# I. Kaluza-Klein reduction on $S^1$

In this section we are working out the dimensional reduction of gravity in D+1 dimension down to D dimensions. As we will see, this provides a unification of the form:

D+1 Gravity $\Longrightarrow$ Gravity + Maxwell + scalar in D

We will describe gravity in D+1 dimensions:

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2} \int d^{D+1}x \, \sqrt{-|\hat{g}|} \, \hat{R}$$

with $\hat{g}_{MN}$ and $\hat{R}_{MN}$ being the metric and Ricci scalar in a (D+1) dimensional space-time $\underbrace{M}_{x^M} = \underbrace{0, 1, \dots, D-1}_{x^\mu}, \underbrace{z}_{z}$.

Let's take the z-coordinate to be $S^1$ $\Longrightarrow$ Fourier expansion

$$\hat{g}_{MN}(x, z) = \sum_{n=0}^{\infty} \hat{g}_{MN}^{(n)}(x) \, e^{i \frac{n}{L} z}$$

Fourier mode

$(z \to z + 2\pi L)$

$\Longrightarrow$ The zero-mode $(n=0)$ is a massless mode whereas $n \neq 0$ correspondes to a tower of massive modes (KK tower).

Example: Scalar field $\hat{\phi}$ in D+1 dimensional flat space-time

$$S_\phi = \int d^{D+1}x \, \partial_M \hat{\phi} \, \partial^M \hat{\phi} \quad \underset{\text{E.O.M}}{\Longrightarrow} \quad \hat{\Box} \hat{\phi} = \partial_M \partial^M \hat{\phi} = 0$$

Fourier expansion along $S^1$ : $\hat{\phi}(x,z) = \sum_{n=0}^{\infty} \phi^{(n)}(x)\, e^{i\frac{n}{L}z}$

so that

$$\hat{\Box}\hat{\phi} = (\underbrace{\partial_\mu\partial^\mu}_{\Box} + \partial_z^2)\hat{\phi} = \sum_{n=0}^{\infty} \underbrace{\left[\Box\phi^{(n)} - \frac{n^2}{L^2}\phi^{(n)}\right]}_{} e^{i\frac{n}{L}z} = 0$$

Each mode must satisfy

$$\Box\phi^{(n)} - \underbrace{\frac{n^2}{L^2}}_{}\phi^{(n)} = 0$$

$m^2 \equiv \frac{n^2}{L^2}$ => Massive modes !!

$$m = \frac{|n|}{L}$$

Important : The KK phylosophy is to assome a very small L (we don't observe $S^1$) so that all the modes with $n \neq 0$ are very massive $m = \frac{|n|}{L}$ and we have to reach very high energies in order to produce such KK modes :

$$L \sim m_P \sim 10^{-33}\,cm \Rightarrow m \sim 10^{-5}\,gr$$

(Higgs mass $\sim m_{top} \sim 10^{-22}\,gr$)

Important : KK reduction $\equiv$ Truncation to $n=0$ massless modes

=> $z$ - independence !!

$$\boxed{\hat{g}_{MN}(x) = \hat{g}^{(0)}_{MN}(x)}$$

The first step to perform a KK reduction is to split the (D+1)-metric in D-dimensional blocks:

$$\hat{g}_{MN}(x) = \begin{bmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{z\nu} & g_{zz} \end{bmatrix} \equiv \begin{bmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix} \Rightarrow \text{Too naive (big mess)}$$

$$\equiv \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix}$$

$\Downarrow$

Much more convenient !!

(see discussion on symmetries)

Therefore we parameterise the (D+1) metric $\hat{g}_{MN}$ as

$$\boxed{d\hat{s}^2_{D+1} = e^{2\alpha\underbrace{\phi}_{\phi \,\equiv\, \text{"Dilaton"}}} ds^2_D + e^{2\beta\phi}\left(dz + A_\mu dx^\mu\right)^2}$$

with $\alpha$ and $\beta$ being constants to be chosen later on.

The associated (D+1) frame then reads:

$$\hat{e}_M{}^A = \begin{bmatrix} e^{\alpha\phi}\, e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \qquad \boxed{\begin{matrix} M = \mu, z \\ A = a, \underline{z} \end{matrix}}$$

Equivalently: $\hat{e}^a = e^{\alpha\phi} \underbrace{e^a}_{e_\mu{}^a dx^\mu}$ and $\hat{e}^{\underline{z}} = e^{\beta\phi}(dz + A)$ with $A \equiv A_\mu dx^\mu$

Ex: Check that $\hat{e}_M{}^A \, \hat{e}_N{}^B \, \hat{\eta}_{AB} = \hat{g}_{MN}$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \underbrace{\begin{bmatrix} \eta_{ab} & 0_{4\times 1} \\ 0_{1\times 4} & +1 \end{bmatrix} \begin{bmatrix} e^{\alpha\phi} e_\nu{}^b & 0_{4\times 1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}_{\begin{bmatrix} e^{\alpha\phi} e_{\nu a} & 0_{4\times 1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}$$

$$= \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix} = \hat{g}_{MN}(x)$$

In the following our goal will be to compute $S_{D+1}$ using the $(D+1)$-dimensional frame $\hat{e}_M{}^A$ given above:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \; \hat{e} \; \hat{e}_A{}^M \, \hat{e}_B{}^N \, \hat{R}_{MN}{}^{AB}(\hat{e})$$

- $\hat{e} = e^{(\alpha D + \beta)\phi} \, e$

- We need the inverse $(D+1)$-dim frame $\hat{e}_A{}^N$

$$\hat{e}_M{}^A \cdot \hat{e}_A{}^N = \delta_M{}^N \;\Rightarrow\; \hat{e}_A{}^N = \begin{bmatrix} e^{-\alpha\phi} e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{bmatrix}$$

$A_a = e_a{}^\nu A_\nu$

Ex: check that $\hat{e}_M{}^A \, \hat{e}_A{}^N = \delta_M{}^N$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \begin{bmatrix} e^{-\alpha\phi} e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{bmatrix} = \begin{bmatrix} \delta_\mu{}^\nu & 0_{4\times 1} \\ 0_{1\times 4} & 1 \end{bmatrix}$$

⊙ Now we perform the computation of the Ricci scalar $\hat{R}$.

▲ First we compute the aholonomy coefficients $\hat{\Omega}$:

$$\hat{\Omega}_{[MN]P} = \left(\partial_M \hat{e}_N{}^A - \partial_N \hat{e}_M{}^A\right) \hat{e}_{PA}$$

• $\hat{\Omega}_{[\mu\nu]\rho} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\, \hat{e}_{\rho A}$

$$= (\partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a)\, \hat{e}_{\rho a} + (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\, \hat{e}_{\rho \underline{z}}$$

$$= \left[\partial_\mu (e^{\alpha\phi} e_\nu{}^a) - \partial_\nu (e^{\alpha\phi} e_\mu{}^a)\right] (e^{\alpha\phi} e_{\rho a})$$

$$+ \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right] (e^{\beta\phi} A_\rho)$$

$$= e^{2\alpha\phi} \left[ (\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a)\, e_{\rho a} + \alpha\, (\partial_\mu \phi\, e_\nu{}^a - \partial_\nu \phi\, e_\mu{}^a)\, e_{\rho a} \right]$$

$$+ e^{2\beta\phi} \left[ F_{\mu\nu} A_\rho + \beta\, (\partial_\mu \phi\, A_\nu - \partial_\nu \phi\, A_\mu)\, A_\rho \right]$$

$$= e^{2\alpha\phi} \left[ \Omega_{[\mu\nu]\rho} + 2\alpha\, \partial_{[\mu} \phi\, e_{\nu]}{}^a e_{\rho a} \right]$$

$$+ e^{2\beta\phi} \left[ F_{\mu\nu} A_\rho + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} A_\rho \right]$$

• $\hat{\Omega}_{[\mu\nu]z} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\, \hat{e}_{zA} = (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\, \hat{e}_{z\underline{z}}$

$$= \left[ \partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu) \right] e^{\beta\phi}$$

$$= e^{2\beta\phi} \left[ F_{\mu\nu} + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} \right]$$

• $\hat{\Omega}_{[\mu z]\rho} = \partial_\mu \hat{e}_z{}^A\, \hat{e}_{\rho A} = \partial_\mu \hat{e}_z{}^{\underline{z}}\, \hat{e}_{\rho \underline{z}} = \partial_\mu (e^{\beta\phi})\, (e^{\beta\phi} A_\rho)$

$$= e^{2\beta\phi}\, \beta\, \partial_\mu \phi\, A_\rho$$

- $\hat{\Omega}_{[\mu z]z} = \partial_\mu \hat{e}_z{}^A \hat{e}_{zA} = \partial_\mu \hat{e}_z{}^{\underline{z}} \hat{e}_{z\underline{z}} = \partial_\mu (e^{\beta\phi})\, e^{\beta\phi}$
$= e^{2\beta\phi} \beta \partial_\mu \phi$

- $\hat{\Omega}_{[z\nu]\rho} = -\partial_\nu \hat{e}_z{}^A \hat{e}_{\rho A} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \hat{e}_{\rho\underline{z}} = -\partial_\nu (e^{\beta\phi})(e^{\beta\phi} A_\rho)$
$= -e^{2\beta\phi} \beta \partial_\nu \phi A_\rho$

- $\hat{\Omega}_{[z\nu]z} = -\partial_\nu \hat{e}_z{}^A \hat{e}_{zA} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \hat{e}_{z\underline{z}} = -\partial_\nu (e^{\beta\phi})(e^{\beta\phi})$
$= -e^{2\beta\phi} \beta \partial_\nu \phi$

- $\hat{\Omega}_{[zz]\rho} = \hat{\Omega}_{[zz]z} = 0$

- Using $\hat{\Omega}$ we compute the spin connection with all indices curved

$$\hat{\omega}_{M[NP]}(\hat{e}) = \frac{1}{2}\left(\hat{\Omega}_{[MN]P} - \hat{\Omega}_{[NP]M} + \hat{\Omega}_{[PM]N}\right)$$
$$= \hat{\omega}_M{}^{BC}(\hat{e})\, \hat{e}_{NB}\, \hat{e}_{PC}$$

- $\hat{\omega}_{\mu[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]\rho} - \hat{\Omega}_{[\nu\rho]\mu} + \hat{\Omega}_{[\rho\mu]\nu}\right)$

$$= \frac{1}{2}\Big[e^{2\alpha\phi}\Big(2\,\omega_{\mu[\nu\rho]} + 2\alpha\big(\partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a} - \partial_{[\nu}\phi\, e_{\rho]}{}^a e_{\mu a} + \partial_{[\rho}\phi\, e_{\mu]}{}^a e_{\nu a}\big)\Big)$$
$$+ e^{2\beta\phi}\Big(F_{\mu\nu} A_\rho - F_{\nu\rho} A_\mu + F_{\rho\mu} A_\nu + 2\beta\big(\partial_{[\mu}\phi A_{\nu]} A_\rho - \partial_{[\nu}\phi A_{\rho]} A_\mu$$
$$+ \partial_{[\rho}\phi A_{\mu]} A_\nu\big)\Big)\Big]$$

- $\hat{\omega}_{z[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]\rho} - \hat{\Omega}_{[\nu\rho]z} + \hat{\Omega}_{[\rho z]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi\, A_\rho - (F_{\nu\rho} + 2\beta\,\partial_{[\nu}\phi\, A_{\rho]}) + \beta\,\partial_\rho\phi\, A_\nu\right)\right]$$

$$= \frac{1}{2} e^{2\beta\phi}\left(-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi\, A_{\rho]}\right)$$

- $\hat{\omega}_{\mu[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]z} - \hat{\Omega}_{[\nu z]\mu} + \hat{\Omega}_{[z\mu]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left((F_{\mu\nu} + 2\beta\,\partial_{[\mu}\phi\, A_{\nu]}) - \beta\,\partial_\nu\phi\, A_\mu - \beta\,\partial_\mu\phi\, A_\nu\right)\right]$$

$$= \frac{1}{2} e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi\, A_\mu\right)$$

- $\hat{\omega}_{z[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]z} - \hat{\Omega}_{[\nu z]z} + \overbrace{\hat{\Omega}_{[zz]\nu}}^{0}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi - \beta\,\partial_\nu\phi\right)\right] = -e^{2\beta\phi}\beta\,\partial_\nu\phi$$

- $\hat{\omega}_{\mu[z\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu z]\rho} - \hat{\Omega}_{[z\rho]\mu} + \hat{\Omega}_{[\rho\mu]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(\beta\,\partial_\mu\phi\, A_\rho + \beta\,\partial_\rho\phi\, A_\mu + (F_{\rho\mu} + 2\beta\,\partial_{[\rho}\phi\, A_{\mu]})\right)\right]$$

$$= \frac{1}{2} e^{2\beta\phi}\left(F_{\rho\mu} + 2\beta\,\partial_\rho\phi\, A_\mu\right)$$

- $\hat{\omega}_{z[z\rho]} = \frac{1}{2}\left(\underbrace{\hat{\Omega}_{[zz]\rho}}_{0} - \hat{\Omega}_{[z\rho]z} + \hat{\Omega}_{[\rho z]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(+\beta\,\partial_\rho\phi + \beta\,\partial_\rho\phi\right)\right] = e^{2\beta\phi}\beta\,\partial_\rho\phi$$

- $\hat{\omega}_{\mu[zz]} = \hat{\omega}_{z[zz]} = 0$

▲ Using the above resuelts we compute the standard spin connection

$$\hat{\omega}_M{}^{BC} = \hat{\omega}_{M[NP]}\,\hat{e}^{BN}\,\hat{e}^{CP}$$

- $\hat{\omega}_\mu{}^{bc} = \hat{\omega}_{\mu[NP]}\,\hat{e}^{bN}\,\hat{e}^{cP}$

$$= \hat{\omega}_{\mu[\nu\rho]}\hat{e}^{b\nu}\hat{e}^{c\rho} + \hat{\omega}_{\mu[\nu z]}\hat{e}^{b\nu}\hat{e}^{cz} + \hat{\omega}_{\mu[z\rho]}\hat{e}^{bz}\hat{e}^{c\rho} + \overbrace{\hat{\omega}_{\mu[zz]}}^{0}\hat{e}^{bz}\hat{e}^{cz}$$

$$= \hat{\omega}_{\mu[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat{\omega}_{\mu[\nu z]} e^{-2\alpha\phi} e^{b\nu} A^c - \hat{\omega}_{\mu[z\rho]} e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2}\Big[2\,\omega_\mu{}^{[bc]} + 2\alpha\big(\partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a} e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi\, e_{\rho]}{}^a e_{\mu a} e^{b\nu} e^{c\rho}$$
$$+ \partial_{[\rho}\phi\, e_{\mu]}{}^a e_{\nu a} e^{b\nu} e^{c\rho}\big) + e^{2(\beta-\alpha)\phi}\big(F_{\mu\nu} A_\rho e^{b\nu} e^{c\rho} -$$
$$- F_{\nu\rho} A_\mu e^{b\nu} e^{c\rho} + F_{\rho\mu} A_\nu e^{b\nu} e^{c\rho}\big) +$$
$$+ 2\beta e^{2(\beta-\alpha)\phi}\big(\partial_{[\mu}\phi A_{\nu]} A_\rho e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi A_{\rho]} A_\mu e^{b\nu} e^{c\rho}$$
$$+ \partial_{[\rho}\phi A_{\mu]} A_\nu e^{b\nu} e^{c\rho}\big)\Big]$$
$$- \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\mu\nu} - 2\beta\partial_\nu\phi A_\mu) e^{b\nu} A^c\Big] - \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\rho\mu} + 2\beta\,\partial_\rho\phi A_\mu) A^b e^{c\rho}\Big]$$

$$= (*)$$

**Note 1:** $2\alpha\,\frac{1}{2}\big(\partial_\mu\phi\, e_\nu{}^a e_{\rho a} e^{b\nu} e^{c\rho} - \partial_\nu\phi\, e_\mu{}^a e_{\rho a} e^{b\nu} e^{c\rho}$
$- \partial_\nu\phi\, e_\rho{}^a e_{\mu a} e^{b\nu} e^{c\rho} + \partial_\rho\phi\, e_\nu{}^a e_{\mu a} e^{b\nu} e^{c\rho}$
$+ \partial_\rho\phi\, e_\mu{}^a e_{\nu a} e^{b\nu} e^{c\rho} - \partial_\mu\phi\, e_\rho{}^a e_{\nu a} e^{b\nu} e^{c\rho}\big)$

$$= \alpha\big(\underline{\partial_\mu\phi\,\eta^{bc}} - \partial_\nu\phi\, e^{b\nu} e_\mu{}^c - \partial_\nu\phi\, e_\mu{}^c e^{b\nu} + \partial_\rho\phi\, e_\mu{}^b e^{c\rho}$$
$$+ \partial_\rho\phi\, e_\mu{}^b e^{c\rho} - \underline{\partial_\mu\phi\,\eta^{bc}}\big)$$

$$= \alpha\big(2\,\partial_\rho\phi\, e_\mu{}^b e^{c\rho} - 2\,\partial_\nu\phi\, e_\mu{}^c e^{b\nu}\big) = \quad [\partial^a \equiv e^{a\rho}\partial_\rho]$$

$$= 2\alpha\big(e_\mu{}^b\,\partial^c\phi - e_\mu{}^c\,\partial^b\phi\big) = 4\alpha\, e_\mu{}^{[b}\,\partial^{c]}\phi$$
$$= 4\alpha\, \partial^{[c}\phi\, e_\mu{}^{b]}$$

NOTE 2: $e^{2(\beta-\alpha)\phi}\left(F_{\mu\nu}A_\rho e^{b\nu}e^{c\rho} - F_{\nu\rho}A_\mu e^{b\nu}e^{c\rho} + F_{\rho\mu}A_\nu e^{b\nu}e^{c\rho}\right)$

$$= e^{2(\beta-\alpha)\phi}\left(F_\mu{}^{b}A^c - F^{bc}A_\mu + F^c{}_\mu A^b\right)$$

$$= e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]} - F^{bc}A_\mu\right)$$

NOTE 3: $2\beta e^{2(\beta-\alpha)\phi}\frac{1}{2}\Big(\partial_\mu\phi A_\nu A_\rho e^{b\nu}e^{c\rho} - \partial_\nu\phi A_\mu A_\rho e^{b\nu}e^{c\rho} - \partial_\nu\phi A_\rho A_\mu e^{b\nu}e^{c\rho} + \partial_\rho\phi A_\nu A_\mu e^{b\nu}e^{c\rho} + \partial_\rho\phi A_\mu A_\nu e^{b\nu}e^{c\rho} - \partial_\mu\phi A_\rho A_\nu e^{b\nu}e^{c\rho}\Big)$

$$= \beta e^{2(\beta-\alpha)\phi}\Big(\partial_\mu\phi A^b A^c - \partial^b\phi A_\mu A^c - \partial^b\phi A^c A_\mu + \partial^c\phi A^b A_\mu + \partial^c\phi A_\mu A^b - \partial_\mu\phi A^c A^b\Big)$$

$$= \beta e^{2(\beta-\alpha)\phi}\left(-2A_\mu\partial^b\phi A^c + 2A_\mu\partial^c\phi A^b\right)$$

$$= -4\beta e^{2(\beta-\alpha)\phi} A_\mu \partial^{[b}\phi A^{c]}$$

$$(*) = \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi + e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]} - F^{bc}A_\mu\right) - 4\beta\, e^{2(\beta-\alpha)\phi}A_\mu\partial^{[b}\phi A^{c]}\Big]$$

$$-\frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[\underbrace{F_\mu{}^{b}A^c - F_\mu{}^{c}A^b}_{2F_\mu{}^{[b}A^{c]}} - 2\beta\big(\underbrace{\partial^b\phi A^c A_\mu - \partial^c\phi A^b}_{2\partial^{[b}\phi A^{c]}}\big)A_\mu\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi - e^{2(\beta-\alpha)\phi}F^{bc}A_\mu + e^{2(\beta-\alpha)\phi}\big(\underline{2F_\mu{}^{[b}A^{c]}} - \underline{4\beta A_\mu\partial^{[b}\phi A^{c]}} - \underline{2F_\mu{}^{[b}A^{c]}} + \underline{4\beta\partial^{[b}\phi A^{c]}}\big)\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} + 4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]} - e^{2(\beta-\alpha)\phi}F^{bc}A_\mu\Big] =$$

$$= \omega_\mu{}^{[bc]} + \alpha\,(\partial^c\phi\, e_\mu{}^b - \partial^b\phi\, e_\mu{}^c) - \frac{1}{2}\, e^{2(\beta-\alpha)\phi} F^{bc} A_\mu .$$

- $\hat\omega_z{}^{bc} = \hat\omega_{z[NP]}\,\hat e^{bN}\,\hat e^{cP}$

$$= \hat\omega_{z[\nu\rho]}\hat e^{b\nu}\hat e^{c\rho} + \hat\omega_{z[\nu z]}\hat e^{b\nu}\hat e^{cz} + \hat\omega_{z[z\rho]}\hat e^{bz}\hat e^{c\rho} + \overbrace{\hat\omega_{z[zz]}}^{0}\hat e^{bz}\hat e^{cz}$$

$$= \hat\omega_{z[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat\omega_{z[\nu z]}\, e^{-2\alpha\phi} e^{b\nu} A^c - \hat\omega_{z[z\rho]}\, e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2} e^{2(\beta-\alpha)\phi}\left[-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right] e^{b\nu} e^{c\rho} + e^{2(\beta-\alpha)\phi}\beta\,\partial_\nu\phi\, e^{b\nu} A^c$$

$$- e^{2(\beta-\alpha)\phi}\beta\,\partial_\rho\phi\, A^b e^{c\rho} =$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} + e^{2(\beta-\alpha)\phi}\beta\left[\underline{-\partial^b\phi A^c} + \underline{\partial^c\phi A^b} + \underline{\partial^b\phi A^c} - \underline{\partial^c\phi A^b}\right]$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc}.$$

Therefore, using compact notation, we find that

$$\boxed{\hat\omega^{bc} = \omega^{[bc]} + \alpha e^{-\alpha\phi}\,(\partial^c\phi\,\hat e^b - \partial^b\phi\,\hat e^c) - \frac{1}{2} F^{bc} e^{(\beta-2\alpha)\phi}\,\hat e^{\underline{z}}}$$

- $\hat\omega_\mu{}^{b\underline{z}} = \hat\omega_{\mu[NP]}\,\hat e^{bN}\,\hat e^{\underline{z}P}$

$$= \hat\omega_{\mu[\nu\rho]}\hat e^{b\nu}\underbrace{\hat e^{\underline{z}\rho}}_{0} + \hat\omega_{\mu[\nu z]}\hat e^{b\nu}\hat e^{\underline{z}z} + \hat\omega_{\mu[z\rho]}\overbrace{\hat e^{bz}}^{0}\hat e^{\underline{z}\rho} + \overbrace{\hat\omega_{\mu[zz]}}^{0}\hat e^{bz}\hat e^{\underline{z}z}$$

$$= \hat\omega_{\mu[\nu z]}\, e^{-(\alpha+\beta)\phi} e^{b\nu} = \frac{1}{2} e^{2\beta\phi}(F_{\mu\nu} - 2\beta\,\partial_\nu\phi A_\mu)\, e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= \frac{1}{2} e^{(\beta-\alpha)\phi}(F_\mu{}^b - 2\beta\,\partial^b\phi A_\mu) \quad \to F^b{}_c\, e_\mu{}^c$$

$$= -\beta e^{(\beta-\alpha)\phi}\partial^b\phi A_\mu - \frac{1}{2} e^{(\beta-\alpha)\phi} F^b{}_\mu$$

$$= -e^{(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi A_\mu + \frac{1}{2} F^b{}_\mu\right] = -\hat\omega_\mu{}^{\underline{z}b}$$

• $\hat{\omega}_{\underline{z}}{}^{b\underline{z}} = \hat{\omega}_{\underline{z}[NP]}\,\hat{e}^{bN}\,\hat{e}^{\underline{z}P}$

$$= \hat{\omega}_{\underline{z}[\nu\rho]}\hat{e}^{b\nu}\underbrace{\hat{e}^{\underline{z}\rho}}_{0} + \hat{\omega}_{\underline{z}[\nu\underline{z}]}\hat{e}^{b\nu}\hat{e}^{\underline{z}\underline{z}} + \hat{\omega}_{\underline{z}[\underline{z}\rho]}\overbrace{\hat{e}^{b\underline{z}}}^{0}\hat{e}^{\underline{z}\rho} + \overbrace{\hat{\omega}_{\underline{z}[\underline{z}\underline{z}]}}^{0}\hat{e}^{b\underline{z}}\hat{e}^{\underline{z}\underline{z}}$$

$$= \hat{\omega}_{\underline{z}[\nu\underline{z}]}\, e^{-(\alpha+\beta)\phi}\, e^{b\nu} = -e^{2\beta\phi}\beta\partial_\nu\phi\, e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\,\partial^b\phi = -\hat{\omega}_{\underline{z}}{}^{\underline{z}b}$$

Using again compact notation we find:

$$\boxed{\hat{\omega}^{b\underline{z}} = -\omega^{\underline{z}b} = -\beta e^{-\alpha\phi}\partial^b\phi\,\hat{e}^{z} - \frac{1}{2}{}^{(\beta-2\alpha)\phi}F^b{}_c\,\hat{e}^c}$$

• $\hat{\omega}_\mu{}^{\underline{z}\underline{z}} = \hat{\omega}_{\underline{z}}{}^{\underline{z}\underline{z}} = 0$

▲ Once we have the spin connection we compute the Riemann tensor

$$\hat{R}_{MN}{}^{BC} = \partial_M\hat{\omega}_N{}^{BC} - \partial_N\hat{\omega}_M{}^{DC} + \hat{\omega}_M{}^B{}_D\,\hat{\omega}_N{}^{DC} - \hat{\omega}_N{}^B{}_D\,\hat{\omega}_M{}^{DC}$$

• $\hat{R}_{\mu\nu}{}^{bc} = \partial_\mu\hat{\omega}_\nu{}^{bc} + \hat{\omega}_\mu{}^b{}_D\,\hat{\omega}_\nu{}^{Dc} - \partial_\nu\hat{\omega}_\mu{}^{bc} - \hat{\omega}_\nu{}^b{}_D\,\hat{\omega}_\mu{}^{Dc}$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}\partial_\mu\left(e^{2(\beta-\alpha)\phi}F^{bc}A_\nu\right)$$

$$+ \hat{\omega}_\mu{}^b{}_d\,\hat{\omega}_\nu{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\,\hat{\omega}_\nu{}^{\underline{z}c} - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}\left[2(\beta-\alpha)\partial_\mu\phi F^{bc}A_\nu\right.$$

$$\left. + \partial_\mu F^{bc}A_\nu + F^{bc}\partial_\mu A_\nu\right]$$

$$+ \left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d A_\mu\right]$$

$$\left[\omega_\nu{}^{dc} + \alpha(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\nu\right]$$

$$
\begin{aligned}
& - e^{2(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi\, A_\mu + \tfrac{1}{2}F^b{}_\mu\right]\left[\beta\,\partial^c\phi\, A_\nu + \tfrac{1}{2}F^c{}_\nu\right] - (\mu\leftrightarrow\nu) \\
&= \overbrace{\partial_\mu \omega_\nu{}^{bc} + \omega_\mu{}^b{}_d\, \omega_\nu{}^{dc}}^{R_{\mu\nu}{}^{bc}} + \alpha\, \partial_\mu\left(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c\right) \\
&\quad - e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} A_\nu + \tfrac{1}{2}\partial_\mu F^{bc} A_\nu + \tfrac{1}{2}F^{bc}\partial_\mu A_\nu \\
&\qquad + \beta^2\, \underline{\partial^b\phi\,\partial^c\phi\, A_\mu A_\nu} + \tfrac{1}{2}\beta\,\partial^b\phi\, A_\mu F^c{}_\nu + \tfrac{1}{2}\beta\,\partial^c\phi\, A_\nu F^b{}_\mu + \tfrac{1}{4}F^b{}_\mu F^c{}_\nu\Big] \\
&\quad + \alpha\, \omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) - \tfrac{1}{2}e^{2(\beta-\alpha)\phi}\, \omega_\mu{}^b{}_d\, F^{dc} A_\nu \\
&\quad + \alpha\, \omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right) - \tfrac{1}{2}\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right)F^{dc} A_\nu \\
&\quad + \alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - \underline{\partial^b\phi\,\partial^c\phi\, g_{\mu\nu}} + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c\right) \\
&\quad - \tfrac{1}{2}e^{2(\beta-\alpha)\phi}\, \omega_\nu{}^{dc}\, F^b{}_d\, A_\mu + \underline{\tfrac{1}{4}e^{4(\beta-\alpha)\phi}\, F^b{}_d\, F^{dc} A_\mu A_\nu} \\
&\quad - \tfrac{1}{2}\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right)F^b{}_d\, A_\mu - \underline{(\mu\leftrightarrow\nu)}
\end{aligned}
$$

NOTE: Underlined terms vanish because they are $\mu\leftrightarrow\nu$ symmetric

$$
\begin{aligned}
&= \tfrac{1}{2}R_{\mu\nu}{}^{bc} + \alpha\left(\partial_\mu\partial^c\phi\, e_\nu{}^b + \partial^c\phi\,\partial_\mu e_\nu{}^b - \partial_\mu\partial^b\phi\, e_\nu{}^c - \partial^b\phi\,\partial_\mu e_\nu{}^c\right) \\
&\quad - e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} A_\nu + \tfrac{1}{2}\beta\,\partial^b\phi\, F^c{}_\nu A_\mu + \tfrac{1}{2}\beta\,\partial^c\phi\, F^b{}_\mu A_\nu \\
&\qquad + \tfrac{1}{2}\alpha\,\partial_d\phi\, F^{dc} A_\nu e_\mu{}^b - \tfrac{1}{2}\alpha\,\partial^b\phi\, F_\mu{}^c A_\nu \\
&\qquad + \tfrac{1}{2}\alpha\,\partial^c\phi\, F^b{}_\nu A_\mu - \tfrac{1}{2}\alpha\,\partial^d\phi\, F^b{}_d\, A_\mu e_\nu{}^c \\
&\qquad + \tfrac{1}{2}\partial_\mu F^{bc} A_\nu + \tfrac{1}{2}F^{bc}\partial_\mu A_\nu + \tfrac{1}{4}F^b{}_\mu F^c{}_\nu \\
&\qquad + \tfrac{1}{2}\omega_\mu{}^b{}_d\, F^{dc} A_\nu + \tfrac{1}{2}\omega_\nu{}^{dc}\, F^b{}_d\, A_\mu\Big] \\
&\quad + \alpha\, \omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) + \alpha\, \omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right) \\
&\quad + \alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c\right) - \underline{(\mu\leftrightarrow\nu)}
\end{aligned}
$$

$$\bullet\quad \hat R_{\mu\nu}{}^{b\underline{z}} = \partial_\mu \hat\omega_\nu{}^{b\underline{z}} + \hat\omega_\mu{}^{b}{}_{D}\,\hat\omega_\nu{}^{D\underline{z}} - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu \hat\omega_\nu{}^{b\underline{z}} + \hat\omega_\mu{}^{b}{}_{c}\,\hat\omega_\nu{}^{c\underline{z}} + \hat\omega_\mu{}^{b}{}_{\underline{z}}\,\overbrace{\hat\omega_\nu{}^{\underline{z}\underline{z}}}^{0} - (\mu\leftrightarrow\nu)$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi\, A_\nu + \partial_\mu\partial^b\phi\, A_\nu + \partial^b\phi\,\partial_\mu A_\nu\right]$$

$$-\frac{1}{2} e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \partial_\mu F^b{}_\nu\right]$$

$$-\left[\omega_\mu{}^b{}_c + \alpha\left(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\right]$$

$$e^{(\beta-\alpha)\phi}\left[\beta\,\partial^c\phi\, A_\nu + \frac{1}{2} F^c{}_\nu\right] - (\mu\leftrightarrow\nu)$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi\, A_\nu + \beta\,\partial_\mu\partial^b\phi\, A_\nu + \beta\,\partial^b\phi\,\partial_\mu A_\nu$$

$$+\frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \frac{1}{2}\partial_\mu F^b{}_\nu + \beta\,\omega_\mu{}^b{}_c\,\partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu$$

$$+\alpha\beta\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\,\partial_c\phi\, e_\mu{}^b F^c{}_\nu$$

$$-\alpha\beta\,\partial_\mu\phi\,\partial^b\phi\, A_\nu - \frac{1}{2}\alpha\,\partial^b\phi\, F_{\mu\nu}\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\left[\underline{\frac{1}{2}\beta F^b{}_c\,\partial^c\phi\, A_\mu A_\nu} + \frac{1}{4} F^b{}_c F^c{}_\nu A_\mu\right] - \underline{(\mu\leftrightarrow\nu)}$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta^2\partial_\mu\phi\,\partial^b\phi\, A_\nu - 2\alpha\beta\,\partial_\mu\phi\,\partial^b\phi\, A_\nu + \beta\,\partial_\mu\partial^b\phi\, A_\nu$$

$$+\alpha\beta\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\,\partial_c\phi\, F^c{}_\nu e_\mu{}^b + \frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu$$

$$+\beta\,\partial^b\phi\,\partial_\mu A_\nu - \frac{1}{2}\alpha\,\partial^b\phi\, F_{\mu\nu} + \frac{1}{2}\partial_\mu F^b{}_\nu$$

$$+\beta\,\omega_\mu{}^b{}_c\,\partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\,\frac{1}{4} F^b{}_c F^c{}_\nu A_\mu - (\mu\leftrightarrow\nu) = -\hat R_{\mu\nu}{}^{\underline{z}b}$$

$$\bullet\quad \hat{R}_{\mu z}{}^{bc} = \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^{b}{}_{D}\, \hat{\omega}_z{}^{Dc} - (\mu \leftrightarrow z)$$

$$= \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^{b}{}_{d}\, \hat{\omega}_z{}^{dc} + \hat{\omega}_\mu{}^{b}{}_{\underline{z}}\, \hat{\omega}_z{}^{\underline{z}c}$$

$$- \underbrace{\partial_z \hat{\omega}_\mu{}^{bc}}_{0} - \hat{\omega}_z{}^{b}{}_{d}\, \hat{\omega}_\mu{}^{dc} - \hat{\omega}_z{}^{b}{}_{\underline{z}}\, \hat{\omega}_\mu{}^{\underline{z}c}$$

$$= -\tfrac{1}{2} e^{2(\beta-\alpha)\phi} \left[ 2(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \partial_\mu F^{bc} \right]$$

$$- \left[ \omega_\mu{}^{b}{}_{d} + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_d A_\mu \right] \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^{dc}$$

$$- e^{(\beta-\alpha)\phi}\left[ \beta \partial^b\phi A_\mu + \tfrac{1}{2} F^b{}_\mu \right] \beta e^{(\beta-\alpha)\phi} \partial^c\phi$$

$$+ \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_d \left[ \omega_\mu{}^{dc} + \alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c) - \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^{dc} A_\mu \right]$$

$$+ \beta e^{(\beta-\alpha)\phi} \partial^b\phi\;\; e^{(\beta-\alpha)\phi}\left[ \beta\partial^c\phi A_\mu + \tfrac{1}{2} F^c{}_\mu \right]$$

$$= - e^{2(\beta-\alpha)\phi} \Big[ (\beta-\alpha)\partial_\mu\phi\, F^{bc} + \tfrac{1}{2}\partial_\mu F^{bc} + \tfrac{1}{2}\omega_\mu{}^b{}_d F^{dc}$$

$$+ \tfrac{1}{2}\alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) F^{dc} + \beta^2 \partial^b\phi\,\partial^c\phi\, A_\mu + \tfrac{1}{2}\beta F^b{}_\mu \partial^c\phi$$

$$- \tfrac{1}{2}\omega_\mu{}^{dc} F^b{}_d - \tfrac{1}{2}\alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c) F^b{}_d - \beta^2 \partial^b\phi\,\partial^c\phi\, A_\mu$$

$$- \tfrac{1}{2}\beta\partial^b\phi\, F^c{}_\mu \Big]$$

$$+ e^{4(\beta-\alpha)\phi}\left[ \tfrac{1}{4} F^b{}_d F^{dc} A_\mu - \tfrac{1}{4} F^b{}_d F^{dc} A_\mu \right]$$

$$= - e^{2(\beta-\alpha)\phi} \Big[ (\beta-\alpha)\partial_\mu\phi\, F^{bc} - \tfrac{1}{2}\alpha\partial^d\phi\, F^c{}_d e_\mu{}^b + \tfrac{1}{2}\alpha\partial^b\phi\, F^c{}_\mu$$

$$- \tfrac{1}{2}\alpha\partial^c\phi\, F^b{}_\mu + \tfrac{1}{2}\alpha\partial^d\phi\, F^b{}_d e_\mu{}^c + \tfrac{1}{2}\beta\partial^c\phi\, F^b{}_\mu - \tfrac{1}{2}\beta\partial^b\phi\, F^c{}_\mu$$

$$- \tfrac{1}{2}\omega_\mu{}^b{}_d F^{cd} + \tfrac{1}{2}\omega_\mu{}^c{}_d F^{bd} + \tfrac{1}{2}\partial_\mu F^{bc} \Big]$$

$$
\begin{aligned}
&= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \alpha\, \partial^{[b}\phi\, F^{c]}{}_\mu + \alpha\, \partial^d\phi\, F^{[b}{}_d\, e_\mu{}^{c]} \\
&\qquad\qquad + \beta\, F^{[b}{}_\mu\, \partial^{c]}\phi - \omega_\mu{}^{[b}{}_d\, F^{c]d} + \tfrac{1}{2}\partial_\mu F^{bc}\Big] \\
&= -\hat{R}_{z\mu}{}^{bc}
\end{aligned}
$$

- $$\hat{R}_{\mu z}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{D\underline{z}} - (\mu \leftrightarrow z)$$

$$
\begin{aligned}
&= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \overbrace{\hat{\omega}_z{}^{\underline{z}\underline{z}}}^{0} \\
&\quad - \underbrace{\partial_z \hat{\omega}_\mu{}^{b\underline{z}}}_{0} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}} - \omega_z{}^b{}_{\underline{z}}\, \underbrace{\hat{\omega}_\mu{}^{\underline{z}\underline{z}}}_{0} \\
&= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}} \\
&= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi\Big] \\
&\quad - \Big[\omega_\mu{}^b{}_c + \alpha(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}) - \tfrac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\Big]\beta\, e^{(\beta-\alpha)\phi}\partial^c\phi \\
&\quad - \tfrac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_c \cdot e^{(\beta-\alpha)\phi}\Big[\beta\partial^c\phi\, A_\mu + \tfrac{1}{2}F^c{}_\mu\Big] \\
&= -\beta e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \omega_\mu{}^b{}_c\,\partial^c\phi \\
&\qquad\qquad + \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b - \alpha\,\partial^b\phi\,\partial_\mu\phi\Big] \\
&\quad + e^{3(\beta-\alpha)\phi}\Big[\underline{\tfrac{1}{2}\beta F^b{}_c\,\partial^c\phi\, A_\mu} - \underline{\tfrac{1}{2}\beta F^b{}_c\,\partial^c\phi\, A_\mu} \\
&\qquad\qquad - \tfrac{1}{4}F^b{}_c\, F^c{}_\mu\Big] \\
&= -\beta e^{(\beta-\alpha)\phi}\Big[(\beta-2\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b + \omega_\mu{}^b{}_c\,\partial^c\phi\Big] \\
&\quad - \tfrac{1}{4}e^{3(\beta-\alpha)\phi} F^b{}_c\, F^c{}_\mu = -\hat{R}_{\mu z}{}^{\underline{z}b} = -\hat{R}_{z\mu}{}^{b\underline{z}} = \hat{R}_{z\mu}{}^{\underline{z}b}
\end{aligned}
$$

▲ With the Riemann tensor we compute now the curved/flat Ricci tensor

$$\hat R_{NC} = \hat R_{MN}{}^{B}{}_{C}\;\hat e_B{}^M$$

• $\hat R_{\nu c} = \hat R_{M\nu}{}^{B}{}_{c}\;\hat e_B{}^M$

$$= \hat R_{\mu\nu}{}^{b}{}_{c}\,\hat e_b{}^\mu + \hat R_{\mu\nu}{}^{\underline z}{}_{c}\,\overbrace{\hat e_{\underline z}{}^\mu}^{0} + \hat R_{z\nu}{}^{b}{}_{c}\,e_b{}^z + \hat R_{z\nu}{}^{\underline z}{}_{c}\,\hat e_{\underline z}{}^z$$

$$= e^{-\alpha\phi}\,e_b{}^\mu\,\hat R_{\mu\nu}{}^{b}{}_{c} - e^{-\alpha\phi}A_b\,\hat R_{z\nu}{}^{b}{}_{c} + e^{-\beta\phi}\,R_{z\nu}{}^{\underline z}{}_{c}$$

$$= e^{-\alpha\phi}R_{\nu c} + e^{-\alpha\phi}\alpha\Big(\partial_b\partial_c\phi\, e_\nu{}^b - \partial_\nu\partial_c\phi\, D + \partial_c\phi\,\partial_b e_\nu{}^b - \partial_c\phi\,\partial_\nu e_\mu{}^b e_b{}^\mu$$

$$- \partial^2\phi\, e_{\nu c} + \partial_\nu\partial^b\phi\,\eta_{bc} - \partial^b\phi\,\partial_b e_{\nu c}$$

$$+ \partial^b\phi\,\partial_\nu e_{\mu c}\, e_b{}^\mu\Big)$$

$$- e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\left(\partial_b\phi\, F^b{}_c A_\nu - \partial_\nu\phi\, F^b{}_c A_b\right)$$

$$+ \tfrac12\beta\left(\partial^b\phi\, F_{c\nu} A_b + \partial_b\phi\, F^b{}_c A_\nu\right)$$

$$+ \tfrac12\beta\left(\cancelto{0}{\partial_c\phi\, F^b{}_b A_\nu} - \partial_c\phi\, F^b{}_\nu A_b\right)$$

$$+ \tfrac12\alpha\,\partial_b\phi\, F^b{}_c\Big(A_\nu \underbrace{\delta_d{}^d}_{D} - \underbrace{A_d\, e_\nu{}^d}_{A_\nu}\Big)$$

$$- \tfrac12\alpha\,\partial_b\phi\left(F^b{}_c A_\nu - F_{\nu c} A^b\right)$$

$$+ \tfrac12\alpha\,\partial_c\phi\left(F^b{}_\nu A_b - \cancelto{0}{F^b{}_b A_\nu}\right)$$

$$+ \tfrac12\alpha\,\partial_b\phi\, F^{bd}\left(A_d\, e_{\nu c} - A_\nu\,\eta_{dc}\right)$$

$$+ \tfrac12\left(\partial_b F^b{}_c A_\nu - \partial_\nu F^b{}_c A_b\right) + \tfrac12 F^b{}_c F_{b\nu}$$

$$+ \tfrac14\left(\cancelto{0}{F^b{}_b F_{c\nu}} - F^b{}_\nu F_{cb}\right)$$

$$+ \tfrac12 F^d{}_c\left(\omega_b{}^b{}_d A_\nu - \omega_\nu{}^b{}_d A_b\right)$$

$$+ \tfrac12 F^b{}_d\left(\omega_\nu{}^d{}_c A_b - \omega_b{}^d{}_c A_\nu\right)\Big]$$

SO(1, D-1) generators are antisymmetric

$$+\, e^{-\alpha\phi}\Big[\alpha\, \omega_b{}^b{}_d\left(\partial_c\phi\, e_\nu{}^d - \partial^d\phi\, e_{\nu c}\right) - \alpha\, \omega_\nu{}^b{}_d\big(\underbrace{\cancel{\partial_c\phi\, \delta_b^d}}_{\to\, 0} - \partial^d\phi\, \eta_{bc}\big)$$

$$+\,\alpha\, \omega_\nu{}^d{}_c\big(\partial_d\phi\, \overset{D}{\delta_b^b} - \partial^b\phi\, \eta_{bd}\big) - \alpha\, \omega_b{}^d{}_c\left(\partial_d\phi\, e_\nu{}^b - \partial^b\phi\, e_{\nu d}\right)$$

$$+\,\alpha^2\big(\partial_\nu\phi\, \partial_c\phi\, \overset{D}{\delta_b^b} - \partial_b\phi\, \partial_c\phi\, e_\nu{}^b + \partial^b\phi\, \partial_b\phi\, e_{\nu c} - \partial^b\phi\, \partial_\nu\phi\, \eta_{bc}$$

$$-\,\partial_d\phi\, \partial^d\phi\, \underset{D}{\delta_b^b}\, e_{\nu c} + \partial_d\phi\, \partial^d\phi\, e_\nu{}^b\, \eta_{bc}\big)\Big]$$

$$-\, A_b\, e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\, \partial_\nu\phi\, F^b{}_c - \tfrac{1}{2}\alpha\, \partial^d\phi\, F_{cd}\, e_\nu{}^b + \tfrac{1}{2}\alpha\, \partial^b\phi\, F_{c\nu}$$

$$-\,\tfrac{1}{2}\alpha\, \partial_c\phi\, F^b{}_\nu + \tfrac{1}{2}\alpha\, \partial^d\phi\, F^b{}_d\, e_{\nu c} + \tfrac{1}{2}\beta\, \partial_c\phi\, F^b{}_\nu$$

$$-\,\tfrac{1}{2}\beta\, \partial^b\phi\, F_{c\nu} - \tfrac{1}{2}\omega_\nu{}^b{}_d\, F_c{}^d + \tfrac{1}{2}\omega_{\nu c d}\, F^{bd}$$

$$+\,\tfrac{1}{2}\partial_\nu F^b{}_c\Big]$$

$$-\,\beta\, e^{-\alpha\phi}\Big[(\beta-2\alpha)\, \partial_\nu\phi\, \partial_c\phi + \partial_\nu\partial_c\phi + \alpha\, \partial_d\phi\, \partial^d\phi\, e_{\nu c} + \omega_{\nu c d}\, \partial^d\phi\Big]$$

$$-\,\tfrac{1}{4}\, e^{(2\beta-3\alpha)\phi}\, F_{cd}\, F^d{}_\nu$$

$$= e^{-\alpha\phi}\Big[R_{\nu c} + \alpha\big(\partial_b\partial_c\phi\, e_\nu{}^b - \partial^2\phi\, e_{\nu c} - (D-1)\, \partial_\nu\partial_c\phi$$

$$+\,\partial_c\phi\, \partial_b e_\nu{}^b - \partial^b\phi\, \partial_b e_{\nu c} - \partial_c\phi\, \partial_\nu e_\mu{}^b\, e_b{}^\mu + \partial^b\phi\, \partial_\nu e_{\mu c}\, e_b{}^\mu\big)$$

$$+\,\omega_b{}^b{}_\nu\, \partial_c\phi - \omega_b{}^{bd}\, \partial_d\phi\, e_{\nu c} + (3-D)\, \omega_{\nu c}{}^d\, \partial_d\phi$$

$$+\,\omega^d{}_{\nu c}\, \partial_d\phi\big) + \alpha^2 (D-2)\left(\partial_\nu\phi\, \partial_c\phi - \partial^d\phi\, \partial_d\phi\, e_{\nu c}\right)$$

$$-\,\beta^2\, \partial_\nu\phi\, \partial_c\phi + \alpha\beta\left(2\, \partial_\nu\phi\, \partial_c\phi - \partial_d\phi\, \partial^d\phi\, e_{\nu c}\right)$$

$$-\,\beta\left(\partial_\nu\partial_c\phi + \omega_{\nu c}{}^d\, \partial_d\phi\right)\Big]$$

$$-\, e^{(2\beta-3\alpha)\phi}\Big[\alpha\, \frac{D-4}{2}\, \partial_b\phi\, F^b{}_c\, A_\nu + \beta\, \frac{3}{2}\, \partial_b\phi\, F^b{}_c\, A_\nu + \tfrac{1}{2}\partial_b F^b{}_c\, A_\nu + \tfrac{1}{2} F_{b\nu}\, F^b{}_c$$

$$+\,\tfrac{1}{2}\omega_b{}^b{}_d\, F^d{}_c\, A_\nu - \tfrac{1}{2}\omega_b{}^d{}_c\, F^b{}_d\, A_\nu\Big]$$

$$
\begin{aligned}
\bullet\ \hat R_{zc} &= \hat R_{Mz}{}^{B}{}_{c}\, \hat e_B{}^M \\
&= \hat R_{\mu z}{}^{b}{}_{c}\, \hat e_b{}^\mu + \hat R_{\mu z}{}^{\underline z}{}_{c}\, \overbrace{\hat e_{\underline z}{}^\mu}^{0} + \overbrace{\hat R_{zz}{}^{b}{}_{c}}^{0}\, \hat e_b{}^z + \overbrace{\hat R_{zz}{}^{\underline z}{}_{c}}^{0}\, \hat e_{\underline z}{}^z \\
&= e^{-\alpha\phi} e_b{}^\mu\, \hat R_{\mu z}{}^{b}{}_{c}
\end{aligned}
$$

$$
= -e^{(2\beta-3\alpha)\phi}\Big[\alpha\Big(-\partial_b\phi F^b{}_c + \frac{D}{2}\partial^d\phi F_{dc} - \partial^b\phi F_{bc}\Big) + \frac{1}{2}\partial_b F^b{}_c
$$

$$
+ \frac{3}{2}\partial_b\phi\, F^b{}_c + \frac{1}{2}\omega_b{}^b{}_d F^d{}_c - \frac{1}{2}\omega_{bcd} F^{db}\Big]
$$

$$
\begin{aligned}
\bullet\ \hat R_{v\underline z} &= \hat R_{Mv}{}^{B}{}_{\underline z}\, \hat e_B{}^M \\
&= \hat R_{\mu v}{}^{b}{}_{\underline z}\, \hat e_b{}^\mu + \hat R_{\mu v}{}^{\underline z}{}_{\underline z}\, \overbrace{\hat e_{\underline z}{}^\mu}^{0} + \hat R_{zv}{}^{b}{}_{\underline z}\, \hat e_b{}^z + \overbrace{\hat R_{zv}{}^{\underline z}{}_{\underline z}}^{0}\, \hat e_{\underline z}{}^z \\
&= e^{-\alpha\phi} e_b{}^\mu\, \hat R_{\mu v}{}^{b}{}_{\underline z} - e^{-\alpha\phi} A_b\, \hat R_{zv}{}^{b}{}_{\underline z}
\end{aligned}
$$

$$
= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\big(\partial_b\phi\,\partial^b\phi\, A_v - \partial_v\phi\,\partial^b\phi\, A_b\big)
$$

$$
+\alpha\beta\big(-2\partial_b\phi\,\partial^b\phi\, A_v + 2\partial_v\phi\,\partial^b\phi\, A_b + (D-1)\partial_c\phi\,\partial^c\phi\, A_v\big)
$$

$$
+\alpha\Big(\frac{1}{2}\partial_c\phi\,(D-1)F^c{}_v - \frac{1}{2}\partial_b\phi\, F^b{}_v - \partial^b\phi\, F_{bv}\Big)
$$

$$
+\beta\Big(\partial^2\phi\, A_v - \partial_v\partial^b\phi\, A_b + \frac{1}{2}\partial_b\phi\, F^b{}_v + \partial^b\phi\, F_{bv}
$$

$$
+\omega_b{}^b{}_c\,\partial^c\phi\, A_v - \omega_v{}^b{}_c\,\partial^c\phi\, A_b\Big) + \frac{1}{2}\partial_b F^b{}_v
$$

$$
-\frac{1}{2}\partial_v F^b{}_\mu e_b{}^\mu + \frac{1}{2}\omega_b{}^b{}_c F^c{}_v - \frac{1}{2}\omega_v{}^b{}_c F^c{}_b\Big]
$$

$$
+e^{(3\beta-4\alpha)\phi}\Big[\frac{1}{4}F^b{}_c F^c{}_v A_b - \frac{1}{4}F^b{}_c F^c{}_b A_v\Big]
$$

$$
-\beta e^{(\beta-2\alpha)\phi} A_b\Big[\alpha\big(-2\partial_v\phi\,\partial^b\phi + \partial_c\phi\,\partial^c\phi\, e_v{}^b\big) + \beta\,\partial_v\phi\,\partial^b\phi
$$

$$
+\partial_v\partial^b\phi + \omega_v{}^b{}_c\,\partial^c\phi\Big] - \frac{1}{4}e^{(3\beta-4\alpha)\phi} A_b F^b{}_c F^c{}_v
$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\,\partial_b\phi\,\partial^b\phi\,A_\nu + \alpha\beta\,(D-2)\,\partial_b\phi\,\partial^b\phi\,A_\nu$$

$$+\,\alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_\nu + \beta\Big(\partial^2\phi\,A_\nu + \frac{3}{2}\,\partial_b\phi\,F^b{}_\nu + \omega_b{}^b{}_c\,\partial^c\phi\,A_\nu\Big)$$

$$+\,\frac{1}{2}\,\partial_b F^b{}_\nu - \frac{1}{2}\,\partial_\nu F^b{}_\mu\,e_b{}^\mu + \frac{1}{2}\,\omega_b{}^b{}_c\,F^c{}_\nu - \frac{1}{2}\,\omega_\nu{}^b{}_c\,F^c{}_b\Big]$$

$$-\,\frac{1}{4}\,e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b\,A_\nu$$

- $$\hat{R}_{z\underline{z}} = \hat{R}_{Mz}{}^B{}_{\underline{z}}\,\hat{e}_B{}^M$$

$$= \hat{R}_{\mu z}{}^b{}_{\underline{z}}\,\hat{e}_b{}^\mu + \hat{R}_{\mu z}{}^{\underline{z}}{}_{\underline{z}}\,\overbrace{\hat{e}_{\underline{z}}{}^\mu}^{0} + \overbrace{\hat{R}_{zz}{}^b{}_{\underline{z}}}^{0}\,\hat{e}_b{}^z + \overbrace{\hat{R}_{zz}{}^{\underline{z}}{}_{\underline{z}}}^{0}\,e_{\underline{z}}{}^z$$

$$= e^{-\alpha\phi}\,e_b{}^\mu\,\hat{R}_{\mu z}{}^b{}_{\underline{z}}$$

$$= -\beta\,e^{(\beta-2\alpha)\phi}\Big[(\beta-2\alpha)\,\partial_b\phi\,\partial^b\phi + \partial^2\phi + \alpha D\,\partial_c\phi\,\partial^c\phi + \omega_b{}^b{}_c\,\partial^c\phi\Big]$$

$$-\,\frac{1}{4}\,e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\,\partial_b\phi\,\partial^b\phi + \alpha\beta\,(D-2)\,\partial_b\phi\,\partial^b\phi + \beta\big(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\big)\Big]$$

$$-\,\frac{1}{4}\,e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b$$

Now we compute the vielbein components of the Ricci tensor

$$\hat{R}_{AC} = \hat{e}_A{}^N\,\hat{R}_{NC}$$

- $$\hat{R}_{ac} = \hat{e}_a{}^N\,\hat{R}_{Nc} = \hat{e}_a{}^\nu\,\hat{R}_{\nu c} + \hat{e}_a{}^z\,\hat{R}_{zc}$$

$$= e^{-\alpha\phi}\,e_a{}^\nu\,\hat{R}_{\nu c} - e^{-\alpha\phi}\,A_a\,\hat{R}_{zc}$$

$$= e^{-2\alpha\phi}\Big[ R_{ac} + \alpha\Big( \partial_a\partial_c\phi - \partial^2\phi\,\eta_{ac} - (D-1)\,\partial_a\partial_c\phi$$
$$+ \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu$$
$$+ \omega_b{}^b{}_a\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\,\eta_{ca} + (3-D)\,\omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi\Big)$$
$$+ \alpha^2 (D-2)\Big(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\Big) - \beta^2\partial_a\phi\,\partial_c\phi$$
$$+ \alpha\beta\,\big(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\big) - \beta\big(\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi\big)\Big]$$

$$- e^{(2\beta-4\alpha)\phi}\Big[\alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_c\,A_a + \beta\,\frac{3}{2}\,\partial_b\phi\,F^b{}_c\,A_a + \frac{1}{2}\partial_b F^b{}_c A_a$$
$$+ \frac{1}{2}F_{ba}F^b{}_c + \frac{1}{2}\omega_b{}^b{}_d\,F^d{}_c\,A_a - \frac{1}{2}\omega_b{}^d{}_c\,F^b{}_d\,A_a\Big]$$

$$+ e^{(2\beta-4\alpha)\phi}\Big[\alpha\Big(-\partial_b\phi\,F^b{}_c\,A_a + \frac{D}{2}\partial^d\phi\,F_{dc}A_a - \partial^b\phi\,F_{bc}A_a\Big)$$
$$+ \frac{1}{2}\partial_b F^b{}_c\,A_a + \beta\,\frac{3}{2}\,\partial_b\phi\,F^b{}_c\,A_a$$
$$+ \frac{1}{2}\omega_b{}^b{}_d\,F^d{}_c\,A_a - \frac{1}{2}\omega_{bcd}\,F^{db}A_a\Big]$$

$$= e^{-2\alpha\phi}\Big[ R_{ac} + \alpha\Big( -(D-2)\,\partial_a\partial_c\phi - \partial^2\phi\,\eta_{ac}$$
$$+ \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu$$
$$+ \omega_b{}^b{}_a\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\,\eta_{ca} + (3-D)\,\omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi\Big)$$
$$+ \alpha^2 (D-2)\Big(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\Big) - \beta^2\partial_a\phi\,\partial_c\phi$$
$$+ \alpha\beta\,\big(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\big) - \beta\big(\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi\big)\Big]$$
$$- e^{(2\beta-4\alpha)\phi}\Big[\partial_b\phi\,F^b{}_c\,A_a\,\frac{(D-4)\alpha+3\beta}{2} + \frac{1}{2}\partial_b F^b{}_c\,A_a + \frac{1}{2}F_{ba}F^b{}_c$$
$$+ \frac{1}{2}\omega_b{}^b{}_d\,F^d{}_c\,A_a - \frac{1}{2}\omega_b{}^d{}_c\,F^b{}_d\,A_a$$
$$- \alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_c\,A_a - \frac{1}{2}\partial_b F^b{}_c\,A_a$$

$$-\beta\,\frac{3}{2}\,\partial_b\phi\, F^b{}_c A_a$$

$$-\frac{1}{2}\,\omega_b{}^b{}_d\, F^d{}_c A_a + \frac{1}{2}\,\omega_b{}^d{}_c\, F^b{}_d A_a \Big]$$

$$= e^{-2\alpha\phi}\Big[ R_{ac} + \alpha\Big( -(D-2)\,\partial_a\partial_c\phi - \underbrace{\partial^2\phi}_{\Box\phi}\,\eta_{ac}$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c} e_b{}^\mu$$

$$+\omega_b{}^b{}_a\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\,\eta_{ac} - \underbrace{(D-3)}_{-(D-2)+1}\,\omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi\Big)$$

$$+\alpha^2(D-2)\Big(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\Big) - \beta^2\,\partial_a\phi\,\partial_c\phi$$

$$+\alpha\beta\Big(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\Big) - \beta\Big(\underbrace{\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi}_{\nabla_a\nabla_c\phi}\Big)\Big]$$

$$-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b F_{cb} = (*)$$

NOTE 4: We will see later that one must set $\beta = -(D-2)\alpha$

$$(*) = -\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b F_{cb} + e^{-2\alpha\phi}\Big[R_{ac} + \overbrace{(D-2)\alpha}^{-\beta}\,\nabla_a\nabla_c\phi$$

$$+\partial_a\phi\,\partial_c\phi\Big(\underbrace{\alpha^2(D-2) - \beta^2 + 2\alpha\beta}_{\alpha\beta-\beta^2 = -(D-2)(D-1)\alpha^2}\Big) - \partial^b\phi\,\partial_b\phi\,\eta_{ac}\Big(\underbrace{\alpha^2(D-2)+\alpha\beta}_{0}\Big)$$

$$+\alpha\Big(-\Box\phi\,\eta_{ac} - (D-2)\,\nabla_a\nabla_c\phi + \omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi$$

$$+\omega_b{}^b{}_a\,\partial_c\phi + \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial_d\phi\,\partial^d e_{\nu c} e_a{}^\nu$$

$$-\partial_c\phi\,\partial_a e_\nu{}^b e_b{}^\nu + \partial_d\phi\,\partial_a e_{\nu c} e^{d\nu}\Big)\Big]$$

NOTE 5: $\alpha^2 = \frac{1}{2(D-2)(D-1)}$ [We will see later]

$$= -\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b F_{cb} + e^{-2\alpha\phi}\Big[R_{ac} - \frac{1}{2}\,\partial_a\phi\,\partial_c\phi - \alpha\,\Box\phi\,\eta_{ac}$$

$$+\partial_d\phi\Big(\underbrace{\omega_{ac}{}^d + \omega^d{}_{ac} - \partial^d e_{\nu c} e_a{}^\nu + \partial_a e_{\nu c} e^{d\nu}}_{0}\Big) + \partial_c\phi\Big(\underbrace{\omega_b{}^b{}_a + \partial_b e_\nu{}^b e_a{}^\nu - \partial_a e_\nu{}^b e_b{}^\nu}_{0}\Big) = (*)$$

## Remark 1

$$
\begin{aligned}
\omega_b{}^{ba} &= e_b{}^\mu \omega_\mu{}^{ba} = -e_b{}^\mu \omega_\mu{}^{ab}(e) \\
&= -e_b{}^\mu \tfrac{1}{2}\Big[e^{\nu a}\partial_\mu e_\nu{}^b - e^{\nu b}\partial_\mu e_\nu{}^a - e^{\nu a}\partial_\nu e_\mu{}^b + e^{\nu b}\partial_\nu e_\mu{}^a \\
&\qquad - e^{\nu a} e^{b\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c + e^{\nu b} e^{a\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c\Big] \\
&= -\tfrac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial_b e_\nu{}^a e^{\nu b} - \partial^a e_\mu{}^b e_b{}^\mu + \partial^b e_\mu{}^a e_b{}^\mu \\
&\qquad - e^{\nu a} e^{b\sigma}\partial_\nu e_{\sigma b} + e^{\nu b} e^{a\sigma}\partial_\nu e_{\sigma b}\Big] \\
&= -\tfrac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial^\nu e_\nu{}^a - \partial^a e_\mu{}^b e_b{}^\mu + \partial^\mu e_\mu{}^a \\
&\qquad - \partial^a e_{\sigma b} e^{b\sigma} + \partial^b e_{\sigma b} e^{a\sigma}\Big] \\
&= -\tfrac{1}{2}\Big[2\partial_b e_\nu{}^b e^{\nu a} - \partial^a e_{\nu b} e^{b\nu}\Big] = \partial^a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e^{\nu a} \\
\Rightarrow \omega_b{}^b{}_a &= \partial_a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e_a{}^\nu
\end{aligned}
$$

## Remark 2

[We can compute faster by using anholonomy coefficients]

$$
\begin{aligned}
\omega_{acd} + \omega_{dac} &= \tfrac{1}{2}\Big[\underline{\Omega_{[ac]d}} - \underline{\Omega_{[cd]a}} + \Omega_{[da]c} \\
&\qquad + \Omega_{[da]c} - \underline{\Omega_{[ac]d}} + \underline{\Omega_{[cd]a}}\Big] \\
&= \Omega_{[da]c}
\end{aligned}
$$

see note below

$$
\begin{aligned}
\Rightarrow \omega_{ac}{}^d + \omega^d{}_{ac} &= \Omega_{[ba]c}\,\eta^{bd} = -(\partial_b e_a{}^\rho - \partial_a e_b{}^\rho)\, e_{\rho c}\,\eta^{bd} \\
&= -\partial^d e_a{}^\nu e_{\nu c} + \partial_a e^{d\nu} e_{\nu c} \\
&= \partial^d e_{\nu c} e_a{}^\nu - \partial_a e_{\nu c} e^{d\nu}
\end{aligned}
$$

NOTE: $\Omega_{[\mu\nu]\rho} = (\partial_\mu e_\nu{}^d - \partial_\nu e_\mu{}^d)\, e_{d\rho}$

$$
\begin{aligned}
\Omega_{[ab]c} &= e_a{}^\mu e_b{}^\nu e_c{}^\rho\, \Omega_{[\mu\nu]\rho} = (\partial_a e_\nu{}^d e_b{}^\nu - \partial_b e_\mu{}^d e_a{}^\mu)\,\eta_{cd} \\
&= -(\partial_a e_b{}^\rho - \partial_b e_a{}^\rho)\, e_{\rho c}
\end{aligned}
$$

Important

$$(*) = e^{-2\alpha\phi}\left[R_{ac} - \frac{1}{2}\partial_a\phi\partial_c\phi - \alpha\,\Box\phi\,\eta_{ac}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^b F_{cb}$$

- $$\hat{R}_{\underline{z}\underline{z}} = \hat{e}_{\underline{z}}{}^N \hat{R}_{N\underline{z}} = \underbrace{\hat{e}_{\underline{z}}{}^0}^{0}\hat{R}_{0\underline{z}} + \hat{e}_{\underline{z}}{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\beta\phi}\hat{R}_{z\underline{z}}$$

$$= -e^{-2\alpha\phi}\left[\partial_b\phi\partial^b\phi\left(\beta^2 + (D-2)\alpha\beta\right) + \beta\overbrace{\left(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\right)}^{\eta^{ab}\nabla_a\nabla_b\phi = \Box\phi}\right]$$

$$+ \frac{1}{4}e^{(2\beta-4\alpha)\phi}F_c{}^b F^c{}_b$$

$$= e^{-2\alpha\phi}\left[-\underbrace{(\beta^2 + (D-2)\alpha\beta)}_{0\ (\text{see note } 4)}\partial_b\phi\partial^b\phi - \beta\,\Box\phi\right] + \frac{1}{4}e^{(2\beta-4\alpha)\phi}F^2$$

▲ Using these two vielbein components of the Ricci tensor it is finally straightforward to compute the Ricci scalar

$$\hat{R} = \hat{R}_{AB}\,\eta^{AB}$$

We find:

$$\hat{R} = \hat{R}_{ac}\eta^{ac} + \hat{R}_{\underline{z}\underline{z}} = e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \overbrace{(D\alpha+\beta)}^{D\alpha-(D-2)\alpha = 2\alpha}\Box\phi\right]$$

$$-\frac{1}{2}e^{-2D\alpha\phi}\left(F^2 - \frac{1}{2}F^2\right) =$$

$$= e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \underbrace{2\alpha\,\Box\phi}_{\text{boundary term}} - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]$$

The full $(D+1)$-dimensional action then reduces to

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2} \int d^{D+1}x \, \hat{e} \, \hat{R}$$

$$= \frac{1}{2\kappa_{D+1}^2} \int_0^{2\pi L} dz \int d^D x \; e^{(\alpha D + \beta)\phi} e \, \hat{R}$$

$$= \frac{1}{2 \underbrace{\frac{\kappa_{D+1}^2}{2\pi L}}_{\kappa_D^2 = \frac{\kappa_{D+1}^2}{2\pi L}}} \int d^D x \; \underbrace{e^{[(D-2)\alpha + \beta]\phi}}_{\substack{\text{Canonical E-H if} \\ \beta = -(D-2)\alpha \\ \text{(see note 4)}}} e \left[ R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} \underbrace{e^{-2(D-1)\alpha\phi}}_{\substack{\text{Proper normalisation if} \\ \alpha^2 = \frac{1}{2(D-2)(D-1)}}} F^2 \right]$$

(see note 5)

$$= \frac{1}{2\kappa_D^2} \int d^D x \, e \left[ R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right] = S_D$$

Therefore we recover an Einstein – Maxwell – Dilaton theory !!

$$\boxed{S_{D+1} = \frac{1}{2\kappa_D^2} \int d^D x \; e \left[ R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right]}$$

with $\kappa_D^2 = \dfrac{\kappa_{D+1}^2}{2\pi L}$

<u>Example</u> : If $D=4 \Rightarrow e^{-2(D-1)\alpha\phi} = e^{-\sqrt{3}\,\phi}$

Exercise: Compute the $\hat{R}_{bz}$ component of the Ricci tensor

$$\hat{R}_{bz} = \hat{e}_b{}^N \hat{R}_{Nz} = \hat{e}_b{}^0 \hat{R}_{0z} + \hat{e}_b{}^z \hat{R}_{zz}$$

$$= e^{-\alpha\phi} e_b{}^0 \hat{R}_{0z} - e^{-\alpha\phi} A_b \hat{R}_{zz}$$

$$= -e^{(\beta-3\alpha)\phi}\Big[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \alpha\frac{D-4}{2}\partial_c\phi F^c{}_b$$
$$+\beta\left(\partial^2\phi A_b + \tfrac{3}{2}\partial_c\phi F^c{}_b + \omega_c{}^c{}_d\partial^d\phi A_b\right) + \tfrac{1}{2}\partial_c F^c{}_0 e_b{}^0$$
$$-\tfrac{1}{2}\partial_b F^c{}_0 e_c{}^0 + \tfrac{1}{2}\omega_c{}^c{}_d F^d{}_b - \tfrac{1}{2}\omega_b{}^c{}_d F^d{}_c\Big]$$

$$+\frac{1}{4}e^{(3\beta-5\alpha)\phi}F^2 A_b$$

$$+e^{(\beta-3\alpha)\phi}\Big[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \beta\Box\phi A_b\Big]$$

$$-\frac{1}{4}e^{(3\beta-5\alpha)\phi}F^2 A_b$$

$$= -\frac{1}{2}\underbrace{e^{(\beta-3\alpha)\phi}}_{e^{-(D+1)\alpha}}\Big[-\overbrace{\big((D-4)\alpha + 3\beta\big)}^{-2(D-1)\alpha}\partial_c\phi F_b{}^c$$
$$+\partial_c F^c{}_0 e_b{}^0 - \partial_b F^c{}_0 e_c{}^0$$
$$+\omega_c{}^c{}_d F^d{}_b + \omega_b{}^c{}_d F_c{}^d\Big]$$

NOTE: $-(D+1)\alpha = (D-3)\alpha - 2(D-1)$

$$= \frac{1}{2}e^{(D-3)\alpha\phi}\times e^{-2(D-1)\alpha\phi}\Big[-2(D-1)\alpha\,\partial_c\phi F_b{}^c$$
$$\underbrace{-\partial_c F^c{}_0 e_b{}^0 + \partial_b F^c{}_0 e_c{}^0}_{\partial_c F_b{}^c - F_0{}^c\partial_c e_b{}^0 + F_0{}^c\partial_b e_c{}^0} + \omega_c{}^c{}_d F_b{}^d + \omega_b{}^c{}_d F^d{}_c\Big]$$

NOTE: $\nabla_c F_b{}^c = \partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi}\Big[-2(D-1)\alpha \overbrace{\partial_c\phi}^{\nabla_c\phi} F_b{}^c$$

$$+ \underbrace{\partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c}_{\nabla_c F_b{}^c} + \omega_{cdb}F^{dc} - F_a{}^c\partial_c e_b{}^a + F_a{}^c \partial_b e_c{}^a$$

$$+ \omega_{bcd}F^{dc}\Big] = (*)$$

## Remark 3

$$\omega_{cdb} + \omega_{bcd} = \frac{1}{2}\Big[\underline{\Omega_{[cd]b}} - \underline{\Omega_{[db]c}} + \Omega_{[cb]d}$$

$$+ \Omega_{[cb]d} - \underline{\Omega_{[cd]b}} + \underline{\Omega_{[db]c}}\Big]$$

$$= \Omega_{[cb]d} = -(\partial_b e_c{}^a - \partial_c e_b{}^a)\, e_{ad}$$

$$\Rightarrow (\omega_{cdb}+\omega_{bcd})F^{dc} = -\partial_b e_c{}^a e_{ad}F^{dc} + \partial_c e_b{}^a e_{ad} F^{dc}$$

$$= F_a{}^c\partial_c e_b{}^a - F_a{}^c \partial_b e_c{}^a$$

$$(*) = \frac{1}{2} e^{(D-3)\alpha\phi}\,\nabla_c\Big[e^{-2(D-1)\alpha\phi}F_b{}^c\Big] = \hat{R}_{\underline{z}b}$$

## II. (D+1)-dimensional vs D-dimensional EOMs and symmetries

In this section we discuss the equations of motion (EOMs) that result from $S_{D+1}$ and $S_D$.

i) (D+1)-dimensional EOMs

$$S_{D+1} = \frac{1}{2K_{D+1}^2}\int d\overset{D+1}{x}\, \hat{e}\, \hat{R} \implies \hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = 0$$

NOTE: $\hat{g}^{MN}\hat{G}_{MN} = \hat{R} - \frac{1}{2}(D+1)\hat{R} = \left(1-\frac{1}{2}(D+1)\right)\hat{R} = 0$

$\implies \hat{R} = 0 \implies \hat{R}_{MN} = 0$

$\hookrightarrow D \neq 1$ $\qquad \implies \hat{R}_{AB} = 0$

$$\hat{R}_{AB} = 0 \begin{cases} \hat{R}_{ab} = e^{-2\alpha\phi}\left[R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\,\Box\phi\,\eta_{ab}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^c F_{bc} = 0 \\ \hat{R}_{a\underline{z}} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla_c\left[e^{-2(D-1)\alpha\phi}F_b{}^c\right] = 0 \\ \hat{R}_{\underline{z}\underline{z}} = (D-2)\alpha\, e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}F^2 = 0 \end{cases}$$

- It is important to notice that

$$\phi = 0 \implies \hat{R}_{\underline{z}\underline{z}} = \frac{1}{4}F^2 = 0 \implies \underline{F=0}$$

$\hookrightarrow$ Trivial Maxwell !!

ii) D-dimensional EOMs

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx \, e \left[ R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4} e^{-2(D-1)\alpha\phi} F_{\mu\nu}F^{\mu\nu} \right]$$

The EOMs that follow from the above action are:

- $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) + \frac{1}{2} e^{-2(D-1)\alpha\phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^2 g_{\mu\nu}\right)$$

- $$\nabla^\mu\left(e^{-2(D-1)\alpha\phi} F_{\mu\nu}\right) = 0$$

- $$\Box\phi = -\frac{1}{2}(D-1)\alpha\, e^{-2(D-1)\alpha\phi} F^2$$

It is important to notice again that

$$\phi = 0 \implies F^2 = 0 \implies \underline{F = 0}$$

Trivial Maxwell !!

IMPORTANT : Having set $\phi = 0$ in the Ansatz for the $(D+1)$-dimensional metric would have been inconsistent !! [common mistake]

[Einstein - Maxwell - DILATON]

iii) (D+1)-dimensional symmetries

The symmetry group is (D+1)-dimensional general coordinate transformations. At the infinitesimal level we have

$$\delta_{\hat{\xi}} \hat{g}_{MN} = \hat{\xi}^P \partial_P \hat{g}_{MN} + \hat{g}_{PN} \partial_M \hat{\xi}^P + \hat{g}_{MP} \partial_N \hat{\xi}^P$$

with

$$\hat{\xi}^M(x,z) = \left( \hat{\xi}^\mu(x,z) \, , \, \hat{\xi}^z(x,z) \right)$$

▲ However, in order to preserve the KK Ansatz of the (D+1)-dimensional metric, there are the restrictions:

$$\text{Diffeom}: \quad \hat{\xi}^\mu = \xi^\mu(x) \quad , \quad \hat{\xi}^z = \lambda(x) + \underbrace{c\, z}_{\substack{\text{linear dependence} \\ \text{on } S^1}}$$

▲ On the other hand, the (D+1)-dimensional EOMs have a scaling symmetry [not the E-H action!!] of the form:

$$\left.\begin{aligned} \hat{g}_{MN} &\to a^2 \hat{g}_{MN} \\ \hat{e} &\to a^{(D+1)} \hat{e} \\ \hat{R} &\to a^{-2} \hat{R} \end{aligned}\right\} \quad \hat{e}\hat{R} \to \underbrace{a^{(D-1)}}_{a \in \mathbb{R}} \hat{e}\hat{R} \implies \delta_a \hat{g}_{MN} = 2a\, \hat{g}_{MN}$$

infinitesim.

iv) D-dimensional symmetries

Starting from (D+1)-dimensional diffeomorphisms we will obtain D-dimensional diff + U(1) gauge symmetry + Global symmetries.

Ex : Using $\left\{ \begin{array}{l} \hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu \\ \hat{g}_{\mu z} = \hat{g}_{z\mu} = e^{2\beta\phi} A_\mu \\ \hat{g}_{zz} = e^{2\beta\phi} \end{array} \right\}$ with $\beta = -(D-2)\alpha$

show that $\delta \hat{g}_{MN} = (\delta\xi + \delta a)\, \hat{g}_{MN}$ gives rise to :

$$\delta\phi = \xi^\rho \partial_\rho \phi - \frac{1}{(D-2)\alpha}(c+a)$$

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda - c A_\mu$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \frac{2}{(D-2)} \left[ c + a(D-1) \right] g_{\mu\nu}$$

- Setting $a = -\frac{c}{(D-1)}$ one finds :

$$\delta\phi = \underbrace{\delta_\xi \phi} - \overbrace{\frac{c}{(D-1)\alpha}}^{\text{shift}}$$

$$\delta A_\mu = \underbrace{\delta_\xi A_\mu} + \underbrace{\partial_\mu \lambda} - \overbrace{c A_\mu}^{\text{scaling}}$$

$$\delta g_{\mu\nu} = \underbrace{\delta_\xi g_{\mu\nu}}$$

→ Global symmetry $\equiv \mathbb{R}$ (real parameter)

→ U(1) gauge symmetry

→ D-dimensional diffeomorphisms

- Setting $a = -c$ one finds : $\qquad$ $\underbrace{n\text{-legs} \Rightarrow nc}$

$$\delta\phi = \delta_\xi \phi \qquad \text{(0-legs)}$$

$$\delta A_\mu = \delta_\xi A_\mu + \partial_\mu \lambda - \underbrace{c A_\mu} \qquad \text{(1-leg)}$$

$$\delta g_{\mu\nu} = \delta_\xi g_{\mu\nu} - \underbrace{2c\, g_{\mu\nu}} \qquad \text{(2-legs)}$$

$\longrightarrow$ Real scaling $\mathbb{R}$ symmetry of the D-dimensional EOMs known as "trombone" scaling symmetry.

Important : There are two inequivalent $\mathbb{R}$ global symmetries. One is an actual symmetry of the D-dimens action whereas the other is only of the EOMs.

Important : In modern language, the global symmetries of the action upon dimensional reduction are referred to as "dualities". In this case the duality group is just $G_{global} = \mathbb{R}$ symmetry and affects scalar and vector fields in the reduced theory.