{}_\rho{}^{\nu\rho}}_{\text{tensor}}$ is a tensor.

* **Tensor densities** : In special relativity there are tensors, like η_{ab} and ϵ_{abcd}, which are invariant due to Lorentz transformations being $\Lambda \in O(1,3)$ and "proper" so that $|\Lambda| = 1$. In general relativity

$$\underbrace{g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}}_{(*)} \quad \text{and} \quad \left|\frac{\partial x'}{\partial x}\right| = \left|\frac{\partial x}{\partial x'}\right|^{-1} \neq 0$$

we assume $\left|\frac{\partial x'}{\partial x}\right| > 0$

which translates into the existence of tensor densities transforming with some power of $\left|\frac{\partial x'}{\partial x}\right|$

$$\mathfrak{T}'^\mu{}_\nu = \left|\frac{\partial x'}{\partial x}\right|^w \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \mathfrak{T}^\rho{}_\sigma \qquad [\, w \equiv \text{"weight"} \,]$$

Ex. 1 : $|g'| = \left|\frac{\partial x}{\partial x'}\right|^2 |g|$ [from (*)]

[w = -2]

Ex. 2 : $\epsilon^{\lambda\epsilon\psi\tau} \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\epsilon} \frac{\partial x'^\rho}{\partial x^\psi} \frac{\partial x'^\sigma}{\partial x^\tau} = \left|\frac{\partial x'}{\partial x}\right| \epsilon'^{\mu\nu\rho\sigma}$ (definition of det)

[w = -1]

Ex. 3 : $d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x$ $\equiv>$ Fundamental theorem of integral calculus

[w = 1] ↳ volume element under g.c.t

Starting from tensor densities one can construct regular tensors

- $$\sqrt{|g'|}\, d^4x' = \underbrace{\left|\frac{\partial x}{\partial x'}\right| \left|\frac{\partial x'}{\partial x}\right|}_{1} \sqrt{|g|}\, d^4x = \sqrt{|g|}\, d^4x$$

$$\Rightarrow \sqrt{|g|}\, d^4x \equiv \text{Volume tensor (no indices) invariant under g.c.t}$$

• $$\frac{1}{\sqrt{|g'|}}\varepsilon'^{\mu\nu\rho\sigma} = \underbrace{\left|\frac{\partial x'}{\partial x}\right|\left|\frac{\partial x}{\partial x'}\right|}_{1} \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\epsilon}}\frac{\partial x'^{\rho}}{\partial x^{\psi}}\frac{\partial x'^{\sigma}}{\partial x^{\tau}}\frac{1}{\sqrt{|g|}}\varepsilon^{\lambda\epsilon\psi\tau}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\epsilon}}\frac{\partial x'^{\rho}}{\partial x^{\psi}}\frac{\partial x'^{\sigma}}{\partial x^{\tau}}\frac{1}{\sqrt{|g|}}\varepsilon^{\lambda\epsilon\psi\tau}$$

$\Rightarrow \frac{1}{\sqrt{|g|}}\varepsilon^{\mu\nu\rho\sigma}$ is a tensor and not a density

For a tensor density $\mathcal{J}'^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}$ of weight w one finds

$$\left(\sqrt{|g'|}\right)^{w}\mathcal{J}'^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\right)\cdots\left(\frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}}\right)\left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\right)\cdots\left(\frac{\partial x^{\sigma_m}}{\partial x'^{\nu_m}}\right)\left(\sqrt{|g|}\right)^{w}\mathcal{J}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}$$

$\Rightarrow \left(\sqrt{|g|}\right)^{w}\mathcal{J}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}$ is a tensor and not a density

II. Affine connection and covariant derivatives

Unlike for special relativity $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and V^ν are tensors but $\partial_\mu V^\nu$ is NOT a tensor

$$\frac{\partial}{\partial x'^{\mu}}V'^{\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\rho}}\left[\frac{\partial x'^{\nu}}{\partial x^{\sigma}}V^{\sigma}\right] =$$

$$= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial}{\partial x^\rho} V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\rho \partial x^\sigma} V^\sigma$$

$\underbrace{\qquad}_{\text{like a regular tensor}}$ $\underbrace{\qquad}_{\text{bad term !!}}$

<u>Important</u> : A new object $\Gamma_{\mu\nu}{}^\rho \equiv$ "affine connexion" [or Christoffel symbols] is introduced in order to construct a $\nabla_\mu \equiv$ "covariant derivative" that transforms covariantly under g.c.t

$$\Gamma_{\mu\nu}{}^\rho(g) \equiv \frac{1}{2} g^{\rho\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right]$$

so that $\Gamma_{\mu\nu}{}^\rho = \Gamma_{\nu\mu}{}^\rho$ [No torsion in space-time]

One can now compute how $\Gamma_{\mu\nu}{}^\rho$ transforms under a g.c.t and finds

$$\Gamma'_{\mu\nu}{}^\rho = \underbrace{\frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\epsilon}{\partial x'^\nu} \Gamma_{\lambda\epsilon}{}^\sigma}_{\text{like a regular tensor}} + \underbrace{\frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}}_{\text{bad term !!}}$$

The two bad terms above turn to conspire so that

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \underbrace{\Gamma_{\mu\rho}{}^\nu}_{\text{space-time connexion}} V^\rho$$

transforms as a tensor

$$\nabla'_\mu V'^\nu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \nabla_\rho V^\sigma$$

Similarly one finds that

$$\nabla_\mu V_\nu \equiv \partial_\mu V_\nu - \underbrace{\Gamma_{\mu\nu}{}^{\rho}}_{\text{space-time connexion}} V_\rho$$

transforms as a tensor

$$\nabla'_\mu V'_\nu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \nabla_\rho V_\sigma$$

For a mixed tensor the covariant derivatives acts as

$$\nabla_\mu T^\nu{}_\rho = \partial_\mu T^\nu{}_\rho + \underbrace{\Gamma_{\mu\lambda}{}^{\nu}}_{\text{connexion}} T^\lambda{}_\rho - \underbrace{\Gamma_{\mu\rho}{}^{\lambda}}_{\text{connexion}} T^\nu{}_\lambda$$

Finally, for a tensor density $\mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}$ of weight w, one had that $\left(\sqrt{|g|}\right)^w \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}$ transforms as a tensor.

Then

$$\nabla_\mu \left[\left(\sqrt{|g|}\right)^w \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m} \right] \overset{\text{definition}}{\equiv} \left(\sqrt{|g|}\right)^w \nabla_\mu \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}$$

$$\Rightarrow \nabla_\mu \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m} \equiv \left(\sqrt{|g|}\right)^{-w} \nabla_\mu \left[\left(\sqrt{|g|}\right)^w \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m} \right]$$

$$\equiv \partial_\mu \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m} + \underbrace{\Gamma\,\mathfrak{T} - \Gamma\,\mathfrak{T}}_{\Gamma\text{-terms}} + \underbrace{\frac{w}{2|g|} \frac{\partial |g|}{\partial x^\mu} \mathfrak{T}^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}}_{\text{density term}}$$

NOTE: $\partial_\mu \left(\sqrt{|g|}\right)^w = \frac{w}{2} \left(\sqrt{|g|}\right)^{w-2} \partial_\mu |g|$

* **Properties of covariant derivatives** : The covariant derivative ∇_μ satisfies

- **Linearity** : $\nabla_\rho (\alpha R^\mu{}_\nu + \beta S^\mu{}_\nu) = \alpha \nabla_\rho R^\mu{}_\nu + \beta \nabla_\rho S^\mu{}_\nu$

- **Leibniz rule** : $\nabla_\rho (A^\mu{}_\nu B^\lambda) = (\nabla_\rho A^\mu{}_\nu) B^\lambda + A^\mu{}_\nu (\nabla_\rho B^\lambda)$

- **Compatibility with contraction** :

$$\nabla_\rho T^{\mu\nu}{}_\nu = \partial_\rho T^{\mu\nu}{}_\nu + \Gamma_{\rho\lambda}{}^\mu T^{\lambda\nu}{}_\nu \underbrace{+ \Gamma_{\rho\lambda}{}^\nu T^{\mu\lambda}{}_\nu - \Gamma_{\rho\nu}{}^\lambda T^{\mu\nu}{}_\lambda}_{\text{cancellation !!}}$$

$$= \partial_\rho T^{\mu\nu}{}_\nu + \Gamma_{\rho\lambda}{}^\mu T^{\lambda\nu}{}_\nu$$

- **Metric postulate** : A tensor that vanishes in a frame will vanish in any other frame

Local frame : $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma_{\mu\nu}{}^\rho = 0$ $\Rightarrow$ $\nabla_\rho g_{\mu\nu} = \partial_\rho \eta_{\mu\nu} = 0$

Any frame : $\nabla_\rho g_{\mu\nu} = 0$ "Metric postulate"

Important : The metric postulate can be used to derive the expression for $\Gamma_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho(g)$

Important : $\nabla_\mu V^\nu = \nabla_\mu (g^{\nu\rho} V_\rho) = \underbrace{(\nabla_\mu g^{\nu\rho})}_{0} V_\rho + g^{\nu\rho} (\nabla_\mu V_\rho)$

$= g^{\nu\rho} \nabla_\mu V_\rho$ $\Rightarrow$ Raising/lowering indices commutes with the covariant derivative

Important : There is a prescription to write physical laws in a theory of gravity. Write the laws in Minkowski space-time (no gravity) and replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$.

NOTE : For scalar quantities (no indices) one has $S' = S$ and therefore $\nabla_\mu S = \partial_\mu S$. The Lagrangian describing a theory of gravity + particles is a scalar quantity and thus invariant under g.c.t (up to boundary terms).

* Covariant gradient, curl and divergence

- Covariant gradient of a scalar function : $\nabla_\mu S = \dfrac{\partial S}{\partial x^\mu}$
- Covariant curl of a (covariant) vector :

$\Gamma_{\nu\mu}{}^{\lambda} = \Gamma_{\mu\nu}{}^{\lambda}$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma_{\mu\nu}{}^{\lambda} A_\lambda - \partial_\nu A_\mu + \Gamma_{\nu\mu}{}^{\lambda} A_\lambda$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(as in flat space-time)}$$

- Covariant divergence of a (contravariant) vector :

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma_{\mu\lambda}{}^{\mu} V^\lambda = \partial_\mu V^\mu + \tfrac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho} V^\lambda$$

NOTE : $\Gamma_{\mu\lambda}{}^{\mu} = \frac{1}{2} g^{\mu\rho} \left[\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda} \right] = \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho}$

$g^{\mu\rho}$: symmetric $(\mu\rho)$; $\partial_\mu g_{\rho\lambda} - \partial_\rho g_{\mu\lambda}$: antisymmetric $[\mu\rho]$

NOTE: Let us consider a generic metric $M(x)$

$$\delta \ln[\mathrm{Det}\, M] \equiv \ln[\mathrm{Det}\,(M+\delta M)] - \ln[\mathrm{Det}\, M]$$

$\ln a - \ln b = \ln\left[\frac{a}{b}\right]$ →

$$= \ln\left[\frac{\mathrm{Det}(M+\delta M)}{\mathrm{Det}\, M}\right] = \ln\left[(\mathrm{Det}\, M^{-1})\ \mathrm{Det}\,(M+\delta M)\right]$$

$$= \ln\left[\mathrm{Det}\left[M^{-1}(M+\delta M)\right]\right] = \ln\left[\mathrm{Det}\,\underbrace{(\mathbb{I}+M^{-1}\delta M)}_{e^{A}}\right]$$

Jacobi's formula →

$$= \ln\left[e^{\mathrm{Tr}\,\ln(\mathbb{I}+M^{-1}\delta M)}\right] = \mathrm{Tr}\,\ln(\mathbb{I}+M^{-1}\delta M)$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ ↑

$$\approx \mathrm{Tr}\,(M^{-1}\delta M) + \dots$$

subnote: Jacobi's formula : $\det[e^{A}] = e^{\mathrm{Tr}\,A}$

Taking $\delta M = \frac{\partial M}{\partial x^\lambda}\delta x^\lambda$ one arrives at

$$\frac{\partial}{\partial x^\lambda}\ln \mathrm{Det}\, M = \mathrm{Tr}\left[M^{-1}\frac{\partial}{\partial x^\lambda}M\right]$$

Therefore, the covariant divergence is given by

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{2}\overbrace{g^{\mu\rho}\partial_\lambda g_{\mu\rho}}^{\mathrm{Tr}(M^{-1}\partial_\lambda M)} V^\lambda = \partial_\mu V^\mu + \underbrace{\frac{1}{2}\frac{\partial}{\partial x^\lambda}(\ln|g|)}_{\Gamma_{\mu\lambda}{}^{\mu}} V^\lambda$$

with

$$\Gamma_{\mu\lambda}{}^{\mu} = \frac{1}{2}\partial_\lambda \ln|g| = \partial_\lambda \ln(\sqrt{|g|}) = \frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|})$$

This is

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|})\, V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|}\, V^\mu \right]$$

* Boundary terms in a space-time without torsion :

Let us consider the covariant divergence of a vector V^μ. Then

$$\underbrace{\int d^4x \sqrt{|g|}}_{\text{volume element invariant under g.c.t}} \nabla_\mu V^\mu = \int d^4x \sqrt{|g|} \frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|}\, V^\mu \right] \equiv \text{Boundary term}$$

NOTE : Stoke's theorem : $\int_U \vec{\nabla} \cdot \vec{F}\, dV = \int_{\partial U} \vec{F} \cdot \vec{n}\, dS$ ($\vec{n}$: normal unit vector)

Let us consider the covariant divergence of a tensor $F^{\mu\nu}$. Then

$$\nabla_\mu F^{\mu\nu} = \underbrace{\frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|}\, F^{\mu\nu} \right]}_{\text{contracted index}} + \Gamma_{\mu\lambda}{}^{\nu} F^{\mu\lambda}$$

$\Gamma_{\mu\lambda}{}^{\nu} = \Gamma_{\lambda\mu}{}^{\nu}$ [no torsion]

if $F^{\mu\nu} = -F^{\nu\mu}$ one has that

$$\int d^4x \sqrt{|g|}\, \nabla_\mu F^{\mu\nu} \equiv \text{Boundary term}$$

Important : Boundary terms can be added to an action without modifying the equations of motion.

III. Curvature and Bianchi identities

To describe the dynamics of the metric we need quantities built from $g_{\mu\nu}$ and its derivatives that transform properly under g.c.t.

Criterium : Using the metric, first and second derivatives [as in Euler-Lagrange] one can construct the Riemann-Christoffel tensor

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} \equiv \partial_\mu \Gamma_{\nu\sigma}{}^{\rho} - \partial_\nu \Gamma_{\mu\sigma}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\sigma}{}^{\lambda} - \Gamma_{\nu\lambda}{}^{\rho}\Gamma_{\mu\sigma}{}^{\lambda}$$

that transforms as a regular tensor under g.c.t

$$R'_{\mu\nu}{}^{\rho}{}_{\sigma} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\epsilon}{\partial x'^\nu}\frac{\partial x'^\rho}{\partial x^\psi}\frac{\partial x^\tau}{\partial x'^\sigma} R_{\lambda\epsilon}{}^{\psi}{}_{\tau}$$

Physical meaning : If we are given a space-time metric $g_{\mu\nu}(x)$, how do we know if there is a non-trivial gravitational field or, on the contrary, there are special coordinates $\xi^a(x)$ such that

$$\text{Minkowski} \leftarrow \eta^{ab} = \frac{\partial \xi^a}{\partial x^\mu}\frac{\partial \xi^b}{\partial x^\nu} g^{\mu\nu} \quad ??$$

Example : $g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix}$ with $x^\mu = (t, r, \theta, \varphi)$

NOTE : Black holes corresponds with $g_{\mu\nu}$ of the above type with $-g_{tt} = g_{rr}^{-1} = f(r)$

There is a new set of coordinates $\xi^a = (\xi^0, \xi^1, \xi^2, \xi^3)$ with

$$\xi^0 = t \ , \quad \xi^1 = r \sin\theta \cos\varphi \ , \quad \xi^2 = r\sin\theta \sin\varphi \ , \quad \xi^3 = r\cos\theta$$

such that

$$\eta^{ab} = \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} g^{\mu\nu}$$

The answer to the question is two-fold:

- $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ must be zero for ξ^a coordinates to exist (a vanishing tensor does vanish for any observer)
- There must exist a point X at which the metric $g_{\mu\nu}(X)$ has one negative and three positive eigenvalues like η_{ab}

Ricci identity: In a space-time without torsion one has that

$$[\nabla_\mu, \nabla_\nu] V^\rho = \underbrace{R_{\mu\nu}{}^{\rho}{}_{\sigma}}_{\text{Riemann-Christoffel tensor}} V^\sigma$$

$\Rightarrow$ If $R_{\mu\nu}{}^{\rho}{}_{\sigma} = 0$ then $[\nabla_\mu, \nabla_\nu] = 0$ as would be expected for a coordinate system that can be transformed into a Minkowski coordinate system via a g.c.t.

* Algebraic properties of $R_{\mu\nu\rho\sigma} = g_{\rho\lambda} R_{\mu\nu}{}^{\lambda}{}_{\sigma}$

- Symmetry : $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
- Antisymmetry : $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\nu\mu\sigma\rho}$
- Cyclicity : $R_{\mu\nu\rho\sigma} + R_{\rho\mu\nu\sigma} + R_{\nu\rho\mu\sigma} = 0$

* Ricci tensor : $R_{\mu\nu} \equiv R_{\lambda\mu}{}^{\lambda}{}_{\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$ (only possibility)

- Symmetry : Since $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \Rightarrow R_{\mu\nu} = R_{\nu\mu}$

* Ricci scalar : $R \equiv R_{\lambda}{}^{\lambda} = g^{\mu\nu} R_{\mu\nu}$ (only possibility)

Note that no other scalar (0-index tensor) can be formed as

$$\frac{1}{\sqrt{|g|}} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0$$

by virtue of the cyclicity of $R_{\mu\nu\rho\sigma}$.

* Bianchi identities : In addition to the algebraic identities, the Rimann-Christoffel tensor, Ricci tensor and Ricci scalar satisfy a set of differential equations

- $\nabla_{\mu} R_{\nu\rho}{}^{\lambda}{}_{\sigma} + \nabla_{\rho} R_{\mu\nu}{}^{\lambda}{}_{\sigma} + \nabla_{\nu} R_{\rho\mu}{}^{\lambda}{}_{\sigma} = 0 \iff \nabla_{[\mu} R_{\nu\rho]}{}^{\lambda}{}_{\sigma} = 0$

NOTE : This is the analogue of the Bianchi form of Gauss-Faraday law in classical electrodynamics

- Tracing over (σ, λ) gives

$$\nabla_\mu R_{\rho\sigma} - \nabla_\rho R_{\mu\sigma} + \nabla_\lambda R_{\rho\mu}{}^{\lambda}{}_{\sigma} = 0$$

- Tracing over (ρ, σ) gives

$$-R_{\rho\mu}{}^{\rho\lambda} = -R_\mu{}^\lambda$$

$$\nabla_\mu R - \nabla_\rho R_\mu{}^\rho + \nabla_\lambda R_{\rho\mu}{}^{\lambda\rho} = \nabla_\mu R - 2\nabla_\rho R_\mu{}^\rho$$

$$= -2\nabla_\rho \left[R_\mu{}^\rho - \frac{1}{2}\delta^\rho_\mu R \right]$$

$$= 0$$

raising μ

$$\Rightarrow \quad \nabla_\nu \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right] = 0$$

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \qquad \text{"Einstein tensor"}$$

$$\Rightarrow \quad \boxed{\nabla_\mu G^{\mu\nu} = 0}$$

$\Rightarrow$ The Einstein tensor is conserved due to symmetries !!

NOTE : Recall that the covariant divergence of a symmetric tensor is <u>not</u> a boundary term upon integration.

IV. Einstein's field equations $[\, c = 1 \,, \; \epsilon_0 \cdot \mu_0 = \frac{1}{c^2} = 1 \,]$

The equations of motion governing the dynamics of a gravitating system are the so-called Einstein's field equations

$\kappa^2 = 8\pi G_N$ with $\kappa^{-1} = m_P = 2.4 \times 10^{18}$ GeV

reduced Plank mass

$$G_{\mu\nu} = \kappa^2 \, T_{\mu\nu}$$

Einstein tensor [Geometry] — Energy-momentum tensor [Matter content]

$T^{\mu\nu} = T^{\nu\mu}$

* Energy-momentum tensor : It depends on what kind of matter and space-time geometry one is considering. Some examples are :

- Free particle of mass m in Minkowski space-time :

$$T^{ab} = \frac{m}{\gamma} u^a u^b \, \delta(\vec{x} - \vec{x}_p(t)) = \frac{E}{\gamma^2} u^a u^b \, \delta(\vec{x} - \vec{x}_p(t))$$

$u^a = \gamma\,(1, \vec{v})$ — γ: Lorentz factor

$E^2 = |\vec{p}|^2 + m^2$

- Perfect fluid (in the inertial frame) in Minkowski space-time :

Inertial frame $[\,\vec{v} = 0\,]$: $u^a = (1, \vec{0}) \Rightarrow \eta_{ab} u^a u^b = -1$

$$T^{ab} = \begin{bmatrix} \rho & \\ & P\,\delta^{ij} \end{bmatrix} \quad \text{with} \quad \begin{array}{l} \rho \equiv \text{energy density} \\ P \equiv \text{isotropic pressure} \end{array}$$

- Perfect fluid in Minkowski space-time

$$T^{ab} = P\,\eta^{ab} + (P+\rho)\,u^a u^b$$

with a normalisation given by $\eta_{ab} u^a u^b = -1$

- Perfect fluid in a gravitational field

"frame field" $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$

$$T^{\mu\nu} = P\,g^{\mu\nu} + (P+\rho)\,u^\mu u^\nu \quad \text{with} \quad u^a = e_\mu{}^a u^\mu$$

with a normalisation given by $g_{\mu\nu} u^\mu u^\nu = \eta_{ab} u^a u^b = -1$

NOTE: Taking the trace one finds

$$T \equiv g_{\mu\nu} T^{\mu\nu} = 4P - (P+\rho) = 3P - \rho$$

Statistical Physics

NOTE: Usually matter satisfies an equation of state $f(P,\rho) = 0$

- Cosmological constant: It is modelled as a perfect fluid with a equation of state $P = -\rho < 0$

$\Rightarrow$ $T^{\mu\nu} = -\Lambda\,g^{\mu\nu}$ with $-\Lambda = P < 0$ $\Rightarrow$ Exotic form of energy / matter !!

- Classical electrodynamics in a gravitational field

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} \right]$$

which takes the form

$$T^{\mu\nu} = \begin{bmatrix} \frac{1}{2}\left[\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right] & S_x & S_y & S_z \\ S_x & & & \\ S_y & & -\sigma^{ij} & \\ S_z & & & \end{bmatrix}$$

where

- $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ => Poynting vector

- $\sigma^{ij} = \epsilon_0 E^i E^j + \frac{1}{\mu_0} B^i B^j - \frac{1}{2}\left[\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right]\delta^{ij}$

  => Maxwell stress tensor

NOTE: Taking the trace one finds

$$T \equiv g_{\mu\nu} T^{\mu\nu} = \frac{1}{\mu_0}\left[F_{\mu\nu} F^{\mu\nu} - F_{\mu\nu} F^{\mu\nu} \right] = 0$$

- Vacuum : There is no matter in the space-time so that

$$T^{\mu\nu} = 0$$

* **Alternative form of Einstein equations:** Starting from the Einstein equation and taking a trace one finds

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \quad \Rightarrow \quad G = \kappa^2 T$$

with

$$G = g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu}\left[R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right] = R - 2R = -R$$

$$\Rightarrow \quad R = -\kappa^2 T$$

Substituting back into the Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} \kappa^2 T = \kappa^2 T_{\mu\nu}$$

$$\Rightarrow \quad R_{\mu\nu} = \kappa^2\left[T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right]$$

Alternative form in terms of $R_{\mu\nu}$!!

At the vacuum one has that

$$T^{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0 \qquad \text{"Ricci-flat manifolds"}$$

Important: In $D = 1+1$ and $D = 1+2$ it can be proven that $R_{\mu\nu} = 0 \Rightarrow R_{\mu\nu\rho\sigma} = 0$ so there is no gravitational field. In $D = 1+3$ this is not the case: Black holes, wormholes, gravitational waves, ...

V. Diffeomorphisms and Lie derivatives

We have seen before that the Einstein tensor is conserved due to symmetries. This is

$$\nabla_\mu G^{\mu\nu} = 0$$

Then, the Einstein equations imply

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \text{Conserved current !!}$$

Important : Noether theorem states that a symmetry in a theory implies a conserved current. In the case of GR, the energy-momentum tensor $T^{\mu\nu}$ is the conserved current associated to space-time translations or diffeomorphisms

- g.c.t : $dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu$
- Infinitesimal g.c.t : $x'^\mu = x^\mu + \xi^\mu(x)$
  [diffeomorphism]

* Lie derivative : Diffeomorphisms acts on tensor in the form of Lie derivatives $\mathcal{L}_\xi$ along a contravariant vector field $\xi^\mu(x)$

- Scalar S : $\delta_\xi S = \mathcal{L}_\xi S = \xi^\rho \partial_\rho S$

- Vector V^μ : $\delta_\xi V^\mu = \mathcal{L}_\xi V^\mu = \xi^\rho \partial_\rho V^\mu - V^\rho \partial_\rho \xi^\mu$
[contravariant]
$= [\xi, V]$ "Lie bracket"
↳ $\xi = \xi^\rho \partial_\rho$ and $V = V^\mu \partial_\mu$

- Vector V_μ : $\delta_\xi V_\mu = \mathcal{L}_\xi V_\mu = \xi^\rho \partial_\rho V_\mu + V_\rho \partial_\mu \xi^\rho$
[covariant]
[1-form]

Important: The Lie derivative constitutes an infinite-dimensional representation of the diffeomorphisms Lie group

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}]\, T = \mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} T - \mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1} T$$

$$= \mathcal{L}_{[\xi_1, \xi_2]}\, T$$

where $T \equiv$ arbitrary tensor and $[\xi_1, \xi_2]$ is the Lie bracket.

NOTE: Taylor expanding the diffeomorphism vector field

$$\xi^\mu(x) = \xi^\mu(x_p) + \underbrace{\left.\frac{\partial \xi^\mu}{\partial x^\nu}\right|_{x_p}}_{M^\mu{}_\nu \,\in\, GL(4)} x^\nu + \dots$$

- In D dimensions one gets $GL(D)$.

* Killing vector : $\xi^\mu(x)$ is a Killing vector if $\mathcal{L}_\xi g_{\mu\nu} = 0$
[isometries]

Appendix : $T_{\mu\nu}$ for a gravitating electromagnetic field

$$S_A = -\frac{1}{4}\int d^4x \sqrt{-|g|}\, F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}\int d^4x \sqrt{-|g|}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$(\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho})$

with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

Then :

$$\delta S_A = -\frac{1}{4}\int d^4x \Big[\underbrace{\delta(\sqrt{-|g|})}_{\frac{1}{2}\sqrt{-|g|}\, g^{\lambda\epsilon}\delta g_{\lambda\epsilon}} F_{\mu\nu} F^{\mu\nu} + \sqrt{-|g|}\, \overbrace{\delta g^{\mu\rho}}^{-g^{\mu\lambda} g^{\rho\epsilon}\delta g_{\lambda\epsilon}} F_{\mu\nu} F_\rho{}^\nu + \sqrt{-|g|}\, \underbrace{\delta g^{\nu\sigma}}_{-g^{\nu\lambda} g^{\sigma\epsilon}\delta g_{\lambda\epsilon}} F^\rho{}_\nu F_{\rho\sigma} \Big]$$

$$= -\frac{1}{4}\int d^4x \sqrt{-|g|} \Big[\frac{1}{2} g^{\lambda\epsilon} F_{\mu\nu} F^{\mu\nu} - F^\lambda{}_\nu F^{\epsilon\nu} - \underbrace{F^{\rho\lambda} F_\rho{}^\epsilon}_{F^\lambda{}_\rho F^{\epsilon\rho}} \Big] \delta g_{\lambda\epsilon}$$

$$= -\frac{1}{4}\int d^4x \sqrt{-|g|} \Big[\frac{1}{2} g^{\lambda\epsilon} F_{\mu\nu} F^{\mu\nu} - 2 F^\lambda{}_\rho F^{\epsilon\rho} \Big] \delta g_{\lambda\epsilon}$$

$$= \frac{1}{2}\int d^4x \sqrt{-|g|} \Big[F^\lambda{}_\rho F^{\epsilon\rho} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\lambda\epsilon} \Big] \delta g_{\lambda\epsilon}$$

$$= \frac{1}{2}\int d^4x \sqrt{-|g|}\; T^{\lambda\epsilon}\, \delta g_{\lambda\epsilon}$$

$$\Longrightarrow \boxed{T^{\mu\nu} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4} F_{\rho\lambda} F^{\rho\lambda} g^{\mu\nu}} \qquad (\mu_0 \equiv 1)$$

Appendix: Equation of motion of a gravitating electromagnetic field

$$S_A = -\frac{1}{4}\int d^4x \sqrt{-|g|}\, F_{\mu\nu} F^{\mu\nu}$$

$$= -\frac{1}{4}\int d^4x \sqrt{-|g|}\,(\nabla_\mu A_\nu - \nabla_\nu A_\mu)\, F^{\mu\nu} \quad \rightsquigarrow \text{Only two contributions}$$

$$= -\frac{1}{2}\int d^4x \sqrt{-|g|}\,(\nabla_\mu A_\nu)\, F^{\mu\nu}$$

$$= -\frac{1}{2}\int d^4x \sqrt{-|g|}\,\left\{ \underbrace{\nabla_\mu (A_\nu F^{\mu\nu})} - A_\nu \nabla_\mu F^{\mu\nu} \right\}$$

$$\nabla_\mu T^\mu \Rightarrow \int d^4x \sqrt{-|g|}\, \nabla_\mu T^\mu = \underbrace{\int d^4x\, \partial_\mu T^\mu}_{\text{"boundary term"}}$$

$$\Rightarrow \tilde{S}_A = +\frac{1}{2}\int d^4x \sqrt{-|g|}\, A_\nu \nabla_\mu F^{\mu\nu}$$

↳ The same action as the original one up to a boundary term ⇒ same equations of motion !!

Then :

$$\delta \tilde{S}_A = 2 \times \frac{1}{2}\int d^4x \sqrt{-|g|}\, \delta A_\nu \nabla_\mu F^{\mu\nu} = 0 \Rightarrow \boxed{\nabla_\mu F^{\mu\nu} = 0}$$

↳ $\delta A_\nu \nabla_\mu F^{\mu\nu} + \underbrace{A_\nu \nabla_\mu \delta F^{\mu\nu}}_{\delta A_\nu \nabla_\mu F^{\mu\nu} + \text{boundary terms}}$

Appendix: $T_{\mu\nu}$ for a gravitating scalar field

$$S_\phi = \int d^4x \sqrt{-|g|}\left[-\frac{1}{2}\underbrace{\nabla_\mu\phi}_{\partial_\mu\phi}\underbrace{\nabla^\mu\phi}_{\partial_\mu\phi} - V(\phi)\right] = \int d^4x \sqrt{-|g|}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

Then:

$$\delta S_\phi = \int d^4x \left[\underbrace{\delta\left(\sqrt{-|g|}\right)}_{\frac{1}{2}\sqrt{-|g|}\,g^{\lambda\epsilon}\delta g_{\lambda\epsilon}}\left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right) - \frac{1}{2}\sqrt{-|g|}\,\underbrace{\delta g^{\mu\rho}}_{-g^{\mu\lambda}g^{\rho\epsilon}\delta g_{\lambda\epsilon}}\partial_\mu\phi\,\partial_\rho\phi\right]$$

$$= \frac{1}{2}\int d^4x \sqrt{-|g|}\left[g^{\lambda\epsilon}\left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right) + \partial^\lambda\phi\,\partial^\epsilon\phi\right]\delta g_{\lambda\epsilon}$$

$$\equiv \frac{1}{2}\int d^4x \sqrt{-|g|}\; T^{\mu\nu}$$

$$\Rightarrow \boxed{T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi + g^{\mu\nu}\left(-\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - V(\phi)\right)}$$

Appendix: Equation of motion of a gravitating scalar field

$$S_\phi = \int d^4x \sqrt{-|g|} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right]$$

$$= -\frac{1}{2} \int d^4x \sqrt{-|g|} \left[g^{\mu\nu} \left(\nabla_\mu (\phi \nabla_\nu \phi) - \phi \nabla_\mu \nabla_\nu \phi \right) + 2V(\phi) \right]$$

$$= -\frac{1}{2} \int d^4x \sqrt{-|g|} \left[-\phi \underbrace{\Box}_{} \phi + 2V(\phi) + \underbrace{\nabla_\mu (\phi \nabla^\mu \phi)}_{} \right]$$

$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$

$\nabla_\mu T^\mu \Rightarrow$ "boundary term"

$$\Rightarrow \tilde{S}_\phi = +\frac{1}{2} \int d^4x \sqrt{-|g|} \left[\phi \Box \phi - 2V(\phi) \right]$$

↳ The same action as the original one up to a boundary term ⇒ same equations of motion !!

Then :

$\delta\phi \, \Box\phi$ + boundary terms

↳ $\delta\phi \, \Box\phi + \phi \, \Box \delta\phi$

$$\delta \tilde{S}_\phi = +\frac{1}{2} \int d^4x \sqrt{-|g|} \left[2 \times \delta\phi \, \Box\phi - 2 V'(\phi) \, \delta\phi \right]$$

$$= \int d^4x \sqrt{-|g|} \left[\Box\phi - \frac{\partial V(\phi)}{\partial \phi} \right] \delta\phi = 0$$

$$\Rightarrow \boxed{\Box\phi = \frac{\partial V(\phi)}{\delta \phi}}$$