I. Postulates and general coordinate transformations

There are two complementary approaches to discuss the principles of General Relativity :

1) The Mathematics approach :

Idea 1 : The existence of a LIF is the math way of saying that a space-time locally looks like Minkowski.
[Equivalence principle]

Idea 2 : If you know the laws of Physics in a LIF the you know them in a general space-time.
$\approx>$ Tensorial equations are the same in any coordinate basis [Principle of general covariance]

2) The Physics approach :

- Weak Equivalence Principle : let us consider a particle with charge q in an electric potential V. Then

$$\vec{F} = -q\,\vec{\nabla} V = \underbrace{m_I}_{\text{inertial mass}}\,\vec{a} \implies \vec{a} = -\frac{q}{m_I}\,\vec{\nabla} V$$

Let us consider now the same particle in a gravitational field Φ

$$\vec{F} = -\underbrace{m_G}_{\text{gravitational mass}} \vec{\nabla}\Phi = m_I \vec{a} \Rightarrow \vec{a} = -\underbrace{\frac{m_G}{m_I}}_{\text{Experimentally: } \frac{m_G}{m_I} = 1} \vec{\nabla}\Phi$$

$\Rightarrow \vec{a} = -\nabla\vec{\Phi} \Rightarrow$ Motion of a particle in a gravitational field is independent of its mass (WEP)

- Equivalence Principle: Laws of Nature cannot distinguish between a gravitational field and an observer uniformly accelerated.

Principles of GR: The laws of Nature are covariant under general coordinate transformations (g.c.t)

$\Rightarrow$ General covariance ... but ... what determines the laws of gravity? ... the local Lorentz symmetry!!

[the equivalence principle]

$$\boxed{\text{Gravity} = \text{Geometry}}$$

* Vectors and tensors: Physical quantities are well-defined objects (tensor) in the space-time manifold $\mathcal{M}$

- Contravariant vectors V^μ
- Covariant vectors V_μ

$$\left.\begin{aligned} V_\mu &= g_{\mu\nu} V^\nu \\ V^\mu &= g^{\mu\nu} V_\nu \end{aligned}\right.$$

with $\underbrace{g^{\mu\sigma}}_{\text{inverse metric}} \underbrace{g_{\sigma\rho}}_{\text{metric}} = \delta^\mu_\rho$

Under a g.c.t one has

$$V'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} V_{\nu} \quad \text{and} \quad V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$$

so that full contractions are g.c.t-invariant quantities

$$\underbrace{u'^{\mu} V'_{\mu}}_{V(u)} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} u^{\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} V_{\rho} = \delta^{\rho}_{\nu} u^{\nu} V_{\rho} = \underbrace{u^{\rho} V_{\rho}}_{V(u)}$$

- **Mixed tensors** : General tensors can be constructed transforming multilinearly under g.c.t. They satisfy
  - Linearity : $T^{\mu}{}_{\nu} \equiv \alpha \underbrace{R^{\mu}{}_{\nu}}_{\text{tensor}} + \beta \underbrace{S^{\mu}{}_{\nu}}_{\text{tensor}}$ is a tensor.
  - Direct product : $T_{\mu\nu}{}^{\rho} \equiv \underbrace{A_{\mu}}_{\text{tensor}} \underbrace{B_{\nu}{}^{\rho}}_{\text{tensor}}$ is a tensor $[A_{\mu} \neq \partial_{\mu}]$ IMPORTANT !!
  - Contraction : $T^{\mu\nu} \equiv \underbrace{T^{\mu}{}_{\rho}{}^{\nu\rho}}_{\text{tensor}}$ is a tensor.

II. Covariant derivatives and parallel transport

Unlike for special relativity $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and V^ν are tensors but $\partial_\mu V^\nu$ is **NOT** a tensor

$$\frac{\partial}{\partial x'^\mu} V'^\nu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} \left[\frac{\partial x'^\nu}{\partial x^\sigma} V^\sigma \right]$$

SR: $x'^a = \overset{\text{constant matrix}}{\Lambda^a{}_b}\, x^b$ => no problem !!

$$= \underbrace{\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial}{\partial x^\rho} V^\sigma}_{\text{like a regular tensor}} + \underbrace{\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} V^\sigma}_{\text{bad term !!}}$$

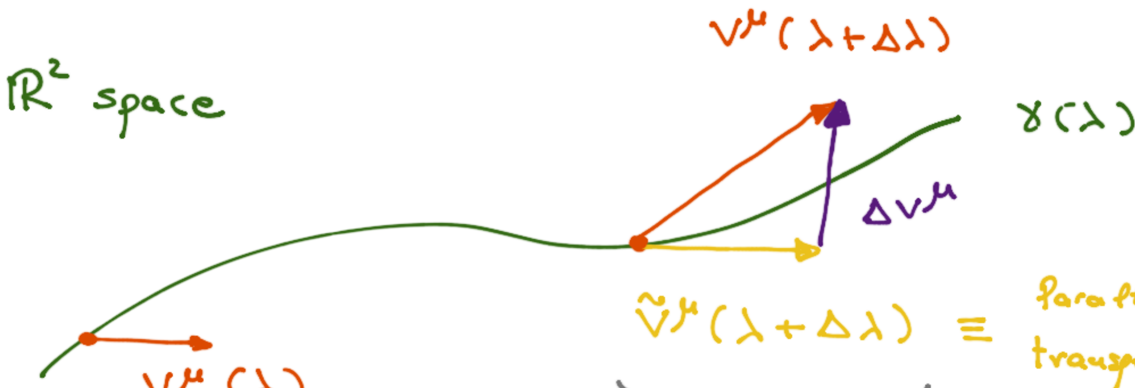
=> We need to change ∂_μ by a "covariant" derivative ∇_μ such that

$$\underbrace{(\nabla_\mu V^\nu)'}_{T'_\mu{}^\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\lambda} \underbrace{(\nabla_\rho V^\lambda)}_{T_\rho{}^\lambda}$$

Q: How do we find an appropriate ∇_μ operation that encodes the "transportation" of objects in a curved space time ?

A: Let us first understand the same question in flat space-time and then covariantise the result in light of the principle of general covariance.

* Parallel transport and $\frac{d}{d\lambda}$ in flat space-time: Let us consider a vector $V^\mu(\lambda)$ along a curve $\gamma(\lambda)$ in $\mathbb{R}^2$



$$\underbrace{\tilde{V}^\mu(\lambda+\Delta\lambda)}_{\substack{\text{to be able to} \\ \text{compare with} \\ V^\mu(\lambda+\Delta\lambda)}} \equiv \text{parallel transported vector}$$

Important: In $\mathbb{R}^2$ we know that

$$\tilde{V}^\mu(\lambda+\Delta\lambda) = V^\mu(\lambda)$$

and then

$$\lim_{\Delta\lambda \to 0} \frac{\Delta V^\mu}{\Delta\lambda} = \frac{V^\mu(\lambda+\Delta\lambda) - \tilde{V}^\mu(\lambda+\Delta\lambda)}{\Delta\lambda}$$

In $\mathbb{R}^2$

$$= \frac{V^\mu(\lambda+\Delta\lambda) - V^\mu(\lambda)}{\Delta\lambda}$$

this is how we define $\frac{d}{d\lambda}$ in $\mathbb{R}^2$

$$\equiv \frac{dV^\mu}{d\lambda} = \underbrace{\frac{\partial V^\mu}{\partial x^\nu}}_{\partial_\nu V^\mu \,\equiv\, \text{ordinary derivative}} \frac{dx^\nu}{d\lambda}$$

Important: If the transported vector $\tilde{V}^\mu(\lambda+\Delta\lambda)$ coincides with $V^\mu(\lambda+\Delta\lambda)$ then

$$\frac{dV^\mu}{d\lambda} = 0 \Rightarrow$$ Parallely-transported vector !!

* Parallel transport and $\frac{D}{D\lambda}$ in curved space-times:

Let us consider a vector $V^\mu(\lambda)$ along a curve $\gamma(\lambda)$ in a curve space time. We then introduce a covariant derivative ∇_μ as

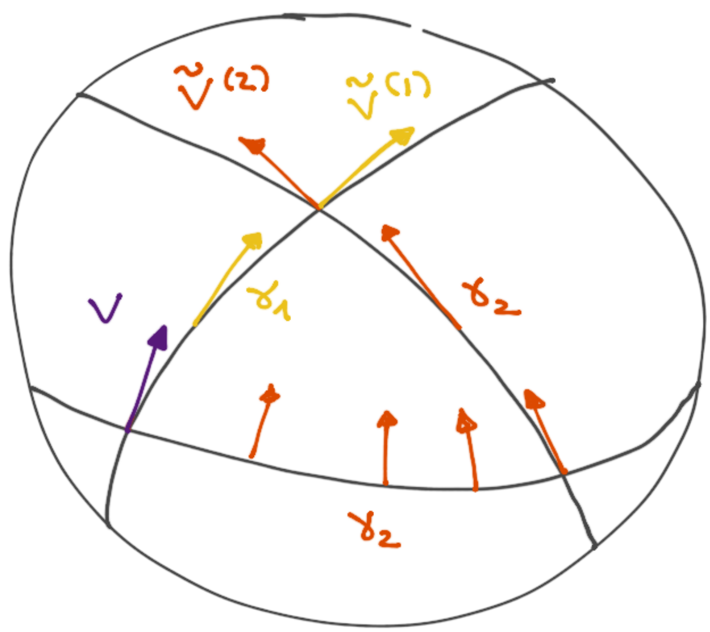
change of a vector V^μ along a curve $\Rightarrow$ $$\frac{DV^\mu}{D\lambda} = \underbrace{\nabla_\nu V^\mu}_{\text{covariant derivative}} \frac{dx^\nu}{d\lambda}$$

If

$$\frac{DV^\mu}{D\lambda} = 0 \Rightarrow \text{Parallely-transported vector !!}$$

NOTE: A curved space-time makes parallel transport non-trivial (it depends on the curve γ followed)

S^2 - space



* Covariant derivative : As for the ordinary derivative in flat space-time, the covariant derivative in a curved space-time must satisfy :

- ∇ : (k,l) - tensor $\Rightarrow$ $(k, l+1)$ - tensor

- Linearity : $\nabla_\mu(\alpha T + \beta S) = \alpha \nabla_\mu T + \beta \nabla_\mu S$

  with $T, S \in$ tensors and $\alpha, \beta \in \mathbb{R}$

- Leibniz rule : $\nabla_\mu(T \otimes S) = \nabla_\mu T \otimes S + T \otimes \nabla_\mu S$

- Compatibility with index contraction :

$$\nabla_\mu \big(\underbrace{\delta^\rho_\sigma \, T_\rho{}^\sigma}_{T_\sigma{}^\sigma} \big) = \delta^\rho_\sigma \, \nabla_\mu T_\rho{}^\sigma$$

$$\Rightarrow \quad \nabla_\mu \delta^\rho_\sigma = 0$$

↳ Leibniz

- Compatibility with the derivative of a function $f \in \mathcal{F}(M)$

  We saw that

$$\frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu f$$

Now we say that, for a $(0,0)$-tensor f

$$\frac{Df}{D\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu f$$

Then, for scalar [(0,0)-tensor] functions, one has

$$\nabla_\mu f = \partial_\mu f$$

After having listed the various properties that a covariant derivative ∇_μ must satisfy, we can provide an explicit expression for it.

- Linearity :

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \underbrace{(\Gamma_\mu)_\rho{}^\nu}_{\text{local (gauge) "rotation"}} V^\rho$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu + \underbrace{(\tilde{\Gamma}_\mu)_\nu{}^\rho}_{\text{in principle different from } \Gamma} V_\rho$$

- Leibniz :

$$\nabla_\mu \underbrace{(T^\nu S_\rho)}_{T^\nu{}_\rho} = (\nabla_\mu T^\nu) S_\rho + T^\nu (\nabla_\mu S_\rho)$$

$$= (\partial_\mu T^\nu + \Gamma_{\mu\lambda}{}^\nu T^\lambda) S_\rho + T^\nu (\partial_\mu S_\rho + \tilde{\Gamma}_{\mu\rho}{}^\lambda S_\lambda)$$

$$= \partial_\mu (T^\nu S_\rho) + \Gamma_{\mu\lambda}{}^\nu T^\lambda S_\rho + T^\nu \tilde{\Gamma}_{\mu\rho}{}^\lambda S_\lambda$$

so that

$$\nabla_\mu T^\nu{}_\rho = \partial_\mu T^\nu{}_\rho + \Gamma_{\mu\lambda}{}^\nu T^\lambda{}_\rho + \tilde{\Gamma}_{\mu\rho}{}^\lambda T^\nu{}_\lambda$$

- Compatibility with index contraction and with the derivative of a function $f \in \mathcal{F}(\mathcal{M})$:

$$\nabla_\mu \underbrace{T^\rho{}_\rho}_{\in \mathcal{F}(\mathcal{M})} = \delta_\rho^\sigma \nabla_\mu T^\rho{}_\sigma$$

$$= \delta_\rho^\sigma \left(\partial_\mu T^\rho{}_\sigma + \Gamma_{\mu\lambda}{}^\rho T^\lambda{}_\sigma + \tilde{\Gamma}_{\mu\sigma}{}^\lambda T^\rho{}_\lambda \right)$$

$$\Downarrow \! |||$$

$$\partial_\mu T^\rho{}_\rho = \partial_\mu T^\rho{}_\sigma + \Gamma_{\mu\lambda}{}^\rho T^\lambda{}_\rho + \underbrace{\tilde{\Gamma}_{\mu\rho}{}^\lambda T^\rho{}_\lambda}_{\tilde{\Gamma}_{\mu\lambda}{}^\rho T^\lambda{}_\rho}$$

$$\Rightarrow \left(\Gamma_{\mu\lambda}{}^\rho + \tilde{\Gamma}_{\mu\lambda}{}^\rho \right) \underbrace{T^\lambda{}_\rho}_{\text{arbitrary tensor}} = 0$$

$$\Rightarrow \tilde{\Gamma}_{\mu\lambda}{}^\rho = - \Gamma_{\mu\lambda}{}^\rho$$

Important: $\Gamma_{\mu\nu}{}^\lambda \equiv$ Connection is NOT a tensor under g.c.t !!

Therefore, the covariant derivative ∇_μ has the following action on tensors :

- $\nabla_\mu f = \partial_\mu f$
- $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}{}^\nu V^\lambda$
- $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^\lambda V_\lambda$

For a (k,l)-tensor one has the multi-linear action

$$\begin{aligned}\nabla_\mu T^{\nu_1 \dots \nu_k}{}_{\rho_1 \dots \rho_l} = \; & \partial_\mu T^{\nu_1 \dots \nu_k}{}_{\rho_1 \dots \rho_l} \\ & + \Gamma_{\mu\lambda}{}^{\nu_1} T^{\lambda \dots \nu_k}{}_{\rho_1 \dots \rho_l} + \dots \\ & - \Gamma_{\mu\rho_1}{}^{\lambda} T^{\nu_1 \dots \nu_k}{}_{\lambda \dots \rho_l} - \dots\end{aligned}$$

* Torsion : It is a tensor $T_{\mu\nu}{}^\lambda$ defined as

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] f &= \nabla_\mu (\nabla_\nu f) - \nabla_\nu (\nabla_\mu f) \\ &= \nabla_\mu (\partial_\nu f) - \nabla_\nu (\partial_\mu f) \\ &= \partial_\mu \partial_\nu f - \Gamma_{\mu\nu}{}^\lambda \partial_\lambda f - \partial_\nu \partial_\mu f + \Gamma_{\nu\mu}{}^\lambda \partial_\lambda f\end{aligned}$$

$$= -\left(\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda}\right) \underbrace{\partial_\lambda f}_{\nabla_\lambda f}$$

$$\equiv -T_{\mu\nu}{}^{\lambda} \nabla_\lambda f$$

so that

$$T_{\mu\nu}{}^{\lambda} = \Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda}$$

Important : Torsionless connection $\Leftrightarrow$ $\Gamma_{\mu\nu}{}^{\lambda} = \Gamma_{\nu\mu}{}^{\lambda}$

[$\mu \leftrightarrow \nu$ symmetric]

* Transformation properties of $\Gamma_{\mu\nu}{}^{\lambda}$ under g.c.t :

If the covariant derivative creates a $(k, \ell+1)$-tensor when acting on a (k,ℓ)-tensor, the connection $\Gamma_{\mu\nu}{}^{\lambda}$ must transform in a very precise manner under a g.c.t. Let us focus on our original example

$$\underbrace{(\nabla_\mu V^\nu)'}_{T'_\mu{}^\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\lambda} \underbrace{(\nabla_\rho V^\lambda)}_{T_\rho{}^\lambda}$$

and work out explicitly what $\Gamma'_{\mu\nu}{}^{\lambda}$ must be.

$$(\nabla_\mu V^\nu)' = \nabla'_\mu V'^\nu = \underbrace{\partial'_\mu}_{\text{tensor}}\ \underbrace{V'^\nu}_{\text{tensor}} + \Gamma'_{\mu\lambda}{}^\nu\ \underbrace{V'^\lambda}_{\text{tensor}}$$

$$= \frac{\partial x^\rho}{\partial x'^\mu}\,\partial_\rho\left(\frac{\partial x'^\nu}{\partial x^\sigma}V^\sigma\right) + \Gamma'_{\mu\lambda}{}^\nu\,\frac{\partial x'^\lambda}{\partial x^\rho}V^\rho$$

$$= \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\sigma}\,\partial_\rho V^\sigma$$

$$+ \Gamma'_{\mu\lambda}{}^\nu\,\frac{\partial x'^\lambda}{\partial x^\rho}V^\rho$$

must be equal → $$\overset{!}{=} \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\lambda}\,(\nabla_\rho V^\lambda)$$

$$= \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\lambda}\,\partial_\rho V^\lambda + \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\lambda}\,\Gamma_{\rho\sigma}{}^\lambda V^\sigma$$

Then (renaming $\rho \to \sigma$ and $\lambda \to \tau$ in the second / third terms)

$$\left(\frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} + \Gamma'_{\mu\lambda}{}^\nu\,\frac{\partial x'^\lambda}{\partial x^\sigma}\right)\underset{\text{arbitrary}}{V^\sigma} = \left(\frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\tau}\,\Gamma_{\rho\sigma}{}^\tau\right)\underset{\text{arbitrary}}{V^\sigma}$$

and multiplying by $\frac{\partial x^\sigma}{\partial x'^\psi}$ we get (for arbitrary V^σ)

$$\Gamma'_{\mu\psi}{}^\nu = \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\psi}\,\frac{\partial x'^\nu}{\partial x^\tau}\,\Gamma_{\rho\sigma}{}^\tau - \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\psi}\,\frac{\partial^2 x'^\nu}{\partial x^\rho\,\partial x^\sigma}}_{\text{bad term !!}}$$

[independent of $\Gamma_{\mu\nu}{}^{\sigma}$]

Important: Since the bad term is independent of the connection chosen, one has that

$$\delta\Gamma_{\mu\nu}{}^{\sigma} \equiv \Gamma_{\mu\nu}{}^{\sigma} - \hat{\Gamma}_{\mu\nu}{}^{\sigma}$$

is a tensor $\Rightarrow$ the difference between two connections is a tensor although each connection is NOT !!

* Geodesic : The equation of parallel transport of a vector V^μ along a curve $x^\sigma(\lambda)$

$$\frac{DV^\mu}{D\lambda} = 0 \quad \Rightarrow \quad \frac{DV^\mu}{D\lambda} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma V^\mu$$

$$= \frac{dx^\sigma}{d\lambda}\left(\partial_\sigma V^\mu + \Gamma_{\sigma\rho}{}^{\mu} V^\rho\right)$$

$$= \frac{dV^\mu}{d\lambda} + \Gamma_{\sigma\rho}{}^{\mu} \frac{dx^\sigma}{d\lambda} V^\rho = 0$$

geodesic $\equiv$ Curve that transports its tangent vector $V^\mu = \dfrac{dx^\mu}{d\lambda}$ parallelly. This is a curve $x^\mu(\lambda)$ satisfying

the geodesic equation :

Trajectories of free particles $\Leftarrow\equiv$ $$\boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}{}^{\mu}\,\frac{dx^\nu}{d\lambda}\,\frac{dx^\rho}{d\lambda} = 0}$$

Example : Flat space time $\Rightarrow \Gamma_{\nu\rho}{}^{\mu} = 0$

$$\Rightarrow \frac{d^2x^\mu}{d\lambda^2} = 0$$

$$\Rightarrow x^\mu(\lambda) = a^\mu \lambda + b^\mu$$

[straight lines]

NOTE : Denoting $\frac{dx^\mu}{d\lambda} \equiv \dot{x}^\mu$, the geodesic equation can be written as $\dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0$

NOTE : The geodesic equation can be obtained as the Euler–Lagrange equations following from the action

$$S[x^\mu(\lambda)] = -m\int\sqrt{-ds^2} = -m\int\sqrt{-g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu}\;d\lambda$$

of a free particle of mass m in a space-time $\mathcal{M}$.

Q : Can we provide an explicit form of the connection $\Gamma_{\mu\nu}{}^{\lambda}$ in terms of the metric $g_{\mu\nu}$?

III. Metric postulate and Christoffel connection

Let us consider a manifold $\mathcal{M}$ endowed with a metric $g_{\mu\sigma}$. We say that a covariant derivative ∇_μ is compatible with the metric if

$$\nabla_\mu g_{\sigma\rho} = 0 \qquad \text{“Metric postulate”}$$

As a consequence

$$\nabla_\mu T_\sigma = \nabla_\mu (g_{\sigma\rho} T^\rho) = \overbrace{(\nabla_\mu g_{\sigma\rho})}^{0} T^\rho + g_{\sigma\rho} (\nabla_\mu T^\rho)$$
$$= g_{\sigma\rho} (\nabla_\mu T^\rho)$$

$\approx>$ Raising / lowering indices commutes with taking a covariant derivative

$*$ Christoffel connection : Let us write the metric postulate in three different fashions

$$\text{i)} \quad \nabla_\lambda g_{\mu\sigma} = \partial_\lambda g_{\mu\sigma} - \Gamma_{\lambda\mu}{}^\rho g_{\rho\sigma} - \Gamma_{\lambda\sigma}{}^\rho g_{\mu\rho} = 0$$
$$\text{ii)} \quad \nabla_\mu g_{\sigma\lambda} = \partial_\mu g_{\sigma\lambda} - \Gamma_{\mu\sigma}{}^\rho g_{\rho\lambda} - \Gamma_{\mu\lambda}{}^\rho g_{\sigma\rho} = 0$$
$$\text{iii)} \quad \nabla_\sigma g_{\lambda\mu} = \partial_\sigma g_{\lambda\mu} - \Gamma_{\sigma\lambda}{}^\rho g_{\rho\mu} - \Gamma_{\sigma\mu}{}^\rho g_{\lambda\rho} = 0$$

and compute the combination i) - ii) - iii)

$$\partial_\lambda g_{\mu\sigma} - \partial_\mu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\mu} - (\Gamma_{\lambda\mu}{}^{\rho} - \Gamma_{\mu\lambda}{}^{\rho})\, g_{\rho\sigma}$$
$$- (\Gamma_{\lambda\sigma}{}^{\rho} - \Gamma_{\sigma\lambda}{}^{\rho})\, g_{\mu\rho}$$
$$+ (\Gamma_{\mu\sigma}{}^{\rho} + \Gamma_{\sigma\mu}{}^{\rho})\, g_{\rho\lambda} = 0$$

If we have a torsion-less connection $[\Gamma_{\mu\sigma}{}^{\lambda} = \Gamma_{\sigma\mu}{}^{\lambda}]$ the above relation reduces to

$$\partial_\lambda g_{\mu\sigma} - \partial_\mu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\mu} = -2\, \Gamma_{\mu\sigma}{}^{\rho}\, g_{\rho\lambda}$$

Multiplying in both sides by $g^{\lambda\sigma}$ we arrive at

$$g^{\sigma\lambda}\left[\partial_\lambda g_{\mu\sigma} - \partial_\mu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\mu}\right] = -2\, \Gamma_{\mu\sigma}{}^{\sigma}$$

and therefore

First-derivatives of $g_{\mu\sigma} \Rightarrow \Gamma_{\mu\sigma}{}^{\rho} = 0$ in L.I.F

$$\boxed{\Gamma_{\mu\sigma}{}^{\sigma} = \frac{1}{2}\, g^{\sigma\lambda}\left[\partial_\mu g_{\sigma\lambda} + \partial_\sigma g_{\lambda\mu} - \partial_\lambda g_{\mu\sigma}\right]}$$

$$\Rightarrow \Gamma_{\mu\sigma}{}^{\sigma} = \Gamma_{\mu\sigma}{}^{\sigma}(g) \equiv \text{“Christoffel connection”}$$

- Metric compatible
- Torsion-less

NOTE: $\underbrace{\frac{n(n+1)}{2}}_{\mu\leftrightarrow\sigma \text{ symmetric}} \times n$ indep. components.

Important: There is a prescription to write physical laws in a theory of gravity. Write the laws in Minkowski space time (no gravity) and replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$.

NOTE: For scalar quantities (no indices) one has $S' = S$ and therefore $\nabla_\mu S = \partial_\mu S$. The Lagrangian describing a theory of gravity + particles is a scalar quantity and thus invariant under g.c.t (up to boundary terms).

* Covariant gradient, curl and divergence

- Covariant gradient of a scalar function : $\nabla_\mu S = \dfrac{\partial S}{\partial x^\mu}$
- Covariant curl of a (covariant) vector :

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma_{\mu\nu}{}^{\lambda} A_\lambda - \partial_\nu A_\mu + \underbrace{\Gamma_{\nu\mu}{}^{\lambda}}_{\Gamma_{\nu\mu}{}^{\lambda} = \Gamma_{\mu\nu}{}^{\lambda}} A_\lambda$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(as in flat space-time)}$$

- Covariant divergence of a (contravariant) vector :

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma_{\mu\lambda}{}^{\mu} V^\lambda = \partial_\mu V^\mu + \tfrac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho} V^\lambda$$

NOTE: $$\Gamma_{\mu\lambda}{}^{\mu} = \frac{1}{2} \underbrace{g^{\mu\rho}}_{\substack{\text{symmetric} \\ (\mu\rho)}} \left[\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda} \right] = \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho}$$

↳ antisymmetric $[\mu\rho]$ and

NOTE: Let us consider a generic metric $M(x)$

$$\delta \operatorname{Ln}[\operatorname{Det} M] \equiv \operatorname{Ln}[\operatorname{Det}(M+\delta M)] - \operatorname{Ln}[\operatorname{Det} M]$$

$\operatorname{Ln} a - \operatorname{Ln} b = \operatorname{Ln}\left[\frac{a}{b}\right]$

$$= \operatorname{Ln}\left[\frac{\operatorname{Det}(M+\delta M)}{\operatorname{Det} M}\right] = \operatorname{Ln}\left[(\operatorname{Det} M^{-1}) \operatorname{Det}(M+\delta M)\right]$$

$$= \operatorname{Ln}\left[\operatorname{Det}\left[M^{-1}(M+\delta M)\right]\right] = \operatorname{Ln}\Big[\operatorname{Det}\underbrace{(\mathbb{I}+M^{-1}\delta M)}_{e^A}\Big]$$

Jacobi's formula

$$= \operatorname{Ln}\left[e^{\operatorname{Tr}\operatorname{Ln}(\mathbb{I}+M^{-1}\delta M)}\right] = \operatorname{Tr}\operatorname{Ln}(\mathbb{I}+M^{-1}\delta M)$$

$\operatorname{Ln}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$\approx \operatorname{Tr}(M^{-1}\delta M) + \dots$$

subnote: Jacobi's formula: $\det[e^A] = e^{\operatorname{Tr} A}$

Taking $\delta M = \frac{\partial M}{\partial x^\lambda}\delta x^\lambda$ one arrives at

$$\frac{\partial}{\partial x^\lambda}\operatorname{Ln}\operatorname{Det} M = \operatorname{Tr}\left[M^{-1}\frac{\partial}{\partial x^\lambda}M\right]$$

Therefore, the covariant divergence is given by

$$D_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{2}\overbrace{g^{\mu\rho}\partial_\lambda g_{\mu\rho}}^{\operatorname{Tr}(M^{-1}\partial_\lambda M)} V^\lambda = \partial_\mu V^\mu + \underbrace{\frac{1}{2}\frac{\partial}{\partial x^\lambda}(\operatorname{Ln}|g|)}_{\Gamma_{\mu\lambda}{}^{\mu}} V^\lambda$$

with

$$\Gamma_{\mu\lambda}{}^{\mu} = \frac{1}{2}\partial_\lambda \operatorname{Ln}|g| = \partial_\lambda \operatorname{Ln}(\sqrt{|g|}) = \frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|})$$

This is

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\right) V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|}\, V^\mu\right]$$

* Boundary terms in a space-time without torsion :

Let us consider the covariant divergence of a vector V^μ. Then

$$\underbrace{\int d^4x \sqrt{|g|}}_{\substack{\text{volume element} \\ \text{invariant under g.c.t}}} \nabla_\mu V^\mu = \int d^4x \sqrt{|g|} \frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|}\, V^\mu\right] \equiv \text{Boundary term}$$

NOTE : Stoke's theorem : $\displaystyle\int_U \vec{\partial}\cdot\vec{F}\, dV = \int_{\partial U} \vec{F}\cdot\vec{n}\, dS$ ← normal unit vector

Let us consider the covariant divergence of a tensor $F^{\mu\nu}$. Then

$$\nabla_\mu F^{\mu\nu} = \underbrace{\frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|}\, F^{\mu\nu}\right]}_{\text{contracted index}} + \underbrace{\Gamma_{\mu\lambda}{}^{\nu}}_{\Gamma_{\mu\lambda}{}^{\nu} = \Gamma_{\lambda\mu}{}^{\nu} \text{ [no torsion]}} F^{\mu\lambda}$$

if $F^{\mu\nu} = -F^{\nu\mu}$ one has that

$$\int d^4x \sqrt{|g|}\, \nabla_\mu F^{\mu\nu} \equiv \text{Boundary term}$$

Important : Boundary terms can be added to an action without modifying the equations of motion .

IV. Curvature and Riemann tensor

In a curved space-time, covariant derivatives do NOT commute

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \underbrace{(\nabla_\nu V^\rho)}_{T_\nu{}^\rho} - \nabla_\nu \underbrace{(\nabla_\mu V^\rho)}_{T_\mu{}^\rho}$$

$$= \nabla_\mu (\partial_\nu V^\rho + \Gamma_{\nu\lambda}{}^\rho V^\lambda) - (\mu \leftrightarrow \nu)$$

$$= \underbrace{\partial_\mu (\partial_\nu V^\rho}_{\text{symmetric } \mu \leftrightarrow \nu} + \Gamma_{\nu\lambda}{}^\rho V^\lambda) - \underbrace{\Gamma_{\mu\nu}{}^\gamma}_{\text{symmetric } \mu \leftrightarrow \nu \text{ (no torsion)}} (\partial_\gamma V^\rho + \Gamma_{\gamma\lambda}{}^\rho V^\lambda)$$

$$+ \Gamma_{\mu\gamma}{}^\rho (\partial_\nu V^\gamma + \Gamma_{\nu\lambda}{}^\gamma V^\lambda)$$

$$- (\mu \leftrightarrow \nu)$$

$$= (\partial_\mu \Gamma_{\nu\lambda}{}^\rho) V^\lambda + \overbrace{\Gamma_{\nu\lambda}{}^\rho (\partial_\mu V^\lambda) + \Gamma_{\mu\gamma}{}^\rho (\partial_\nu V^\gamma)}^{\text{symmetric } \mu \leftrightarrow \nu}$$

$$+ \Gamma_{\mu\gamma}{}^\rho \Gamma_{\nu\lambda}{}^\gamma V^\lambda - (\mu \leftrightarrow \nu)$$

$$= \left[\partial_\mu \Gamma_{\nu\lambda}{}^\rho - \partial_\nu \Gamma_{\mu\lambda}{}^\rho + \Gamma_{\mu\gamma}{}^\rho \Gamma_{\nu\lambda}{}^\gamma - \Gamma_{\nu\gamma}{}^\rho \Gamma_{\mu\lambda}{}^\gamma \right] V^\lambda$$

$$\equiv \underbrace{R_{\mu\nu}{}^\rho{}_\lambda}_{\text{“Riemann-Christoffel tensor”}} V^\lambda$$

Then we arrive at the result

$$[\nabla_\mu, \nabla_\nu]\, V^\rho = R_{\mu\nu}{}^\rho{}_\lambda\, V^\lambda$$

with

$$R_{\mu\nu}{}^\rho{}_\lambda = \partial_\mu \Gamma_{\nu\lambda}{}^\rho - \partial_\nu \Gamma_{\mu\lambda}{}^\rho + \Gamma_{\mu\tau}{}^\rho \Gamma_{\nu\lambda}{}^\tau - \Gamma_{\nu\tau}{}^\rho \Gamma_{\mu\lambda}{}^\tau$$

The Riemann tensor $R_{\mu\nu}{}^\rho{}_\sigma$ is a proper tensor. Then, under a g.c.t. it transforms as

$$R'_{\mu\nu}{}^\rho{}_\sigma = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\epsilon}{\partial x'^\nu} \frac{\partial x'^\rho}{\partial x^\psi} \frac{\partial x^\tau}{\partial x'^\sigma} R_{\lambda\epsilon}{}^\psi{}_\tau$$

Important: The Riemann tensor is constructed from the metric $g_{\mu\nu}$ and its first and second derivatives.

Physical meaning: If we are given a space-time metric $g_{\mu\nu}(x)$, how do we know if there is a non-trivial gravitational field or, on the contrary, there are special coordinates $\xi^a(x)$ such that

$$\text{Minkowski} \leftarrow \eta^{ab} = \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} g^{\mu\nu} \quad ??$$

Example: $g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix}$ with $x^\mu = (t, r, \theta, \varphi)$

NOTE: Black holes corresponds with $g_{\mu\nu}$ of the above type with $-g_{tt} = g_{rr}^{-1} = f(r)$

There is a new set of coordinates $\xi^a = (\xi^0, \xi^1, \xi^2, \xi^3)$ with

$$\xi^0 = t \;, \quad \xi^1 = r \sin\theta \cos\varphi \;, \quad \xi^2 = r \sin\theta \sin\varphi \;, \quad \xi^3 = r\cos\theta$$

such that

$$\eta^{ab} = \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} g^{\mu\nu}$$

The answer to the question is two-fold :

- $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ must be zero for ξ^a coordinates to exist (a vanishing tensor does vanish for any observer)

- There must exist a point X at which the metric $g_{\mu\nu}(X)$ has one negative and three positive eigenvalues like η_{ab}

Ricci identity : In a space-time without torsion one has that

$$[\nabla_\mu, \nabla_\nu]\, V^\rho = \underbrace{R_{\mu\nu}{}^{\rho}{}_{\sigma}}_{\text{Riemann-Christoffel tensor}} V^\sigma$$

$\Rightarrow$ If $R_{\mu\nu}{}^{\rho}{}_{\sigma} = 0$ then $[\nabla_\mu, \nabla_\nu] = 0$ as would be expected for a coordinate system that can be transformed into a Minkowski coordinate system via a g.c.t.

* Algebraic properties of $R_{\mu\nu\rho\sigma} = g_{\rho\lambda} R_{\mu\nu}{}^{\lambda}{}_{\sigma}$

- Symmetry : $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

- Antisymmetry : $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\nu\mu\sigma\rho}$

- Cyclicity : $R_{\mu\nu\rho\sigma} + R_{\rho\mu\nu\sigma} + R_{\nu\rho\mu\sigma} = 0$

Important : The above algebraic properties can be shown to hold in the L.I.F where $\partial_a g_{bc} = 0$ (so $\Gamma_{\mu\nu}{}^{\rho} = 0$) but $\partial_a \partial_b g_{cd} \neq 0$. In this L.I.F :

$$R_{abcd} = \frac{1}{2}\left[\partial_c \partial_b g_{ad} - \partial_c \partial_a g_{bd} - \partial_d \partial_b g_{ac} + \partial_d \partial_a g_{bc}\right]$$

and the algebraic properties follow. Then, by general covariance, these symmetry properties hold in any other reference frame.

* Ricci tensor : $R_{\mu\nu} \equiv R_{\lambda\mu}{}^{\lambda}{}_{\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$ (only possibility)

- Symmetry : Since $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \Rightarrow R_{\mu\nu} = R_{\nu\mu}$

* Ricci scalar : $R \equiv R_{\lambda}{}^{\lambda} = g^{\mu\nu} R_{\mu\nu}$ (only possibility)

Note that no other scalar (0-index tensor) can be formed as

$$\frac{1}{\sqrt{|g|}} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0$$

by virtue of the cyclicity of $R_{\mu\nu\rho\sigma}$.

* Bianchi identities : In addition to the algebraic identities, the Riemann-Christoffel tensor, Ricci tensor and Ricci scalar satisfy a set of differential equations

- $\nabla_\mu R_{\nu\rho}{}^{\lambda}{}_{\sigma} + \nabla_\rho R_{\mu\nu}{}^{\lambda}{}_{\sigma} + \nabla_\nu R_{\rho\mu}{}^{\lambda}{}_{\sigma} = 0 \iff \nabla_{[\mu} R_{\nu\rho]}{}^{\lambda}{}_{\sigma} = 0$

NOTE : This is the analogue of the Bianchi form of Gauss-Faraday law in classical electrodynamics

Important : These differential identities can be again proved in the L.I.F where $\Gamma_{\mu\nu}{}^{\rho} = 0$, and then invoking covariance of tensorial identities.

- Tracing over (ν, λ) gives

$$\nabla_\mu R_{\rho\sigma} - \nabla_\rho R_{\mu\sigma} + \nabla_\lambda R_{\rho\mu}{}^{\lambda}{}_{\sigma} = 0$$

- Tracing over (ρ,σ) gives

$$\underbrace{-R_{\rho\mu}{}^{\rho\lambda}}= -R_{\mu}{}^{\lambda}$$

$$\nabla_\mu R - \nabla_\rho R_\mu{}^\rho + \nabla_\lambda R_{\rho\mu}{}^{\lambda\rho} = \nabla_\mu R - 2\nabla_\rho R_\mu{}^\rho$$

$$= -2\nabla_\rho \left[R_\mu{}^\rho - \tfrac{1}{2}\delta^\rho_\mu R\right] = 0$$

raising μ

$$\Rightarrow \quad \nabla_\nu \underbrace{\left[R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R\right]}_{G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R} = 0$$

"Einstein tensor"

$$\Rightarrow \quad \boxed{\nabla_\mu G^{\mu\nu} = 0} \quad \equiv\!> \quad G^{\mu\nu} \text{ conserved due to symmetries !!}$$

Important : Noether theorem states that a symmetry in a theory implies a conserved current. In the case of GR, the energy-momentum tensor $T^{\mu\nu}$ is the conserved current associated to space-time translations or diffeomorphisms, i.e. $\nabla_\mu T^{\mu\nu} = 0$:

- g.c.t : $dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu$

- Infinitesimal g.c.t : $x'^\mu = x^\mu + \xi^\mu(x)$
[diffeomorphism]

V. Einstein's field equations $[\, c = 1 \,,\; \epsilon_0 \cdot \mu_0 = \frac{1}{c^2} = 1 \,]$

The equations of motion governing the dynamics of a gravitating system are the so-called Einstein's field equations

$$\boxed{G_{\mu\nu} = \kappa^2 \, T_{\mu\nu}}$$

Einstein tensor [Geometry]

Energy-momentum tensor [Matter content]

$$T^{\mu\nu} = T^{\nu\mu}$$

Note: $\kappa^2 \equiv \frac{8\pi G_N}{c^4}$ (see Newtonian limit of GR)

* Energy-momentum tensor: It depends on what kind of matter and space-time geometry one is considering. Some examples are:

- Free particle of mass m in Minkowski space-time:

$$T^{ab} = \frac{m}{\gamma} u^a u^b \, \delta(\vec{x} - \vec{x}_p(t)) = \frac{E}{\gamma^2} u^a u^b \, \delta(\vec{x} - \vec{x}_p(t))$$

$u^a = \gamma(1, \vec{v})$ — γ: Lorentz factor

$E^2 = |\vec{p}|^2 + m^2$

- Perfect fluid (in the inertial frame) in Minkowski space-time:

Inertial frame $[\, \vec{v} = 0 \,]$: $u^a = (1, \vec{0}) \Rightarrow \eta_{ab} u^a u^b = -1$

$$T^{ab} = \begin{bmatrix} \rho & \\ & P\,\delta^{ij} \end{bmatrix} \quad \text{with} \quad \begin{array}{l} \rho \equiv \text{energy density} \\ P \equiv \text{isotropic pressure} \end{array}$$

- Perfect fluid in Minkowski space-time

$$T^{ab} = P\,\eta^{ab} + (P+\rho)\,u^a u^b$$

with a normalisation given by $\eta_{ab} u^a u^b = -1$

- Perfect fluid in a gravitational field

"frame field"
$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$

$$T^{\mu\nu} = P\,g^{\mu\nu} + (P+\rho)\,u^\mu u^\nu \quad \text{with} \quad u^a = \overbrace{e_\mu{}^a}\, u^\mu$$

with a normalisation given by $g_{\mu\nu} u^\mu u^\nu = \eta_{ab} u^a u^b = -1$

NOTE: Taking the trace one finds

$$T \equiv g_{\mu\nu} T^{\mu\nu} = 4P - (P+\rho) = 3P - \rho$$

Statistical Physics

NOTE: Usually matter satisfies an equation of state $f(P,\rho) = 0$

- Cosmological constant : It is modelled as a perfect fluid with a equation of state $P = -\rho \equiv -\frac{\Lambda}{\kappa^2} < 0$

$$\Rightarrow \quad T^{\mu\nu} = -\frac{\Lambda}{\kappa^2}\, g^{\mu\nu} \quad \text{with} \quad P = -\frac{\Lambda}{\kappa^2} < 0 \quad \Rightarrow$$

Exotic form of energy / matter !!

- **Classical electrodynamics in a gravitational field**

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} \right]$$

which takes the form

$$T^{\mu\nu} = \begin{bmatrix} \frac{1}{2}\left[\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right] & S_x & S_y & S_z \\ S_x & & & \\ S_y & & -\sigma^{ij} & \\ S_z & & & \end{bmatrix}$$

where

- $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ => Poynting vector
- $\sigma^{ij} = \epsilon_0 E^i E^j + \frac{1}{\mu_0} B^i B^j - \frac{1}{2}\left[\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right]\delta^{ij}$

  => Maxwell stress tensor

NOTE: Taking the trace one finds

$$T \equiv g_{\mu\nu} T^{\mu\nu} = \frac{1}{\mu_0}\left[F_{\mu\nu} F^{\mu\nu} - F_{\mu\nu} F^{\mu\nu} \right] = 0$$

- **Vacuum** : There is no matter in the space-time so that

$$T^{\mu\nu} = 0$$

* **Alternative form of Einstein equations:** Starting from the Einstein equation and taking a trace one finds

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \quad \Rightarrow \quad G = \kappa^2 T$$

with

$$G = g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu}\left[R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right] = R - 2R = -R$$

$$\Rightarrow \quad R = -\kappa^2 T$$

Substituting back into the Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \kappa^2 T = \kappa^2 T_{\mu\nu}$$

$$\Rightarrow \quad R_{\mu\nu} = \kappa^2 \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]$$

Alternative form in terms of $R_{\mu\nu}$!!

At the vacuum one has that

$$T^{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0 \qquad \text{“Ricci-flat manifolds”}$$

<u>Important</u>: In $D=1+1$ and $D=1+2$ it can be proven that $R_{\mu\nu} = 0 \Rightarrow R_{\mu\nu\rho\sigma} = 0$ so there is no gravitational field. In $D=1+3$ this is not the case: Black holes, wormholes, gravitational waves, ...

VI. An action principle for Einstein gravity

The action governing the dynamics of gravity is the so-called the Einstein-Hilbert action

$$S_g[g] = \frac{1}{2\kappa^2}\int d^4x \sqrt{-|g|}\,\underbrace{R}_{\text{Ricci scalar } R \equiv g^{\mu\nu}R_{\mu\nu}}$$

where $\kappa^2 = 8\pi G_N$ with $\kappa^{-1} = \underbrace{m_P}_{\text{reduced Planck mass}} = 2.4 \times 10^{18}$ GeV

$\hookrightarrow c = 1$

Important: Unlike for other interactions like electromagnetism, the gravitational coupling constant κ^2 has has energy units $[\kappa^2] = E^{-2} = L^2$. This has important consequences as far as quantum renormalisation is concerned.

Comment: Higher-derivative and modified theories of Gravity also exist. An example is $f(R)$ gravity

$$S_g[g] = \frac{1}{2\kappa^2}\int d^4x \sqrt{-|g|}\; f(R) \quad \text{with} \quad f(R) = R + \alpha R^2 + \dots$$

$\Rightarrow$ Intersting for Cosmology: Starobinski model of inflation, ...

From the Einstein-Hilbert action we can compute the equations of motion for the metric in a purely gravitational theory. This equation of motion is derived by applying standard variational principle:

$$\delta S_g = \underbrace{\frac{\delta S}{\delta g_{\mu\nu}}}_{\text{E.O.M}=0} \underbrace{\delta g_{\mu\nu}}_{\text{arbitrary variation } \delta g_{\mu\nu}} = 0$$

Let us vary the Einstein-Hilbert action!

$$\delta S_g = \frac{1}{2\kappa^2}\int d^4x \; \delta\left[\sqrt{-|g|}\, g^{\mu\nu} R_{\mu\nu}\right]$$

$$= \frac{1}{2\kappa^2}\int d^4x \left[\underbrace{\delta\left(\sqrt{-|g|}\right)}_{\text{Note 1}} R + \sqrt{-|g|}\,\underbrace{\delta\left(g^{\mu\nu}\right)}_{\text{Note 2}} R_{\mu\nu} + \sqrt{-|g|}\, g^{\mu\nu}\underbrace{\delta R_{\mu\nu}}_{\text{Note 3}}\right]$$

$$= (*)$$

Note 1: Jacobi's formula: $\delta|M| = |M| \, \mathrm{Tr}(M^{-1}\delta M)$

$$\Rightarrow \delta\left(\sqrt{-|g|}\right) = \frac{1}{2}\frac{1}{\sqrt{-|g|}}\,\delta(-|g|) = \frac{1}{2}\frac{(-|g|)}{\sqrt{-|g|}}\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$= \frac{1}{2}\sqrt{-|g|}\, g^{\mu\nu}\,\delta g_{\mu\nu}$$

NOTE 2: $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda \Rightarrow \delta g_{\mu\nu}\, g^{\nu\lambda} = -g_{\mu\nu}\,\delta g^{\nu\lambda}$ $(\times g^{\mu\rho})$

$$\Rightarrow \delta g^{\rho\lambda} = -g^{\rho\mu} g^{\lambda\nu}\,\delta g_{\mu\nu}$$

Note 3: Starting from the Riemann tensor

$$R_{\mu\nu}{}^{\rho}{}_{\lambda} = \partial_\mu \Gamma_{\nu\lambda}{}^{\rho} - \partial_\nu \Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\tau}{}^{\rho}\Gamma_{\nu\lambda}{}^{\tau} - \Gamma_{\nu\tau}{}^{\rho}\Gamma_{\mu\lambda}{}^{\tau}$$

we can obtain the Ricci tensor

$$R_{\nu\lambda} = \partial_\rho \Gamma_{\nu\lambda}{}^{\rho} - \partial_\nu \Gamma_{\rho\lambda}{}^{\rho} + \Gamma_{\rho\tau}{}^{\rho}\Gamma_{\nu\lambda}{}^{\tau} - \Gamma_{\nu\tau}{}^{\rho}\Gamma_{\rho\lambda}{}^{\tau}$$

Then, under a variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, one gets

$$\delta R_{\nu\lambda} = \partial_\rho \delta\Gamma_{\nu\lambda}{}^{\rho} - \partial_\nu \delta\Gamma_{\rho\lambda}{}^{\rho} + \delta\Gamma_{\rho\tau}{}^{\rho}\Gamma_{\nu\lambda}{}^{\tau} + \Gamma_{\rho\tau}{}^{\rho}\delta\Gamma_{\nu\lambda}{}^{\tau} - \delta\Gamma_{\nu\tau}{}^{\rho}\Gamma_{\rho\lambda}{}^{\tau} - \Gamma_{\nu\tau}{}^{\rho}\delta\Gamma_{\rho\lambda}{}^{\tau}$$

$$= \nabla_\rho \delta\Gamma_{\nu\lambda}{}^{\rho} - \nabla_\nu \delta\Gamma_{\rho\lambda}{}^{\rho}.$$

$$\Rightarrow \quad g^{\nu\lambda}\delta R_{\nu\lambda} = \nabla_\rho \underbrace{\Big(g^{\nu\lambda} \underbrace{\delta\Gamma_{\nu\lambda}{}^{\rho}}_{\text{tensor}}\Big)}_{V^\rho} - \nabla_\nu \underbrace{\Big(g^{\nu\lambda} \underbrace{\delta\Gamma_{\rho\lambda}{}^{\rho}}_{\text{tensor}}\Big)}_{W^\nu} \Rightarrow$$ Boundary terms !!

$$(*) = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\left[\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right]\delta g_{\mu\nu}$$

$$= -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\;G^{\mu\nu}\underbrace{\delta g_{\mu\nu}}_{\text{arbitrary}} = 0$$

Then $\delta S_g = 0 \Rightarrow \boxed{G^{\mu\nu} = 0}$ Einstein equation for pure Gravity !!

VII. Scalar and Maxwell fields in curved space-times

We have presented the Einstein - Hilbert action describing the theory of pure gravity

* Gravity : $S_g[g] = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\,R$

Then

$$\delta S_g = \frac{\delta S_g}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu} = -\frac{1}{2\kappa^2}\,G^{\mu\nu}\,\delta g_{\mu\nu}$$

Let us couple gravity to matter / radiation fields !!

* Scalar field $\phi(x)$: It is governed by the action

$$S_\phi(g,\phi) = \int d^4x\,\sqrt{-|g|}\left[-\frac{1}{2}\,g^{\mu\nu}\,\underbrace{\nabla_\mu\phi}_{\partial_\mu\phi}\,\underbrace{\nabla_\nu\phi}_{\partial_\nu\phi} - V(\phi)\right]$$

Then

$$\delta S_\phi = \underbrace{\frac{\delta S_\phi}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu}}_{} + \underbrace{\frac{\delta S_\phi}{\delta\phi}\,\delta\phi}_{}$$

$$(A) \equiv \frac{1}{2}T^{\mu\nu}_\phi \qquad (B) \equiv \text{E.O.M of } \phi$$

We will compute each contribution separately :

(A) : Energy-momentum tensor of a gravitating field ϕ

$$S_\phi = \int d^4x \sqrt{-|g|}\left[-\frac{1}{2}\underbrace{\nabla_\mu \phi}_{\partial_\mu \phi}\ \underbrace{\nabla^\mu \phi}_{\partial_\mu \phi} - V(\phi)\right] = \int d^4x \sqrt{-|g|}\left[-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

Under variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ one has

$$\delta S_\phi = \int d^4x \left[\underbrace{\delta\left(\sqrt{-|g|}\right)}_{\frac{1}{2}\sqrt{-|g|}\, g^{\lambda\epsilon}\,\delta g_{\lambda\epsilon}}\left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right) - \frac{1}{2}\sqrt{-|g|}\ \underbrace{\delta g^{\mu\rho}}_{-g^{\mu\lambda}g^{\rho\epsilon}\,\delta g_{\lambda\epsilon}}\ \partial_\mu\phi\,\partial_\rho\phi\right]$$

$$= \frac{1}{2}\int d^4x \sqrt{-|g|}\left[g^{\lambda\epsilon}\left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right) + \partial^\lambda\phi\,\partial^\epsilon\phi\right]\delta g_{\lambda\epsilon}$$

$$\equiv \frac{1}{2}\int d^4x \sqrt{-|g|}\ T^{\mu\nu}_\phi$$

$$\Rightarrow \boxed{T^{\mu\nu}_\phi = \partial^\mu\phi\,\partial^\nu\phi + g^{\mu\nu}\left(-\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - V(\phi)\right)}$$

Energy-momentum tensor of ϕ

(B) : Equation of motion of a gravitating field ϕ

$$S_\phi = \int d^4x \sqrt{-|g|} \left[-\frac{1}{2} \nabla_\mu \phi \, \nabla^\mu \phi - V(\phi) \right]$$

Under variation of the scalar field $\phi \to \phi + \delta\phi$ one has

$$\delta S_\phi = \int d^4x \sqrt{-|g|} \left[-\frac{1}{2} \left(\nabla_\mu \delta\phi \, \nabla^\mu \phi + \nabla_\mu \phi \, \nabla^\mu \delta\phi \right) - \frac{dV}{d\phi} \delta\phi \right]$$

$$= \int d^4x \sqrt{-|g|} \left[- \underbrace{\nabla_\mu \phi \, \nabla^\mu \delta\phi}_{\text{Note}} - \frac{dV}{d\phi} \delta\phi \right] = (*)$$

Note :
$$g^{\mu\nu} \nabla_\mu \phi \, \nabla_\nu \delta\phi = \underbrace{\nabla_\nu \left(g^{\mu\nu} \nabla_\mu \phi \, \delta\phi \right)}_{\nabla_\nu V^\nu \,\equiv\, \text{boundary term}} - \underbrace{\left(g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \right)}_{\Box \,\equiv\, g^{\mu\nu} \nabla_\mu \nabla_\nu} \delta\phi$$

(D'Alembertian)

$$= -(\Box \phi) \, \delta\phi$$

$$(*) = \int d^4x \sqrt{-|g|} \underbrace{\left[\Box \phi - \frac{dV}{d\phi} \right]}_{\text{Equation of motion of } \phi} \delta\phi$$

The final result is then

$$\delta S_\phi[g,\phi] = \frac{1}{2} T^{\mu\nu}_\phi \, \delta g_{\mu\nu} + \left[\Box\phi - \frac{dV}{d\phi} \right] \delta\phi$$

* Maxwell field $A_\mu(x)$: It is governed by the action

$$S_A(g,A) = \int d^4x \sqrt{-|g|} \left[-\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right]$$

with

$\Gamma_{\mu\nu}{}^\lambda = \Gamma_{\nu\mu}{}^\lambda$ (no torsion)

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Then

$$\delta S_A = \underbrace{\frac{\delta S_A}{\delta g_{\mu\nu}} \delta g_{\mu\nu}}_{(A) \,\equiv\, \frac{1}{2} T^{\mu\nu}_A} + \underbrace{\frac{\delta S_A}{\delta A_\mu} \delta A_\mu}_{(B) \,\equiv\, \text{E.O.M of } A_\mu}$$

We will compute each contribution separately :

(A) : Energy-momentum tensor of a gravitating field A_μ

$$S_A = -\frac{1}{4}\int d^4x \sqrt{-|g|}\; F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}\int d^4x \sqrt{-|g|}\; g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Under variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ one has

$$\delta S_A = -\frac{1}{4}\int d^4x \Big[\underbrace{\delta\big(\sqrt{-|g|}\big)}_{\frac{1}{2}\sqrt{-|g|}\, g^{\lambda\epsilon}\,\delta g_{\lambda\epsilon}} F_{\mu\nu} F^{\mu\nu} + \sqrt{-|g|}\; \overbrace{\delta g^{\mu\rho}}^{-g^{\mu\lambda} g^{\rho\epsilon}\,\delta g_{\lambda\epsilon}} F_{\mu\nu} F_{\rho}{}^{\nu} + \sqrt{-|g|}\; \underbrace{\delta g^{\nu\sigma}}_{-g^{\nu\lambda} g^{\sigma\epsilon}\,\delta g_{\lambda\epsilon}} F^{\rho}{}_{\nu} F_{\rho\sigma} \Big]$$

$$= -\frac{1}{4}\int d^4x \sqrt{-|g|} \Big[\frac{1}{2} g^{\lambda\epsilon} F_{\mu\nu} F^{\mu\nu} - F^{\lambda}{}_{\nu} F^{\epsilon\nu} - \underbrace{F^{\rho\lambda} F_{\rho}{}^{\epsilon}}_{F^{\lambda}{}_{\rho} F^{\epsilon\rho}} \Big] \delta g_{\lambda\epsilon}$$

$$= -\frac{1}{4}\int d^4x \sqrt{-|g|} \Big[\frac{1}{2} g^{\lambda\epsilon} F_{\mu\nu} F^{\mu\nu} - 2 F^{\lambda}{}_{\rho} F^{\epsilon\rho} \Big] \delta g_{\lambda\epsilon}$$

$$= \frac{1}{2}\int d^4x \sqrt{-|g|} \Big[F^{\lambda}{}_{\rho} F^{\epsilon\rho} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\lambda\epsilon} \Big] \delta g_{\lambda\epsilon}$$

$$= \frac{1}{2}\int d^4x \sqrt{-|g|}\; T_A^{\lambda\epsilon}\, \delta g_{\lambda\epsilon}$$

$$\Longrightarrow \boxed{T_A^{\mu\nu} = F^{\mu}{}_{\rho} F^{\nu\rho} - \frac{1}{4} F_{\rho\lambda} F^{\rho\lambda} g^{\mu\nu}}$$

Energy-momentum tensor of A_μ

(B) : Equation of motion of a gravitating field A_μ

$$S_A = \int d^4x \sqrt{-|g|} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

Under variation of the scalar field $A_\mu \to A_\mu + \delta A_\mu$ one has

$$\delta S_A = \int d^4x \sqrt{-|g|} \left[-\frac{1}{4} \left(\delta F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \delta F^{\mu\nu} \right) \right]$$

$$= -\frac{1}{2} \int d^4x \sqrt{-|g|} \, F^{\mu\nu} \delta F_{\mu\nu} = (*)$$

Note : $F^{\mu\nu} \delta F_{\mu\nu} = F^{\mu\nu} \delta \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu \right)$

$$= F^{\mu\nu} \left(\nabla_\mu \delta A_\nu - \nabla_\nu \delta A_\mu \right)$$

$$= 2 \, F^{\mu\nu} \nabla_\mu \delta A_\nu$$

$$= 2 \underbrace{\nabla_\mu \left(F^{\mu\nu} \delta A_\nu \right)}_{\nabla_\mu V^\mu \,\equiv\, \text{boundary term}} - 2 \left(\nabla_\mu F^{\mu\nu} \right) \delta A_\nu$$

$$(*) = \int d^4x \sqrt{-|g|} \, \underbrace{\nabla_\mu F^{\mu\nu}}_{\text{Equation of motion of } A_\mu} \delta A_\nu$$

Important: $F_{\mu\nu}$ still obeys the Bianchi identity

$$\nabla_{[\mu} F_{\nu\rho]} = 0 \quad \text{by virtue of} \quad R_{\lambda[\mu\nu\rho]} = 0$$

The final result is then

$$\delta S_A[g,A] = \frac{1}{2} T_A^{\mu\nu} \delta g_{\mu\nu} + \left[\nabla_\nu F^{\nu\mu} \right] \delta A_\mu$$

* Gravity + scalar ϕ + Maxwell : This general system is described by the action

$$S[g,\phi,A] = S_g[g] + S_\phi[g,\phi] + S_A[g,A]$$

The dynamics is determined by the variational principle

$$\delta S = \underbrace{\frac{\delta S}{\delta g_{\mu\nu}}}_{(A)} \delta g_{\mu\nu} + \underbrace{\frac{\delta S}{\delta \phi}}_{(B.1)} \delta\phi + \underbrace{\frac{\delta S}{\delta A_\mu}}_{(B.2)} \delta A_\mu = 0$$

Then :

(A) : $G^{\mu\nu} = \kappa^2 T^{\mu\nu} = \kappa^2 \left(T_\phi^{\mu\nu} + T_A^{\mu\nu} \right) \Rightarrow$ Einstein equations

(B.1) : $\Box\phi = \frac{dV}{d\phi} \Rightarrow$ E.O.M of ϕ

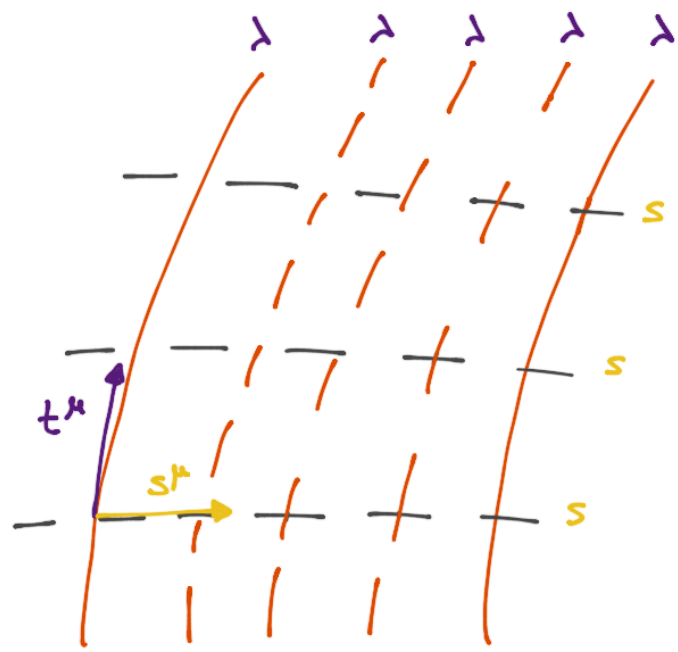
(B.2) : $\nabla_\mu F^{\mu\nu} = 0 \Rightarrow$ E.O.M of A_μ

Appendix : Riemann tensor and gedesics deviation

Let us consider two geodesics $\gamma_0(\lambda)$ and $\gamma_1(\lambda)$ which correspond to the limitting geodesics within a family of geodesics

$$\gamma(s,\lambda) \quad \text{with} \quad s \in [0,1) \quad \text{such that} \quad \begin{cases} \gamma_0(\lambda) \equiv \gamma(0,\lambda) \\ \gamma_1(\lambda) \equiv \gamma(1,\lambda) \end{cases}$$

This is depicted as a family of curves $x^\mu(s,\lambda)$



$$\text{with} \quad \begin{cases} t^\mu \equiv \dfrac{\partial x^\mu}{\partial \lambda} & \text{being the tangent vector} \\ s^\mu \equiv \dfrac{\partial x^\mu}{\partial s} & \text{being the deviation vector} \end{cases}$$

The goal is to compute the change of the deviation vector s^μ when moving along λ. This is

$$v^\mu \equiv \frac{Ds^\mu}{D\lambda} = t^\sigma \nabla_\sigma s^\mu \quad \Rightarrow \text{"velocity of geodesic deviation"}$$

Even more importantly, we will compute the change of this velocity, namely, the acceleration

$$a^\mu \equiv \frac{Dv^\mu}{D\lambda} = \frac{D^2 s^\mu}{D\lambda^2} = t^\rho \nabla_\rho v^\mu = t^\rho \nabla_\rho \left(t^\sigma \nabla_\sigma s^\mu \right) = (*)$$

Note: It can be shown that $t^\rho \nabla_\rho s^\mu = s^\rho \nabla_\rho t^\mu$.
↳ no torsion !!

Proof:

$$t^\rho \nabla_\rho s^\mu = t^\rho \left(\partial_\rho s^\mu + \Gamma_{\rho\lambda}{}^\mu s^\lambda \right)$$

$$s^\rho \nabla_\rho t^\mu = s^\rho \left(\partial_\rho t^\mu + \Gamma_{\rho\lambda}{}^\mu t^\lambda \right)$$

Then

$$t^\rho \nabla_\rho s^\mu - s^\rho \nabla_\rho t^\mu = t^\rho \partial_\rho s^\mu - s^\rho \partial_\rho t^\mu + \Gamma_{\rho\lambda}{}^\mu t^\rho s^\lambda - \underbrace{\Gamma_{\rho\lambda}{}^\mu t^\lambda s^\rho}_{\Gamma_{\lambda\rho}{}^\mu t^\rho s^\lambda}$$

$$= t^\rho \partial_\rho s^\mu - s^\rho \partial_\rho t^\mu + \underbrace{\left(\Gamma_{\rho\lambda}{}^\mu - \Gamma_{\lambda\rho}{}^\mu \right)}_{0 \text{ if no torsion}} t^\rho s^\lambda$$

$$= \frac{\partial x^\rho}{\partial \lambda} \frac{\partial s^\mu}{\partial x^\rho} - \frac{\partial x^\rho}{\partial s} \frac{\partial t^\mu}{\partial x^\rho}$$

$$= \frac{\partial s^\mu}{\partial \lambda} - \frac{\partial t^\mu}{\partial s} = \frac{\partial^2 x^\mu}{\partial s \, \partial \lambda} - \frac{\partial^2 x^\mu}{\partial \lambda \, \partial s} = 0$$

$$(*) = t^\rho \nabla_\rho (s^\sigma \nabla_\sigma t^\mu) = \underbrace{t^\rho \nabla_\rho s^\sigma}_{s^\rho \nabla_\rho t^\sigma} \nabla_\sigma t^\mu + t^\rho s^\sigma \nabla_\rho \nabla_\sigma t^\mu$$

$$= \underbrace{s^\rho \nabla_\rho t^\sigma \nabla_\sigma t^\mu}_{s^\rho \nabla_\rho (\underbrace{t^\sigma \nabla_\sigma t^\mu}_{\text{(geodesic equation) } 0}) - \underbrace{s^\rho t^\sigma \nabla_\rho \nabla_\sigma t^\mu}_{t^\rho s^\sigma \nabla_\sigma \nabla_\rho t^\mu}} + t^\rho s^\sigma \nabla_\rho \nabla_\sigma t^\mu$$

$$= t^\rho s^\sigma (\nabla_\rho \nabla_\sigma - \nabla_\sigma \nabla_\rho) t^\mu$$

$$= t^\rho s^\sigma \underbrace{[\nabla_\rho, \nabla_\sigma] t^\mu}_{R_{\rho\sigma}{}^\mu{}_\lambda \, t^\lambda} = R_{\rho\sigma}{}^\mu{}_\lambda \, t^\rho s^\sigma t^\lambda$$

$$\Rightarrow \quad \boxed{a^\mu = - R_{\sigma\rho}{}^\mu{}_\lambda \, s^\sigma t^\rho t^\lambda}$$

Comment: In flat spacetime the geodesic (straight lines) separation velocity is constant and $a^\mu = 0$.

Appendix : action principle for the geodesic equation

We have seen the geodesic equation

$$\dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0 \quad \text{with} \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

We will now show that this equation follows from an action upon extremisation following the variational principle.

mass of a test particle

a) $$S[x] = -m \int \sqrt{-ds^2} = -m \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda$$

$$\equiv \int d\lambda\, \mathcal{L}(x^\mu, \dot{x}^\mu)$$

$$\approx> \quad \mathcal{L}(x^\mu, \dot{x}^\mu) = -m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

Using the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

the relevant terms are given by

- $$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -m\,\frac{1}{2}\,\frac{1}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\left(-g_{\mu\nu}\dot{x}^\nu\, 2\right) = m\,\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}$$

note: $\quad ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -d\tau^2 < 0$

$$\Rightarrow \sqrt{-g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu} = \frac{d\tau}{d\lambda} = -\frac{\mathcal{L}}{m} .$$

$$= m\, g_{\mu\nu}\, \dot{x}^\nu\, \frac{d\lambda}{d\tau} = m\, g_{\mu\nu}\, \frac{dx^\nu}{d\tau}$$

- $$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{d\tau}{d\lambda}\, \frac{d}{d\tau}\left(m\, g_{\mu\nu}\, \frac{dx^\nu}{d\tau}\right)$$
$$= m\, \frac{d\tau}{d\lambda}\left(\partial_\rho g_{\mu\nu}\, \frac{dx^\rho}{d\tau}\, \frac{dx^\nu}{d\tau} + g_{\mu\nu}\, \frac{d^2 x^\nu}{d\tau^2}\right)$$

- $$\frac{\partial \mathcal{L}}{\partial x^\mu} = -m\left(-\frac{1}{2}\, \frac{\partial_\mu g_{\rho\sigma}\, \dot{x}^\rho\, \dot{x}^\sigma}{\sqrt{-g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu}}\right) = \frac{1}{2}\, m\, \partial_\mu g_{\rho\sigma}\, \dot{x}^\rho\, \dot{x}^\sigma\, \frac{d\lambda}{d\tau}$$
$$= \frac{1}{2}\, m\, \partial_\mu g_{\rho\sigma}\, \dot{x}^\rho\, \frac{dx^\sigma}{d\tau}$$

Then one arrives at

$$\frac{d\tau}{d\lambda}\left(\partial_\rho g_{\mu\nu}\, \frac{dx^\rho}{d\tau}\, \frac{dx^\nu}{d\tau} + g_{\mu\nu}\, \frac{d^2 x^\nu}{d\tau^2}\right) = \frac{1}{2}\, \partial_\mu g_{\rho\sigma}\, \dot{x}^\rho\, \frac{dx^\sigma}{d\tau}$$

$$\Rightarrow g_{\mu\nu}\, \frac{d^2 x^\nu}{d\tau^2} = \frac{1}{2}\, \partial_\mu g_{\rho\sigma}\, \frac{dx^\rho}{d\tau}\, \frac{dx^\sigma}{d\tau} - \partial_\rho g_{\mu\sigma}\, \underbrace{\frac{dx^\rho}{d\tau}\, \frac{dx^\sigma}{d\tau}}_{(\rho,\sigma)\text{-symmetric}}$$

$$\Rightarrow \quad g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} = \frac{1}{2}\left(\partial_\mu g_{\rho\sigma} - \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho} \right) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

$$\Rightarrow \quad \frac{d^2 x^\lambda}{d\tau^2} = - \underbrace{\frac{1}{2} g^{\lambda\mu} \left(\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\mu\rho} - \partial_\mu g_{\rho\sigma} \right)}_{\Gamma_{\rho\sigma}{}^{\lambda}(g)} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

$$\Rightarrow \quad \frac{d^2 x^\lambda}{d\tau^2} + \Gamma_{\rho\sigma}{}^{\lambda} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

" Geodesic equation using $d\tau$ "

If we change to the original parameter λ we find

$$\frac{d}{d\tau}\left(\frac{dx^\lambda}{d\tau} \right) + \Gamma_{\rho\sigma}{}^{\lambda} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

$$= \frac{d}{d\tau} \left(\frac{d\lambda}{d\tau} \, \dot{x}^\lambda \right) + \Gamma_{\rho\sigma}{}^{\lambda} \, \dot{x}^\rho \, \dot{x}^\sigma \left(\frac{d\lambda}{d\tau} \right)^2$$

$$= \dot{x}^\lambda \underbrace{\frac{d^2\lambda}{d\tau^2}}_{\underbrace{\frac{d}{d\tau}}_{\underbrace{\frac{d\lambda}{d\tau}}_{Q}\frac{d}{d\lambda}} \underbrace{\left(\frac{d\lambda}{d\tau}\right)}_{Q}} + \underbrace{\left(\frac{d\lambda}{d\tau} \right)^2}_{Q^2} \left[\ddot{x}^\lambda + \Gamma_{\rho\sigma}{}^{\lambda} \, \dot{x}^\rho \, \dot{x}^\sigma \right]$$

$$= \dot{x}^\lambda \, Q \frac{dQ}{d\lambda} + Q^2 \left[\ddot{x}^\lambda + \Gamma_{\rho\sigma}{}^\lambda \dot{x}^\rho \dot{x}^\sigma \right] = 0$$

Then, provided $Q = \frac{d\lambda}{d\tau} \neq 0$, one gets

$$\underbrace{\ddot{x}^\mu + \Gamma_{\rho\sigma}{}^\mu \dot{x}^\rho \dot{x}^\sigma}_{\dot{x}^\nu \nabla_\nu \dot{x}^\mu} = -\dot{x}^\mu \frac{1}{Q} \frac{dQ}{d\lambda}$$

$$= -\dot{x}^\mu \frac{d}{d\lambda} \ln(Q)$$

$$= \dot{x}^\mu \frac{d}{d\lambda} \ln(Q^{-1})$$

$$= \dot{x}^\mu \frac{d}{d\lambda} \ln\left(\frac{d\tau}{d\lambda}\right)$$

so that

$$\boxed{\dot{x}^\nu \nabla_\nu \dot{x}^\mu = \dot{x}^\mu \frac{d}{d\lambda} \ln\left(\frac{d\tau}{d\lambda}\right)}$$

Important: If using the afine parameter $d\lambda = d\tau$ one recovers $\dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0$.

Important: The above action is not valid when $m = 0$.
(We will investigate an alternative action)

b) $S[x,e] = \frac{1}{2}\int d\lambda \left[e^{-1}(\lambda)\, g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu - \underbrace{m^2}_{m \neq 0} e(\lambda) \right]$

$$\equiv \int d\lambda\, \mathcal{L}(x^\mu, \dot{x}^\mu, e)$$

$$\approx> \quad \mathcal{L}(x^\mu, \dot{x}^\mu, e) = \frac{1}{2}\left[e^{-1} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - m^2 e \right]$$

Important: This action is well-defined when $m = 0$.

Let us proceed with the Euler-Lagrange equations. The relevant quantities are:

→ For $x^\mu(\lambda)$:

- $$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = e^{-1} g_{\mu\nu} \dot{x}^\nu$$
- $$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = -\frac{\dot{e}}{e^2} g_{\mu\nu} \dot{x}^\nu + e^{-1} \partial_\rho g_{\mu\nu} \dot{x}^\rho \dot{x}^\nu + e^{-1} g_{\mu\nu} \ddot{x}^\nu$$
- $$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{2} e^{-1} \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

$$\approx> \quad g_{\mu\nu}\ddot{x}^\nu + \partial_\rho g_{\mu\nu} \underbrace{\dot{x}^\rho \dot{x}^\nu}_{(\rho,\nu)\text{-symmetric}} - \frac{1}{2}\partial_\mu g_{\rho\nu} \dot{x}^\rho \dot{x}^\nu = \frac{d}{d\lambda}\ln(e)\, g_{\mu\nu}\dot{x}^\nu$$

$$\Rightarrow \quad g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left[\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\rho\nu}\right]\dot{x}^\rho \dot{x}^\nu$$

$$= g_{\mu\nu}\dot{x}^\nu \frac{d}{d\lambda} Ln(e)$$

$$\Rightarrow \quad \ddot{x}^\lambda + \frac{1}{2} g^{\lambda\mu}\left[\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\rho\nu}\right]\dot{x}^\rho \dot{x}^\nu$$

$$= \dot{x}^\lambda \frac{d}{d\lambda} Ln(e)$$

$$\Rightarrow \quad \boxed{\ddot{x}^\mu + \Gamma_{\rho\nu}{}^{\mu}\dot{x}^\rho \dot{x}^\nu = \dot{x}^\nu \nabla_\nu \dot{x}^\mu = \dot{x}^\mu \frac{d}{d\lambda} Ln(e)}$$

→ For $e(\lambda)$:

- $\frac{\partial \mathcal{L}}{\partial \dot{e}} = 0 \quad , \quad \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial e}\right) = 0$
- $\frac{\partial \mathcal{L}}{\partial e} = -\frac{1}{2}\frac{1}{e^2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \frac{1}{2} m^2$

$$\Rightarrow \quad \frac{1}{e^2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -m^2$$

$$\Rightarrow \quad \boxed{e = \frac{1}{m}\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = \frac{1}{m}\frac{d\tau}{d\lambda}}$$

Important: The function $e(\lambda)$ simply encodes the relation between the proper time $d\tau$ and the curve parameter $d\lambda$.

- If $e(\lambda) = \text{cte} \Rightarrow \lambda = a\tau + b$ "afin parameter"

$$\Rightarrow \dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0$$

Important: For massive particles $(m \neq 0)$ one has

$$e(\lambda) = \frac{1}{m}\frac{d\tau}{d\lambda} = \frac{1}{m}\sqrt{\underbrace{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}_{\text{Must be } < 0}}$$

$\Rightarrow$ Massive particles follow time-like curves

* The $m=0$ case: In the massless case one has

$$\mathcal{L}(x^\mu, \dot{x}^\mu, e) = \frac{1}{2} e^{-1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

Then the Euler-Lagrange equations reduce to

→ For $x^\mu(\lambda)$: $\dot{x}^\nu \nabla_\nu \dot{x}^\mu = \dot{x}^\mu \frac{d}{d\lambda}\ln(e)$ [same as $m \neq 0$]

→ For $e(\lambda)$: $\frac{1}{e^2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \Rightarrow ds^2 = 0 \Rightarrow$ Massless particles follow null curves