I. Differential manifolds

In GR the space-time is identified with a differential manifold. Let us introduce various concepts that will lead us to define a differential manifold.

* Topological space : It is a set of points (events), along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods. It is the most general type of mathematical space that allows for definition of limits, continuity and connectedness.

* Manifold $\mathcal{M}$: It is the n-dimensional generalisation of a curve (1-manifold) or a surface (2-manifold). More precisely, a manifold is a topological space that "locally resembles $\mathbb{R}^n$" where n is the dimension of the manifold.

Q : What does it mean that "locally resembles $\mathbb{R}^n$" ?

A : We have to introduce more concepts to discuss this.

* Open n-ball of radius r in $\mathbb{R}^n$: Set of points (events) in the manifold around a point $x_0^\mu = (x_0^1, \ldots, x_0^n)$ for which

$$|x - x_0| = \sqrt{\sum_{\mu=1}^{n} (x^\mu - x_0^\mu)^2} < r \qquad [\mu = 1, \ldots, n]$$

* Open set $U \subset \mathbb{R}^n$: For each point belonging to U there is an open n-ball around it which is fully contained in U.

* Coordinate map: Let us consider a manifold $\mathcal{M}$ and a submanifold $O \subset \mathcal{M}$. A coordinate map is then a one-to-one (injective) application (map)

$$\psi : O \longrightarrow \mathbb{R}^n$$

$$p \in O \longrightarrow x_p^\mu \qquad \text{with } \mu = 1, \ldots, n$$

* Coordinate system (chart): Let us consider the subset $U \in \mathbb{R}^n$ corresponding to the image of $O \subset \mathcal{M}$, namely,

$$U = \psi(O) \subset \mathbb{R}^n$$

If $U \subset \mathbb{R}^n$ is an open set then we say that the pair

$$(O, \psi) \equiv \text{coordinate system or chart}$$

and that

$$O \subset \mathcal{M} \equiv \text{open submanifold of } \mathcal{M}$$

* Properties of the coordinate map ψ :

- Bijective : Then it is invertible

$$I = \psi^{-1} \circ \psi : \quad O \subset \mathcal{M} \xrightarrow{\psi} U \subset \mathbb{R}^n \xrightarrow{\psi^{-1}} O \subset \mathcal{M}$$

- Coordinate representation :

$$\psi : \; p \in O \longrightarrow \underbrace{x^\mu_p}_{} \subset U \subset \mathbb{R}^n$$

"Coordinates of point (event) p"

- A point $p \in \mathcal{M}$ may have different coordinates on different charts.

* **Atlas C^∞**: It is a set of charts $\{(O_\alpha, \psi_\alpha\}$ satisfying the two conditions

i) Any $p \in \mathcal{M}$ is at least in a chart O_α
$\Rightarrow \{O_\alpha\}$ fully covers $\mathcal{M}$

ii) At the intersection $O_\alpha \cap O_\beta \neq \emptyset$, the transition function

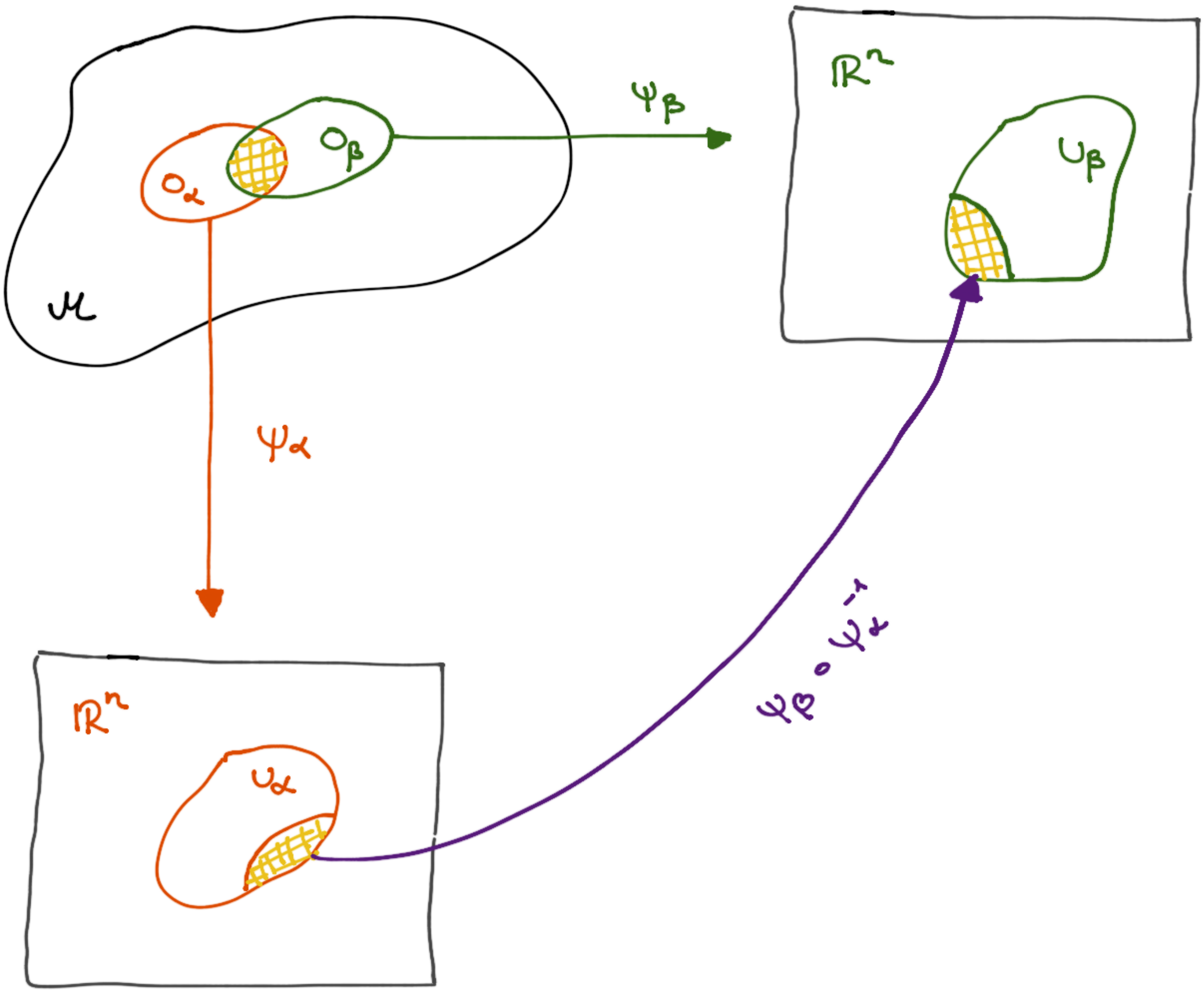
$$\psi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha \subset \mathbb{R}^n \rightarrow \psi_\beta(O_\alpha \cap O_\beta) \subset U_\beta \subset \mathbb{R}^n$$

is C^∞.

* **Maximal atlas C^∞**: It is an atlas C^∞ that contains all possible coordinate systems.

$\Rightarrow$ **Differential manifold**: It is a manifold $\mathcal{M}$ endowed with a maximal atlas C^∞.

* **Scheme of coordinate maps**:



* Examples :

- $\mathbb{R}^n$
- S^n
- Group manifolds : $G = e^{\theta^a t_a}$
  - θ^a ↦ Coordinates on the manifold
  - ↳ $a = 1, \dots, \underbrace{\dim(G)}_{n = \dim(G)}$
- etc

II. Vectors and dual vectors on $\mathcal{M}$

In order to define objects in $\mathcal{M}$ we will have to first introduce the notion of a function on $\mathcal{M}$

$$f : \mathcal{M} \longrightarrow \mathbb{R}$$
$$p \in \mathcal{M} \longrightarrow f(p) \in \mathbb{R}$$

* Smooth $f(p)$: A function is called smooth (or C^∞) if the application

$$f \circ \psi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

is C^∞ for all the coordinate systems (O, ψ)

$$\mathcal{F}(\mathcal{M}) \equiv \text{Set of smooth functions on } \mathcal{M}$$

* Curve γ passing through $p \in \mathcal{M}$: It is defined as an application from an interval to $\mathcal{M}$

$$\gamma : \quad (a, b) \subset \mathbb{R} \longrightarrow \mathcal{M}$$
$$\lambda \in (a,b) \longrightarrow \gamma(\lambda)$$

For convenience we will choose that the interval contains $\lambda = 0$ and will denote p the point at $\lambda = 0$, namely, $p = \gamma(0)$

* Derivative of a function f : It is a differential operator defined along a curve $\gamma \subset \mathcal{M}$

$$
\begin{aligned}
f \circ \gamma : \quad (a,b) \subset \mathbb{R} &\longrightarrow \mathbb{R} \\
\lambda \in (a,b) &\longrightarrow (f \circ \gamma)(\lambda)
\end{aligned}
$$

so that the derivative is defined on a point p as

$$\left.\frac{df}{d\lambda}\right|_p \equiv \left.\frac{d(f \circ \gamma)}{d\lambda}\right|_{\lambda=0}$$

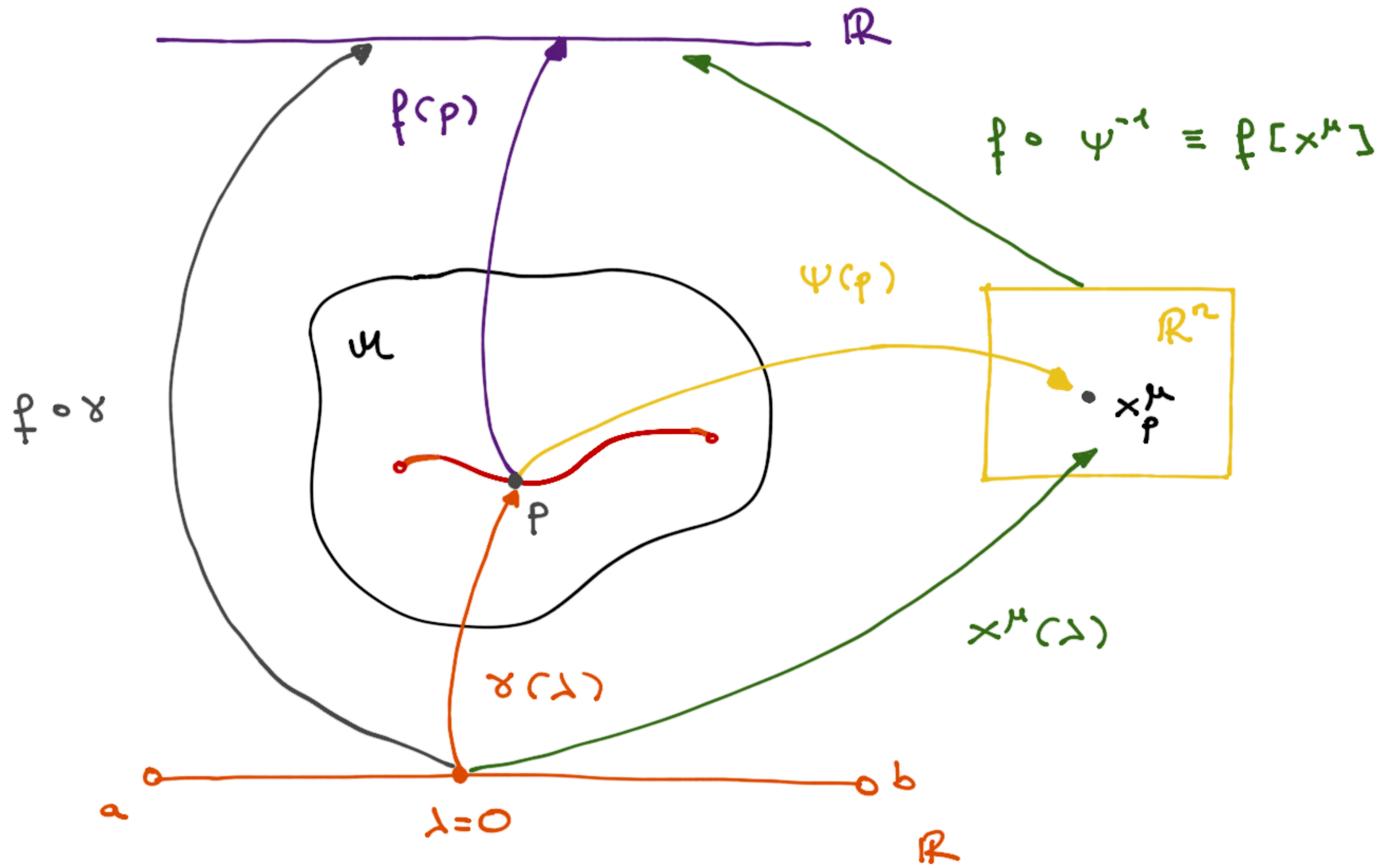
An alternative (more intuitive) way of thinking about this is

$$f \circ \gamma = \underbrace{f \circ \overbrace{\psi^{-1}}^{}}_{\text{"}f\text{-function"}} \overbrace{\circ\ \underbrace{\psi \circ \gamma}_{x^\mu(\lambda)}}^{\mathrm{I}} \equiv f\left[x^\mu(\lambda)\right]$$

so that

$$\left.\frac{df}{d\lambda}\right|_p \equiv \left.\frac{d(f\circ\gamma)}{d\lambda}\right|_{\lambda=0} = \left.\frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\right|_{\lambda=0}$$

* Scheme :



Then :

$$\frac{df}{d\lambda} = \left[\frac{dx^\mu}{d\lambda}\underbrace{\frac{\partial}{\partial x^\mu}}_{\partial_\mu}\right] f \quad \Rightarrow \quad \boxed{\frac{d}{d\lambda} = \underbrace{\frac{dx^\mu}{d\lambda}}_{\text{tangent direction to the curve } \gamma}\partial_\mu}$$

Denoting the tangent direction to the curve

$$V^\mu \equiv \frac{dx^\mu}{d\lambda}$$

one arrives at the concept of vector fields as differential operator acting on $\mathcal{F}(\mathcal{M})$

$$\boxed{V \equiv V^\mu \partial_\mu}$$

Ex: $V^\mu = (1, 0, 0, 0) \Rightarrow V = \partial_0$

$\Rightarrow \partial_\mu \equiv$ tangent vector to the direction x^μ

* Tangent space $T_p(\mathcal{M})$: It is the vector space containing all the curves on $\mathcal{M}$ passing through $p \in \mathcal{M}$.

Then :

- $(V + W)(f) = V(f) + W(f) \quad \forall V, W \in T_p(\mathcal{M}), \; f \in \mathcal{F}(\mathcal{M})$
- $(\alpha V)(f) = \alpha V(f) \quad \forall V \in T_p(\mathcal{M}), \; \alpha \in \mathbb{R}$

→ A tangent vector $V = V^\mu \partial_\mu$ satisfies :

i) Linearity : $V(\alpha f + \beta g) = \alpha V(f) + \beta V(g)$

ii) Leibniz rule : $V(fg) = V(f)\, g + f\, V(g)$

for $f, g \in \mathcal{F}(\mathcal{M})$.

- Coordinate basis on $T_p(\mathcal{M})$: $\{\partial_\mu\} \equiv \{\hat{e}_\mu = \partial_\mu\}$

$$\Downarrow$$

$$\mathrm{Dim}\left[T_p(\mathcal{M})\right] = n$$

≡> In a coordinate basis : $V = V^\mu \partial_\mu$

$$V(f) = V^\mu \partial_\mu f$$

- A tangent space $T_p(\mathcal{M})$ can be defined at any $p \in \mathcal{M}$.
- Choosing a V^μ at each $T_p(\mathcal{M})$ => Vector field on $\mathcal{M}$.

* Lie bracket : It is defined as the commutator of two vector fields

$$[U, V](f) = U^\rho \partial_\rho (V^\lambda \partial_\lambda f) - V^\rho \partial_\rho (U^\lambda \partial_\lambda f)$$

$$= \left[U^\rho \partial_\rho V^\lambda - V^\rho \partial_\rho U^\lambda\right] \partial_\lambda f$$

$$= [U, V]^\lambda \partial_\lambda f$$

with

$$[U, V]^\lambda = U^\rho \partial_\rho V^\lambda - V^\rho \partial_\rho U^\lambda \qquad \text{"Lie bracket"}$$

The Lie bracket satisfies

- $[U, V] = -[V, U]$ (antisymmetry)
- $[U, [V, W]] + [W, [U, V]] + [V, [W, U]] = 0$ (Jacobi)
- $[\alpha U + \beta V, W] = \alpha [U, W] + \beta [V, W]$ (bilinear)

so it defines a Lie algebra.

* Cotangent (dual) space $T_p^*(M)$: It is a vector space dual to the tangent space $T_p(M)$. It consists of applications called "dual vectors"

$$\begin{aligned} \omega : \quad T_p(M) &\longrightarrow \mathbb{R} \\ V &\longrightarrow \omega(V) \end{aligned}$$

such that

i) $(\omega + \eta)(V) = \omega(V) + \eta(V)$ ii) $(\alpha\,\omega)\, V = \alpha\, \omega(V)$

with $\omega, \eta \in T_p^*(\mathcal{M})$ and $\alpha \in \mathbb{R}$.

- Dual coordinate basis on $T_p^*(\mathcal{M})$: $\{ \hat{\theta}^\mu = dx^\mu \}$ $\Rightarrow$ $\mathrm{Dim}[T_p^*(\mathcal{M})] = n$

such that : $\hat{\theta}^\mu (\hat{e}_\nu) = \delta^\mu_\nu$

Any dual vector can then be expanded as

$$\omega = \omega_\mu \hat{\theta}^\mu = \omega_\mu dx^\mu$$

and then one has that

$$\omega(V) = \omega_\rho V^\rho$$

Ex 1 : $dx^\mu = \frac{\partial x^\mu}{dx^\nu} dx^\nu = \delta^\mu_\nu dx^\nu = dx^\mu \Rightarrow \omega_\nu = \delta^\mu_\nu$

Ex 2 : $df = \frac{\partial f}{\partial x^\mu} dx^\mu \Rightarrow \omega_\mu = \partial_\mu f \Rightarrow$ Gradient of f is a dual vector

NOTE : We could have alternatively started by introducing $T_p^*(\mathcal{M})$ and then

$$V : T_p^*(\mathcal{M}) \longrightarrow \mathbb{R}$$
$$\omega \longrightarrow \omega(V) \equiv V(\omega) = V^\rho \omega_\rho$$

Then it follows that $\left(T_p^*(\mathcal{M})\right)^* = T_p(\mathcal{M})$

III. Tensors on $\mathcal{M}$

Tensor fields are well-defined objects living on the manifold $\mathcal{M}$.

* Mixed tensor of type (k,ℓ) : It is an application

$$T : \underbrace{T_p^* \times \dots \times T_p^*}_{k\text{-times}} \times \underbrace{T_p \times \dots \times T_p}_{\ell\text{-times}} \longrightarrow \mathbb{R}$$

$$(\omega_1, \dots, \omega_k, V_1, \dots, V_\ell) \longrightarrow T(\omega_1, \dots, \omega_k, V_1, \dots, V_\ell)$$

so it has a multilinear action on k-copies of $T_p^*(\mathcal{M})$ and ℓ copies of $T_p(\mathcal{M})$.

Using a coordinate basis $\{\partial_\mu, dx^\mu\}$ to describe a tensor of type (k,ℓ) one has

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\ell} \, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_\ell}$$

Important : The properties of T do not depende on the choice of the coordinate basis. However, the tensor components $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\ell}$ do depend on the choice of coordinate basis.

* Tensors and changes of coordinate basis :

Let us consider two different choices of coordinate basis

$$\text{Basis} : \left\{ \frac{\partial}{\partial x^\mu}, dx^\mu \right\} \Rightarrow T = T^{\mu\cdots}{}_{\nu\cdots} \frac{\partial}{\partial x^\mu} \otimes \cdots \otimes dx^\nu \otimes \cdots$$

$$\text{Basis}' : \left\{ \frac{\partial}{\partial x'^\mu}, dx'^\mu \right\} \Rightarrow T = T'^{\mu\cdots}{}_{\nu\cdots} \frac{\partial}{\partial x'^\mu} \otimes \cdots \otimes dx'^\nu \otimes \cdots$$

Important: The basis elements are related by

$$dx^\nu = \frac{\partial x^\nu}{\partial x'^\rho} dx'^\rho \quad ; \quad \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial}{\partial x'^\lambda}$$

It then follows that

$$\begin{aligned} T &= T^{\mu\cdots}{}_{\nu\cdots} \frac{\partial}{\partial x^\mu} \otimes \cdots \otimes dx^\nu \otimes \cdots \\ &= T^{\mu\cdots}{}_{\nu\cdots} \frac{\partial x'^\lambda}{\partial x^\mu} \cdots \frac{\partial x^\nu}{\partial x'^\rho} \cdots \frac{\partial}{\partial x'^\lambda} \otimes \cdots \otimes dx'^\rho \otimes \cdots \\ &= T'^{\lambda\cdots}{}_{\rho\cdots} \frac{\partial}{\partial x'^\lambda} \otimes \cdots \otimes dx'^\rho \otimes \cdots \end{aligned}$$

so that

$$\boxed{T'^{\lambda\cdots}{}_{\rho\cdots}(x') = \frac{\partial x'^\lambda}{\partial x^\mu} \cdots \frac{\partial x^\nu}{\partial x'^\rho} T^{\mu\cdots}{}_{\nu\cdots}(x)}$$

IV. Metric tensor on $\mathcal{M}$

The metric is a tensor of type (0, 2)

$$g = g_{\mu\nu} \; dx^\mu \otimes dx^\nu$$

that plays a central role in the geometric description of space-time.

* **Scalar product**: Being an application it allows as to define the product of two vectors

$$g : \; T_p \times T_p \longrightarrow \mathbb{R}$$

$$(v, w) \longrightarrow g(v, w) = g_{\mu\nu} \, v^\mu \, w^\nu$$

$$\equiv v \cdot w$$

↳ scalar product

and also the norm of a vector u^μ

$$\| u \|^2 = g(u, u) = g_{\mu\nu} \, u^\mu \, u^\nu$$

* **Line element (distance) on $\mathcal{M}$**: A particularly important vector is

$$ds = dx^\mu \, \hat{e}_\mu \quad \Rightarrow \quad \|ds\|^2 \equiv ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

which allows us to define the distance between two points (events) separated by dx^μ. More concretely

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

Line element on a general space-time $\mathcal{M}$

NOTE: Minkowski space-time (Special Relativity) is recovered if setting $g_{\mu\nu} = \eta_{\mu\nu}$.

Comments:

- The definition of variety does NOT need of a metric.
- Different metrics g, $\tilde{g}$, etc. can be asigned to the same variety

- Only when specifying a metric on $\mathcal{M}$ we endow it with a "shape", curvature, etc. Different metrics may give rise to a different curvature.

* Some properties:

- Non-singular : $\det(g_{\mu\nu}) \neq 0$

- Inverse : $(g^{-1})^{\mu\nu} \equiv g^{\mu\nu}$ so that $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$

$$\Rightarrow \det(g^{\mu\nu}) = \frac{1}{\det(g_{\mu\nu})}$$

- Symmetry : $g_{\mu\nu} = g_{\nu\mu}$; $g^{\mu\nu} = g^{\nu\mu}$

[distance from A to B = distance from B to A]

$$\Rightarrow \underbrace{V \cdot W}_{\text{"angle" between } V \text{ and } W} = \underbrace{W \cdot V}_{\text{"angle" between } W \text{ and } V} \quad \forall V, W \in T_p(\mathcal{M})$$

* Types of vectors : Let us consider a vector $V \in T_p(\mathcal{M})$

- $\|V\|^2 < 0$: Time-like
- $\|V\|^2 = 0$: Null
- $\|V\|^2 > 0$: Space-like

* Vector $\leftrightarrow$ Dual vector correspondence : The metric tensor allows us to establish a correspondence between tensors of (k,ℓ)-type and (ℓ,k)-type.

For example

$$V^\mu \longrightarrow V_\mu = g_{\mu\nu} V^\nu$$

$$V_\mu \longrightarrow V^\mu = g^{\mu\nu} V_\nu$$

so that

$$V^\mu = g^{\mu\nu} V_\nu = \underbrace{g^{\mu\nu} g_{\nu\rho}}_{\delta^\mu_\rho} V^\rho = V^\mu \qquad \text{[consistency]}$$

Then, for a (k,ℓ)-tensor, one has

$$\underbrace{T_{\mu_1 \dots \mu_k}{}^{\nu_1 \dots \nu_\ell}}_{(\ell,k)\text{-tensor}} = g_{\mu_1\rho_1} \cdots g_{\mu_k \rho_k} g^{\nu_1 \sigma_1} \cdots g^{\nu_\ell \sigma_\ell} \underbrace{T^{\rho_1 \dots \rho_k}{}_{\sigma_1 \dots \sigma_\ell}}_{(k,\ell)\text{-tensor}}$$

<u>Important</u>: The metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ allow us to lower and raise any number of indices of a tensor $T_{\mu \dots}{}^{\nu \dots}$.

v. Tensor densities on $\mathcal{M}$

In the theory of Special Relativity there are tensors, like η_{ab} and ε_{abcd}, which are invariant due to Lorentz transformations being $\Lambda \in O(1,3)$ and "proper" so that $|\Lambda| = 1$.

However, under a change of coordinate basis, one has

$$\underbrace{g_{\mu\nu}' = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}}_{(*)} \quad \text{and} \quad \left|\frac{\partial x'}{\partial x}\right| = \underbrace{\left|\frac{\partial x}{\partial x'}\right|^{-1}}_{\text{we assume } \left|\frac{\partial x'}{\partial x}\right| > 0} \neq 0$$

which translates into the existence of tensor densities transforming with some power of $\left|\frac{\partial x'}{\partial x}\right|$

$$\mathfrak{T}'^\mu{}_\nu = \left|\frac{\partial x'}{\partial x}\right|^w \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \mathfrak{T}^\rho{}_\sigma \qquad [\, w \equiv \text{"weight"} \,]$$

Ex. 1 : $|g'| = \left|\frac{\partial x}{\partial x'}\right|^2 |g|$ [from $(*)$]
[$w = -2$]

Ex. 2 : $\varepsilon^{\lambda\epsilon\psi\gamma} \dfrac{\partial x'^\mu}{\partial x^\lambda} \dfrac{\partial x'^\nu}{\partial x^\epsilon} \dfrac{\partial x'^\rho}{\partial x^\psi} \dfrac{\partial x'^\sigma}{\partial x^\gamma} \overset{\text{definition of det}}{=} \left|\dfrac{\partial x'}{\partial x}\right| \varepsilon'^{\mu\nu\rho\sigma}$
[$w = -1$]

Ex. 3 : $d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x \Rightarrow$ Fundamental theorem of integral calculus
[$w = 1$]
↳ volume element under a change of coordinate basis

Starting from tensor densities one can construct regular tensors

- $$\sqrt{|g'|}\, d^4x' = \underbrace{\left|\frac{\partial x}{\partial x'}\right| \left|\frac{\partial x'}{\partial x}\right|}_{1} \sqrt{|g|}\, d^4x = \sqrt{|g|}\, d^4x$$

$\Rightarrow$ $\sqrt{|g|}\, d^4x$ $\equiv$ Volume tensor (no indices) invariant under a change of coordinate basis

- $$\frac{1}{\sqrt{|g'|}}\, \varepsilon'^{\mu\nu\rho\sigma} = \underbrace{\left|\frac{\partial x'}{\partial x}\right| \left|\frac{\partial x}{\partial x'}\right|}_{1} \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\epsilon}} \frac{\partial x'^{\rho}}{\partial x} \frac{\partial x'^{\sigma}}{\partial x^{\tau}} \frac{1}{\sqrt{|g|}}\, \varepsilon^{\lambda\epsilon\psi\tau}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\epsilon}} \frac{\partial x'^{\rho}}{\partial x^{\psi}} \frac{\partial x'^{\sigma}}{\partial x^{\tau}} \frac{1}{\sqrt{|g|}}\, \varepsilon^{\lambda\epsilon\psi\tau}$$

$\Rightarrow$ $\frac{1}{\sqrt{|g|}}\, \varepsilon^{\mu\nu\rho\sigma}$ is a tensor and not a density

For a tensor density $\mathcal{T}'^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}$ of weight w one finds

$$\left(\sqrt{|g'|}\right)^{w} \mathcal{T}'^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\right) \cdots \left(\frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}}\right) \left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\right) \cdots \left(\frac{\partial x^{\sigma_m}}{\partial x'^{\nu_m}}\right) \left(\sqrt{|g|}\right)^{w} \mathcal{T}^{\rho_1 \ldots \rho_n}{}_{\sigma_1 \ldots \sigma_m}$$

$\Rightarrow$ $\left(\sqrt{|g|}\right)^{w} \mathcal{T}^{\rho_1 \ldots \rho_n}{}_{\sigma_1 \ldots \sigma_m}$ is a tensor and not a density

V. Locally Inertial Frames (LIF)

Manifold: a manifold is a topological space that "locally resembles $\mathbb{R}^n$" where n is the dimension of the manifold.

This statement can now be made more precise ...

Since the metric tensor $g_{\mu\nu}$ is symmetric, it can always be "locally" (point by point) diagonalised and brought into a canonical form η_{ab}. This is

$$\boxed{g_{\mu\nu}(x) = e_\mu{}^a(x)\, e_\nu{}^b(x)\, \eta_{ab}}$$

[at each point x]

with

$$\boxed{\eta_{ab} = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)}$$

Signature: It is defined as the number of $-$ and $+$ signs.

$\Rightarrow$ Our space-time has signature $(1,3)$

Metrics are classified according to their signature:

- $\eta_{ab} = \delta_{ab}$ => Riemannian or Euclidean
- If η_{ab} contains one and only one − sign => Pseudo-Riemannian or Lorentzian [Minkowski space-time]

Then

$$\text{sign}\left[\det(g_{\mu\nu})\right] = (-1)^{s}$$

of − signs in the signature [Minkowski : s=1]

* LIF : The local change of coordinate basis is encoded into the "frames" or "tetrads" or "vierbein"

$$e_\mu{}^a(x) \quad , \quad e_a{}^\mu(x)$$

These fields specify a Local Inertial Frame (LIF) coordinate basis

$$\hat{\theta}^a = e_\mu{}^a(x)\, dx^\mu \quad , \quad dx^\mu = e_a{}^\mu(x)\, \hat{\theta}^a$$

Important: $e_\mu{}^a(x)$ and $e_a{}^\mu(x)$ depend on the point x !!

so that

$$g^{\mu\nu} e_\mu{}^a e_\nu{}^b = \eta^{ab} \quad , \qquad \eta^{ab} e_a{}^\mu e_b{}^\nu = g^{\mu\nu}$$

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab} \quad , \qquad \eta_{ab} e_\mu{}^a e_\nu{}^b = g_{\mu\nu}$$

As a result one has that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \underbrace{g_{\mu\nu} e_a{}^\mu e_b{}^\nu}_{\eta_{ab}} \hat{\theta}^a \hat{\theta}^b = \eta_{ab} \hat{\theta}^a \hat{\theta}^b$$

Message : There is a coordinate basis change bringing the geometry locally flat. Equivalently, there is a LIF

$$\hat{\theta}^a = e_\mu{}^a(x)\, dx^\mu$$

in which the metric is the Mkw η_{ab} metric.

Q : Is there a general coordinate transformation (g.c.t)

$$\xi^a = \xi^a(x)$$

flattening the geometry ?

A1 : Locally YES

A2 : Globally NO [otherwise : geometry = coordinate artefact]

Justification : If such a g.c.t $\xi^a = \xi^a(x)$ existed globally then

$$\underbrace{\hat{\theta}^a}_{d\xi^a} = e_\mu{}^a \, dx^\mu \implies e_\mu{}^a(x) = \frac{\partial \xi^a}{\partial x^\mu}$$

Applying Schwarz theorem

$$\frac{\partial^2 \xi^a}{\partial x^\nu \partial x^\mu} = \frac{\partial^2 \xi^a}{\partial x^\mu \partial x^\nu} \implies \frac{\partial e_\mu{}^a}{\partial x^\nu} = \frac{\partial e_\nu{}^a}{\partial x^\mu}$$

which is not true for a general geometry at any x.

* Local form of $g_{\mu\nu}$: It is not possible to find a global LIF, but at each $p \in \mathcal{M}$, it is possible to find local coordinates ξ^a such that

$$g_{\mu\nu}(\xi) = \eta_{\mu\nu} + O(\xi^2)$$

so that

$$\boxed{\begin{aligned} g_{\mu\nu}(p) &= \eta_{\mu\nu} \\ \partial_\rho g_{\mu\nu}(p) &= 0 \\ \partial_\rho \partial_\sigma g_{\mu\nu}(p) &\neq 0 \end{aligned}}$$

$\Rightarrow$ The metric differs from Mkw only at quadratic order when using these local coordinates ξ^a.

* Local Lorentz transformations : Let us recall that frame fields $e_\mu{}^a(x)$ are defined by the relation

$$g_{\mu\nu}(x) = e_\mu{}^a(x)\, e_\nu{}^b(x)\, \eta_{ab}$$

Then it is easy to see that local redefinition of the frame fields of the form

$$e_\mu{}^a(x) \longrightarrow \tilde{e}_\mu{}^a(x) = \underbrace{\Lambda^a{}_c(x)}\, e_\mu{}^c(x)$$

$\Lambda^a{}_c(x) \in SO(1,3)$

$\hookrightarrow$ Local !!

yields the same metric

$$\tilde{g}_{\mu\nu} = \tilde{e}_\mu{}^a\, \tilde{e}_\nu{}^b\, \eta_{ab} = \underbrace{\Lambda^a{}_c\, \Lambda^b{}_d\, \eta_{ab}}_{\eta_{cd}}\, e_\mu{}^c\, e_\nu{}^d = g_{\mu\nu}$$

Important : There is a local (gauge) ambiguity when defining $e_\mu{}^a(x)$:

$$\Lambda^a{}_b(x) = e^{\frac{1}{2}\theta^{cd}(x)\,[M_{cd}]^a{}_b}$$

with $\theta^{cd}(x) = -\theta^{dc}(x)$ being 6 independent functions ⟹ 6 gauge fields $\underbrace{\omega_\mu{}^{cd}(x)}_{\text{spin connection}}$ to define a covariant derivative !!

Comment : This is a gauge theory approach to Gravity. We will not develop it any further here. Instead, we will move to a more geometrical approach.