[VII Escuela Mexicana de Cuerdas : July 2021]

Lectures on Type II flux compactifications

I. Kaluza–Klein reduction on S^1

II. (D+1)-dimensional vs D-dimensional EOMs and symmetries

III. Kaluza–Klein reduction of Maxwell and scalar on S^1

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* Scalar kinetic terms and coset spaces

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VI. Type II reduction on T^6

VII. Type II reduction on T^6 with background fluxes

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I. Kaluza-Klein reduction on S^1

In this section we are working out the dimensional reduction of gravity in D+1 dimension down to D dimensions. As we will see, this provides a unification of the form:

D+1 Gravity $\Rightarrow$ Gravity + Maxwell + scalar in D

We will describe gravity in D+1 dimensions:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \, \sqrt{-|\hat{g}|} \; \hat{R}$$

with $\hat{g}_{MN}$ and $\hat{R}_{MN}$ being the metric and Ricci scalar in a (D+1) dimensional space-time $\underbrace{M}_{x^M} = \underbrace{0, 1, \dots, D-1}_{x^\mu}, \underbrace{z}_{z}$.

Let's take the z-coordinate to be S^1 $\Rightarrow$ Fourier expansion

$$\hat{g}_{MN}(x, z) = \sum_{n=0}^{\infty} \hat{g}^{(n)}_{MN}(x) \, e^{i \frac{n}{L} z}$$

$(z \to z + 2\pi L)$

$\Uparrow$ Fourier mode

$\Rightarrow$ The zero-mode $(n=0)$ is a massless mode whereas $n \neq 0$ correspondes to a tower of massive modes (KK tower).

Example: Scalar field $\hat{\phi}$ in D+1 dimensional flat space-time

$$S_\phi = \int d^{D+1}x \, \partial_M \hat{\phi} \, \partial^M \hat{\phi} \quad \underset{\text{E.O.M}}{\Rightarrow} \quad \Box \hat{\phi} = \partial_M \partial^M \hat{\phi} = 0$$

Fourier expansion along S^1 : $\hat{\phi}(x,z) = \sum_{n=0}^{\infty} \phi^{(n)}(x)\, e^{i\frac{n}{L}z}$

so that

$$\hat{\Box}\hat{\phi} = (\underbrace{\partial_\mu \partial^\mu}_{\Box} + \partial_z^2)\hat{\phi} = \sum_{n=0}^{\infty} \underbrace{\left[\Box\phi^{(n)} - \frac{n^2}{L^2}\phi^{(n)}\right]}_{} e^{i\frac{n}{L}z} = 0$$

Each mode must satisfy

$$\Box\phi^{(n)} - \underbrace{\frac{n^2}{L^2}}_{}\phi^{(n)} = 0$$

$m^2 \equiv \frac{n^2}{L^2}$ => Massive modes !!

$m = \frac{|n|}{L}$

Important : The KK phylosophy is to assome a very small L (we don't observe S^1) so that all the modes with $n \neq 0$ are very massive $m = \frac{|n|}{L}$ and we have to reach very high energies in order to produce such KK modes :

$$L \sim m_p \sim 10^{-33}\,cm \Rightarrow m \sim 10^{-5}\,gr$$

(Higgs mass $\sim m_{top} \sim 10^{-22}$ gr)

Important : KK reduction $\equiv$ Truncation to $n=0$ massless modes

=> z - independence !!

$$\boxed{\hat{g}_{MN}(x) = \hat{g}^{(0)}_{MN}(x)}$$

The first step to perform a KK reduction is to split the (D+1)-metric in D-dimensional blocks:

$$\hat{g}_{MN}(x) = \begin{bmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{z\nu} & g_{zz} \end{bmatrix} \equiv \begin{bmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix}$$

$\Rightarrow$ Too naive (big mess)

$$\equiv \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix}$$

$\Downarrow$

Much more convenient !!

(see discussion on symmetries)

Therefore we parameterise the (D+1) metric $\hat{g}_{MN}$ as

$$\boxed{d\hat{s}^2_{D+1} = e^{2\alpha\underbrace{\phi}_{\phi \,\equiv\, \text{"Dilaton"}}} ds^2_D + e^{2\beta\phi} (dz + A_\mu dx^\mu)^2}$$

with α and β being constants to be chosen later on.

The associated (D+1) frame then reads:

$$\hat{e}_M{}^A = \begin{bmatrix} e^{\alpha\phi}\, e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix}$$

$$\boxed{\begin{matrix} M = \mu, z \\ A = a, \underline{z} \end{matrix}}$$

Equivalently: $\hat{e}^a = e^{\alpha\phi} \underbrace{e^a}_{e_\mu{}^a dx^\mu}$ and $\hat{e}^{\underline{z}} = e^{\beta\phi}(dz + A)$ with $A \equiv A_\mu dx^\mu$

Ex: Check that $\hat{e}_M{}^A \; \hat{e}_N{}^B \; \hat{\eta}_{AB} = \hat{g}_{MN}$

$$\begin{bmatrix} e^{\alpha\phi}\, e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \underbrace{\begin{bmatrix} \eta_{ab} & 0_{4\times 1} \\ 0_{1\times 4} & +1 \end{bmatrix} \begin{bmatrix} e^{\alpha\phi}\, e_\nu{}^b & 0_{4\times 1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}_{\begin{bmatrix} e^{\alpha\phi}\, e_{\nu a} & 0_{4\times 1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}$$

$$= \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix} = \hat{g}_{MN}(x)$$

In the following our goal will be to compute S_{D+1} using the $(D+1)$-dimensional frame $\hat{e}_M{}^A$ given above:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \; \hat{e} \; \hat{e}_A{}^M \, \hat{e}_B{}^N \, \hat{R}_{MN}{}^{AB}(\hat{e})$$

- $\hat{e} = e^{(\alpha D + \beta)\phi} \, e$

$A_a = e_a{}^\nu A_\nu$

- We need the inverse $(D+1)$-dim frame $\hat{e}_A{}^N$

$$\hat{e}_M{}^A \cdot \hat{e}_A{}^N = \delta_M{}^N \implies \hat{e}_A{}^N = \begin{bmatrix} e^{-\alpha\phi}\, e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{bmatrix}$$

Ex: check that $\hat{e}_M{}^A \, \hat{e}_A{}^N = \delta_M{}^N$

$$\begin{bmatrix} e^{\alpha\phi}\, e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \begin{bmatrix} e^{-\alpha\phi}\, e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{bmatrix} = \begin{bmatrix} \delta_\mu{}^\nu & 0_{4\times 1} \\ 0_{1\times 4} & 1 \end{bmatrix}$$

⊙ Now we perform the computation of the Ricci scalar $\hat{R}$.

▲ First we compute the anholonomy coefficients $\hat{\Omega}$:

$$\hat{\Omega}_{[MN]P} = \left(\partial_M \hat{e}_N{}^A - \partial_N \hat{e}_M{}^A\right) \hat{e}_{PA}$$

• $\hat{\Omega}_{[\mu\nu]\rho} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\,\hat{e}_{\rho A}$

$$= (\partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a)\,\hat{e}_{\rho a} + (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\,\hat{e}_{\rho\underline{z}}$$

$$= \left[\partial_\mu (e^{\alpha\phi} e_\nu{}^a) - \partial_\nu (e^{\alpha\phi} e_\mu{}^a)\right](e^{\alpha\phi} e_{\rho a})$$

$$+ \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right](e^{\beta\phi} A_\rho)$$

$$= e^{2\alpha\phi}\left[(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a)\, e_{\rho a} + \alpha\,(\partial_\mu\phi\, e_\nu{}^a - \partial_\nu\phi\, e_\mu{}^a)\, e_{\rho a}\right]$$

$$+ e^{2\beta\phi}\left[F_{\mu\nu} A_\rho + \beta\,(\partial_\mu\phi\, A_\nu - \partial_\nu\phi\, A_\mu)\, A_\rho\right]$$

$$= e^{2\alpha\phi}\left[\Omega_{[\mu\nu]\rho} + 2\alpha\, \partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a}\right]$$

$$+ e^{2\beta\phi}\left[F_{\mu\nu} A_\rho + 2\beta\, \partial_{[\mu}\phi\, A_{\nu]} A_\rho\right]$$

• $\hat{\Omega}_{[\mu\nu]z} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\,\hat{e}_{zA} = (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\,\hat{e}_{z\underline{z}}$

$$= \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right] e^{\beta\phi}$$

$$= e^{2\beta\phi}\left[F_{\mu\nu} + 2\beta\, \partial_{[\mu}\phi\, A_{\nu]}\right]$$

• $\hat{\Omega}_{[\mu z]\rho} = \partial_\mu \hat{e}_z{}^A\, \hat{e}_{\rho A} = \partial_\mu \hat{e}_z{}^{\underline{z}}\, \hat{e}_{\rho\underline{z}} = \partial_\mu (e^{\beta\phi})\,(e^{\beta\phi} A_\rho)$

$$= e^{2\beta\phi}\, \beta\, \partial_\mu\phi\, A_\rho$$

- $\hat{\Omega}_{[\mu z]z} = \partial_\mu \hat{e}_z{}^A \hat{e}_{zA} = \partial_\mu \hat{e}_z{}^{\underline{z}} \hat{e}_{z\underline{z}} = \partial_\mu (e^{\beta\phi}) e^{\beta\phi}$
$= e^{2\beta\phi} \beta \partial_\mu \phi$

- $\hat{\Omega}_{[z\nu]\rho} = -\partial_\nu \hat{e}_z{}^A \hat{e}_{\rho A} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \hat{e}_{\rho\underline{z}} = -\partial_\nu (e^{\beta\phi})(e^{\beta\phi} A_\rho)$
$= -e^{2\beta\phi} \beta \partial_\nu \phi A_\rho$

- $\hat{\Omega}_{[z\nu]z} = -\partial_\nu \hat{e}_z{}^A \hat{e}_{zA} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \hat{e}_{z\underline{z}} = -\partial_\nu (e^{\beta\phi})(e^{\beta\phi})$
$= -e^{2\beta\phi} \beta \partial_\nu \phi$

- $\hat{\Omega}_{[zz]\rho} = \hat{\Omega}_{[zz]z} = 0$

Using $\hat{\Omega}$ we compute the spin connection with all indices curved

$$\hat{\omega}_{M[NP]}(\hat{e}) = \frac{1}{2}\left(\hat{\Omega}_{[MN]P} - \hat{\Omega}_{[NP]M} + \hat{\Omega}_{[PM]N}\right)$$
$$= \hat{\omega}_M{}^{BC}(\hat{e})\, \hat{e}_{NB}\, \hat{e}_{PC}$$

- $\hat{\omega}_{\mu[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]\rho} - \hat{\Omega}_{[\nu\rho]\mu} + \hat{\Omega}_{[\rho\mu]\nu}\right)$

$$= \frac{1}{2}\Big[e^{2\alpha\phi}\Big(2\,\omega_{\mu[\nu\rho]} + 2\alpha\left(\partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a} - \partial_{[\nu}\phi\, e_{\rho]}{}^a e_{\mu a} + \partial_{[\rho}\phi\, e_{\mu]}{}^a e_{\nu a}\right)\Big)$$
$$+ e^{2\beta\phi}\Big(F_{\mu\nu} A_\rho - F_{\nu\rho} A_\mu + F_{\rho\mu} A_\nu + 2\beta\big(\partial_{[\mu}\phi A_{\nu]} A_\rho - \partial_{[\nu}\phi A_{\rho]} A_\mu$$
$$+ \partial_{[\rho}\phi A_{\mu]} A_\nu\big)\Big)\Big]$$

- $\hat{\omega}_{z[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]\rho} - \hat{\Omega}_{[\nu\rho]z} + \hat{\Omega}_{[\rho z]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi\,A_\rho - (F_{\nu\rho} + 2\beta\,\partial_{[\nu}\phi\,A_{\rho]}) + \beta\,\partial_\rho\phi\,A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi\,A_{\rho]}\right)$$

- $\hat{\omega}_{\mu[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]z} - \hat{\Omega}_{[\nu z]\mu} + \hat{\Omega}_{[z\mu]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left((F_{\mu\nu} + 2\beta\,\partial_{[\mu}\phi\,A_{\nu]}) - \beta\,\partial_\nu\phi\,A_\mu - \beta\,\partial_\mu\phi\,A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi\,A_\mu\right)$$

- $\hat{\omega}_{z[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]z} - \hat{\Omega}_{[\nu z]z} + \overbrace{\hat{\Omega}_{[zz]\nu}}^{0}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi - \beta\,\partial_\nu\phi\right)\right] = -e^{2\beta\phi}\beta\,\partial_\nu\phi$$

- $\hat{\omega}_{\mu[z\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu z]\rho} - \hat{\Omega}_{[z\rho]\mu} + \hat{\Omega}_{[\rho\mu]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(\beta\,\partial_\mu\phi\,A_\rho + \beta\,\partial_\rho\phi\,A_\mu + (F_{\rho\mu} + 2\beta\,\partial_{[\rho}\phi\,A_{\mu]})\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\rho\mu} + 2\beta\,\partial_\rho\phi\,A_\mu\right)$$

- $\hat{\omega}_{z[z\rho]} = \frac{1}{2}\left(\underbrace{\hat{\Omega}_{[zz]\rho}}_{0} - \hat{\Omega}_{[z\rho]z} + \hat{\Omega}_{[\rho z]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(+\beta\,\partial_\rho\phi + \beta\,\partial_\rho\phi\right)\right] = e^{2\beta\phi}\beta\,\partial_\rho\phi$$

- $\hat{\omega}_{\mu[zz]} = \hat{\omega}_{z[zz]} = 0$

▲ Using the above results we compute the standard spin connection

$$\hat{\omega}_M{}^{BC} = \hat{\omega}_{M[NP]}\,\hat{e}^{BN}\,\hat{e}^{CP}$$

- $$\hat{\omega}_{\mu}{}^{bc} = \hat{\omega}_{\mu[NP]}\,\hat{e}^{bN}\,\hat{e}^{cP}$$

$$= \hat{\omega}_{\mu[\nu\rho]}\hat{e}^{b\nu}\hat{e}^{c\rho} + \hat{\omega}_{\mu[\nu z]}\hat{e}^{b\nu}\hat{e}^{cz} + \hat{\omega}_{\mu[z\rho]}\hat{e}^{bz}\hat{e}^{c\rho} + \overbrace{\hat{\omega}_{\mu[zz]}}^{0}\hat{e}^{bz}\hat{e}^{cz}$$

$$= \hat{\omega}_{\mu[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat{\omega}_{\mu[\nu z]} e^{-2\alpha\phi} e^{b\nu} A^{c} - \hat{\omega}_{\mu[z\rho]} e^{-2\alpha\phi} A^{b} e^{c\rho}$$

$$= \frac{1}{2}\Big[2\,\omega_{\mu}{}^{[bc]} + 2\alpha\Big(\partial_{[\mu}\phi\, e_{\nu]}{}^{a} e_{\rho a} e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi\, e_{\rho]}{}^{a} e_{\mu a} e^{b\nu} e^{c\rho}$$

$$+ \partial_{[\rho}\phi\, e_{\mu]}{}^{a} e_{\nu a} e^{b\nu} e^{c\rho}\Big) + e^{2(\beta-\alpha)\phi}\Big(F_{\mu\nu} A_{\rho} e^{b\nu} e^{c\rho} -$$

$$- F_{\nu\rho} A_{\mu} e^{b\nu} e^{c\rho} + F_{\rho\mu} A_{\nu} e^{b\nu} e^{c\rho}\Big) +$$

$$+ 2\beta e^{2(\beta-\alpha)\phi}\Big(\partial_{[\mu}\phi A_{\nu]} A_{\rho} e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi A_{\rho]} A_{\mu} e^{b\nu} e^{c\rho}$$

$$+ \partial_{[\rho}\phi\, A_{\mu]} A_{\nu} e^{b\nu} e^{c\rho}\Big)\Big]$$

$$- \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\mu\nu} - 2\beta\partial_{\nu}\phi A_{\mu}) e^{b\nu} A^{c}\Big] - \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\rho\mu} + 2\beta\,\partial_{\rho}\phi A_{\mu}) A^{b} e^{c\rho}\Big]$$

$$= (*)$$

NOTE 1: $$2\alpha\,\frac{1}{2}\,(\partial_{\mu}\phi\, e_{\nu}{}^{a} e_{\rho a} e^{b\nu} e^{c\rho} - \partial_{\nu}\phi\, e_{\mu}{}^{a} e_{\rho a} e^{b\nu} e^{c\rho}$$

$$-\partial_{\nu}\phi\, e_{\rho}{}^{a} e_{\mu a} e^{b\nu} e^{c\rho} + \partial_{\rho}\phi\, e_{\nu}{}^{a} e_{\mu a} e^{b\nu} e^{c\rho}$$

$$+\partial_{\rho}\phi\, e_{\mu}{}^{a} e_{\nu a} e^{b\nu} e^{c\rho} - \partial_{\mu}\phi\, e_{\rho}{}^{a} e_{\nu a} e^{b\nu} e^{c\rho})$$

$$= \alpha\,(\underline{\partial_{\mu}\phi\,\eta^{bc}} - \partial_{\nu}\phi\, e^{b\nu} e_{\mu}{}^{c} - \partial_{\nu}\phi\, e_{\mu}{}^{c} e^{b\nu} + \partial_{\rho}\phi\, e_{\mu}{}^{b} e^{c\rho}$$

$$+ \partial_{\rho}\phi\, e_{\mu}{}^{b} e^{c\rho} - \underline{\partial_{\mu}\phi\,\eta^{bc}})$$

$$= \alpha\,(2\,\partial_{\rho}\phi\, e_{\mu}{}^{b} e^{c\rho} - 2\,\partial_{\nu}\phi\, e_{\mu}{}^{c} e^{b\nu}) = \quad [\partial^{a} \equiv e^{a\rho}\partial_{\rho}]$$

$$= 2\alpha\,(e_{\mu}{}^{b}\,\partial^{c}\phi - e_{\mu}{}^{c}\,\partial^{b}\phi) = 4\alpha\, e_{\mu}{}^{[b}\,\partial^{c]}\phi$$

$$= 4\alpha\,\partial^{[c}\phi\, e_{\mu}{}^{b]}$$

NOTE 2 : $$e^{2(\beta-\alpha)\phi}\left(F_{\mu\nu}A_\rho e^{b\nu}e^{c\rho} - F_{\nu\rho}A_\mu e^{b\nu}e^{c\rho} + F_{\rho\mu}A_\nu e^{b\nu}e^{c\rho}\right)$$
$$= e^{2(\beta-\alpha)\phi}\left(F_\mu{}^b A^c - F^{bc}A_\mu + F^c{}_\mu A^b\right)$$
$$= e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]} - F^{bc}A_\mu\right)$$

NOTE 3 : $$2\beta e^{2(\beta-\alpha)\phi}\frac{1}{2}\Big(\partial_\mu\phi A_\nu A_\rho e^{b\nu}e^{c\rho} - \partial_\nu\phi A_\mu A_\rho e^{b\nu}e^{c\rho}$$
$$-\partial_\nu\phi A_\rho A_\mu e^{b\nu}e^{c\rho} + \partial_\rho\phi A_\nu A_\mu e^{b\nu}e^{c\rho}$$
$$+\partial_\rho\phi A_\mu A_\nu e^{b\nu}e^{c\rho} - \partial_\mu\phi A_\rho A_\nu e^{b\nu}e^{c\rho}\Big)$$

$$= \beta e^{2(\beta-\alpha)\phi}\Big(\partial_\mu\phi A^b A^c - \partial^b\phi A_\mu A^c - \partial^b\phi A^c A_\mu + \partial^c\phi A^b A_\mu$$
$$+\partial^c\phi A_\mu A^b - \partial_\mu\phi A^c A^b\Big)$$

$$= \beta e^{2(\beta-\alpha)\phi}\left(-2A_\mu\partial^b\phi A^c + 2A_\mu\partial^c\phi A^b\right)$$

$$= -4\beta e^{2(\beta-\alpha)\phi}A_\mu\partial^{[b}\phi A^{c]}$$

$$(*) = \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi + e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]} - F^{bc}A_\mu\right)$$
$$-4\beta e^{2(\beta-\alpha)\phi}A_\mu\partial^{[b}\phi A^{c]}\Big]$$
$$-\frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[\underbrace{F_\mu{}^b A^c - F_\mu{}^c A^b}_{2F_\mu{}^{[b}A^{c]}} - 2\beta\underbrace{\left(\partial^b\phi A^c A_\mu - \partial^c\phi A^b\right)}_{2\partial^{[b}\phi A^{c]}}A_\mu\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi - e^{2(\beta-\alpha)\phi}F^{bc}A_\mu$$
$$+ e^{2(\beta-\alpha)\phi}\left(\underline{2F_\mu{}^{[b}A^{c]}} - 4\beta A_\mu\partial^{[b}\phi A^{c]} - \underline{2F_\mu{}^{[b}A^{c]}} + 4\beta\partial^{[b}\phi A^{c]}\right)\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} + 4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]} - e^{2(\beta-\alpha)\phi}F^{bc}A_\mu\Big] =$$

$$= \omega_\mu{}^{[bc]} + \alpha\left(\partial^c\phi\, e_\mu{}^b - \partial^b\phi\, e_\mu{}^c\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} A_\mu .$$

- $$\hat\omega_z{}^{bc} = \hat\omega_{z[NP]}\, \hat e^{bN}\, \hat e^{cP}$$

$$= \hat\omega_{z[\nu\rho]}\hat e^{b\nu}\hat e^{c\rho} + \hat\omega_{z[\nu z]}\hat e^{b\nu}\hat e^{cz} + \hat\omega_{z[z\rho]}\hat e^{bz}\hat e^{c\rho} + \overbrace{\hat\omega_{z[zz]}}^{0}\hat e^{bz}\hat e^{cz}$$

$$= \hat\omega_{z[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat\omega_{z[\nu z]}\, e^{-2\alpha\phi} e^{b\nu} A^c - \hat\omega_{z[z\rho]}\, e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2} e^{2(\beta-\alpha)\phi}\left[-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right] e^{b\nu} e^{c\rho} + e^{2(\beta-\alpha)\phi}\beta\,\partial_\nu\phi\, e^{b\nu} A^c$$

$$- e^{2(\beta-\alpha)\phi}\beta\,\partial_\rho\phi\, A^b e^{c\rho} =$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} + e^{2(\beta-\alpha)\phi}\beta\left[-\partial^b\phi A^c + \partial^c\phi A^b + \partial^b\phi A^c - \partial^c\phi A^b\right]$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc}.$$

Therefore, using compact notation, we find that

$$\boxed{\hat\omega^{bc} = \omega^{[bc]} + \alpha e^{-\alpha\phi}\left(\partial^c\phi\, \hat e^b - \partial^b\phi\, \hat e^c\right) - \frac{1}{2} F^{bc} e^{(\beta-2\alpha)\phi}\, \hat e^{\underline{z}}}$$

- $$\hat\omega_\mu{}^{b\underline{z}} = \hat\omega_{\mu[NP]}\, \hat e^{bN}\, \hat e^{\underline{z}P}$$

$$= \hat\omega_{\mu[\nu\rho]}\hat e^{b\nu}\underbrace{\hat e^{\underline{z}\rho}}_{0} + \hat\omega_{\mu[\nu z]}\hat e^{b\nu}\hat e^{\underline{z}z} + \hat\omega_{\mu[z\rho]}\overbrace{\hat e^{bz}}^{0}\hat e^{\underline{z}\rho} + \overbrace{\hat\omega_{\mu[zz]}}^{0}\hat e^{bz}\hat e^{\underline{z}z}$$

$$= \hat\omega_{\mu[\nu z]}\, e^{-(\alpha+\beta)\phi} e^{b\nu} = \frac{1}{2} e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi A_\mu\right) e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= \frac{1}{2} e^{(\beta-\alpha)\phi}\left(F_\mu{}^b - 2\beta\,\partial^b\phi A_\mu\right) \quad \longrightarrow F^b{}_c\, e_\mu{}^c$$

$$= -\beta e^{(\beta-\alpha)\phi}\partial^b\phi A_\mu - \frac{1}{2} e^{(\beta-\alpha)\phi} F^b{}_\mu$$

$$= -e^{(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi A_\mu + \frac{1}{2} F^b{}_\mu\right] = -\hat\omega_\mu{}^{\underline{z}b}$$

- $$\hat\omega_{\underline z}{}^{b\underline z} = \hat\omega_{\underline z[NP]}\,\hat e^{bN}\,\hat e^{\underline z P}$$

$$= \hat\omega_{\underline z[\nu\rho]}\hat e^{b\nu}\underbrace{\hat e^{\underline z\rho}}_{0} + \hat\omega_{\underline z[\nu\underline z]}\hat e^{b\nu}\hat e^{\underline z\underline z} + \hat\omega_{\underline z[\underline z\rho]}\overbrace{\hat e^{b\underline z}}^{0}\hat e^{\underline z\rho} + \overbrace{\hat\omega_{\underline z[\underline z\underline z]}}^{0}\hat e^{b\underline z}\hat e^{\underline z\underline z}$$

$$= \hat\omega_{\underline z[\nu\underline z]}\,e^{-(\alpha+\beta)\phi}\,e^{b\nu} = -e^{2\beta\phi}\beta\partial_\nu\phi\,e^{-(\alpha+\beta)\phi}\,e^{b\nu}$$

$$= -\beta\,e^{(\beta-\alpha)\phi}\,\partial^b\phi = -\hat\omega_{\underline z}{}^{\underline z b}$$

Using again compact notation we find:

$$\boxed{\hat\omega^{b\underline z} = -\omega^{\underline z b} = -\beta e^{-\alpha\phi}\partial^b\phi\,\hat e^{z} - \frac{1}{2}{}^{(\beta-2\alpha)\phi}F^b{}_c\,\hat e^c}$$

- $$\hat\omega_\mu{}^{\underline z\underline z} = \hat\omega_{\underline z}{}^{\underline z\underline z} = 0$$

▲ Once we have the spin connection we compute the Riemann tensor

$$\hat R_{MN}{}^{BC} = \partial_M\hat\omega_N{}^{BC} - \partial_N\hat\omega_M{}^{DC} + \hat\omega_M{}^B{}_D\,\hat\omega_N{}^{DC} - \hat\omega_N{}^B{}_D\,\hat\omega_M{}^{DC}$$

- $$\hat R_{\mu\nu}{}^{bc} = \partial_\mu\hat\omega_\nu{}^{bc} + \hat\omega_\mu{}^b{}_D\,\hat\omega_\nu{}^{Dc} - \partial_\nu\hat\omega_\mu{}^{bc} - \hat\omega_\nu{}^b{}_D\,\hat\omega_\mu{}^{Dc}$$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\,e_\nu{}^b - \partial^b\phi\,e_\nu{}^c) - \frac{1}{2}\partial_\mu\left(e^{2(\beta-\alpha)\phi}F^{bc}A_\nu\right)$$

$$+ \hat\omega_\mu{}^b{}_d\,\hat\omega_\nu{}^{dc} + \hat\omega_\mu{}^b{}_{\underline z}\,\hat\omega_\nu{}^{\underline z c} - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\,e_\nu{}^b - \partial^b\phi\,e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[2(\beta-\alpha)\partial_\mu\phi F^{bc}A_\nu$$

$$+ \partial_\mu F^{bc}A_\nu + F^{bc}\partial_\mu A_\nu\Big]$$

$$+ \left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\,e_\mu{}^b - \partial^b\phi\,e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d A_\mu\right]$$

$$\left[\omega_\nu{}^{dc} + \alpha(\partial^c\phi\,e_\nu{}^d - \partial^d\phi\,e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\nu\right]$$

$$-e^{2(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi\, A_\mu+\tfrac{1}{2}F^b{}_\mu\right]\left[\beta\,\partial^c\phi\, A_\nu+\tfrac{1}{2}F^c{}_\nu\right]-(\mu\leftrightarrow\nu)$$

$$=\overbrace{\partial_\mu\omega_\nu{}^{bc}+\omega_\mu{}^b{}_d\,\omega_\nu{}^{dc}}^{R_{\mu\nu}{}^{bc}}+\alpha\,\partial_\mu\left(\partial^c\phi\, e_\nu{}^b-\partial^b\phi\, e_\nu{}^c\right)$$

$$-e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu+\tfrac{1}{2}\partial_\mu F^{bc}A_\nu+\tfrac{1}{2}F^{bc}\partial_\mu A_\nu$$

$$+\underline{\beta^2\,\partial^b\phi\,\partial^c\phi\, A_\mu A_\nu}+\tfrac{1}{2}\beta\,\partial^b\phi\, A_\mu F^c{}_\nu+\tfrac{1}{2}\beta\,\partial^c\phi\, A_\nu F^b{}_\mu+\tfrac{1}{4}F^b{}_\mu F^c{}_\nu\Big]$$

$$+\alpha\,\omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d-\partial^d\phi\, e_\nu{}^c\right)-\tfrac{1}{2}e^{2(\beta-\alpha)\phi}\,\omega_\mu{}^b{}_d\,F^{dc}A_\nu$$

$$+\alpha\,\omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b-\partial^b\phi\, e_{\mu d}\right)-\tfrac{1}{2}\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial_d\phi\, e_\mu{}^b-\partial^b\phi\, e_{\mu d}\right)F^{dc}A_\nu$$

$$+\alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b-\partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c-\underline{\partial^b\phi\,\partial^c\phi\, g_{\mu\nu}}+\partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c\right)$$

$$-\tfrac{1}{2}e^{2(\beta-\alpha)\phi}\,\omega_\nu{}^{dc}F^b{}_d A_\mu+\underline{\tfrac{1}{4}e^{4(\beta-\alpha)\phi}F^b{}_d F^{dc}A_\mu A_\nu}$$

$$-\tfrac{1}{2}\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial^c\phi\, e_\nu{}^d-\partial^d\phi\, e_\nu{}^c\right)F^b{}_d A_\mu-\underline{(\mu\leftrightarrow\nu)}$$

NOTE: Underlined terms vanish because they are $\mu\leftrightarrow\nu$ symmetric

$$=\tfrac{1}{2}R_{\mu\nu}{}^{bc}+\alpha\left(\partial_\mu\partial^c\phi\, e_\nu{}^b+\partial^c\phi\,\partial_\mu e_\nu{}^b-\partial_\mu\partial^b\phi\, e_\nu{}^c-\partial^b\phi\,\partial_\mu e_\nu{}^c\right)$$

$$-e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu+\tfrac{1}{2}\beta\,\partial^b\phi\, F^c{}_\nu A_\mu+\tfrac{1}{2}\beta\,\partial^c\phi\, F^b{}_\mu A_\nu$$

$$+\tfrac{1}{2}\alpha\,\partial_d\phi\, F^{dc}A_\nu e_\mu{}^b-\tfrac{1}{2}\alpha\,\partial^b\phi\, F_\mu{}^c A_\nu$$

$$+\tfrac{1}{2}\alpha\,\partial^c\phi\, F^b{}_\nu A_\mu-\tfrac{1}{2}\alpha\,\partial^d\phi\, F^b{}_d A_\mu e_\nu{}^c$$

$$+\tfrac{1}{2}\partial_\mu F^{bc}A_\nu+\tfrac{1}{2}F^{bc}\partial_\mu A_\nu+\tfrac{1}{4}F^b{}_\mu F^c{}_\nu$$

$$+\tfrac{1}{2}\omega_\mu{}^b{}_d F^{dc}A_\nu+\tfrac{1}{2}\omega_\nu{}^{dc}F^b{}_d A_\mu\Big]$$

$$+\alpha\,\omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d-\partial^d\phi\, e_\nu{}^c\right)+\alpha\,\omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b-\partial^b\phi\, e_{\mu d}\right)$$

$$+\alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b+\partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c-\partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c-\underline{(\mu\leftrightarrow\nu)}\right.$$

- $$\hat{R}_{\mu\nu}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{D}\, \hat{\omega}_\nu{}^{D\underline{z}} - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{c}\, \hat{\omega}_\nu{}^{c\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{\underline{z}}\, \overbrace{\hat{\omega}_\nu{}^{\underline{z}\underline{z}}}^{0} - (\mu \leftrightarrow \nu)$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \partial_\mu\partial^b\phi\, A_\nu + \partial^b\phi\, \partial_\mu A_\nu\right]$$

$$-\frac{1}{2} e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \partial_\mu F^b{}_\nu\right]$$

$$-\left[\omega_\mu{}^b{}_c + \alpha\left(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\right]$$

$$e^{(\beta-\alpha)\phi}\left[\beta\, \partial^c\phi\, A_\nu + \frac{1}{2} F^c{}_\nu\right] - (\mu \leftrightarrow \nu)$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \beta\, \partial_\mu\partial^b\phi\, A_\nu + \beta\, \partial^b\phi\, \partial_\mu A_\nu$$

$$+ \frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \frac{1}{2}\partial_\mu F^b{}_\nu + \beta\, \omega_\mu{}^b{}_c\, \partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu$$

$$+ \alpha\beta\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\, \partial_c\phi\, e_\mu{}^b F^c{}_\nu$$

$$- \alpha\beta\, \partial_\mu\phi\, \partial^b\phi\, A_\nu - \frac{1}{2}\alpha\, \partial^b\phi\, F_{\mu\nu}\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\left[\underline{\frac{1}{2}\beta F^b{}_c\, \partial^c\phi\, A_\mu A_\nu} + \frac{1}{4} F^b{}_c F^c{}_\nu A_\mu\right] - \underline{(\mu \leftrightarrow \nu)}$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta^2 \partial_\mu\phi\, \partial^b\phi\, A_\nu - 2\alpha\beta\, \partial_\mu\phi\, \partial^b\phi\, A_\nu + \beta\, \partial_\mu\partial^b\phi\, A_\nu$$

$$+ \alpha\beta\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\, \partial_c\phi\, F^c{}_\nu e_\mu{}^b + \frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu$$

$$+ \beta\, \partial^b\phi\, \partial_\mu A_\nu - \frac{1}{2}\alpha\, \partial^b\phi\, F_{\mu\nu} + \frac{1}{2}\partial_\mu F^b{}_\nu$$

$$+ \beta\, \omega_\mu{}^b{}_c\, \partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\, \frac{1}{4} F^b{}_c F^c{}_\nu A_\mu - (\mu \leftrightarrow \nu) = -\hat{R}_{\mu\nu}{}^{\underline{z}b}$$

$$\bullet\ \hat{R}_{\mu z}{}^{bc} = \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{Dc} - (\mu \leftrightarrow z)$$

$$= \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_d\, \hat{\omega}_z{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \hat{\omega}_z{}^{\underline{z}c}$$
$$\underbrace{-\partial_z \hat{\omega}_\mu{}^{bc}}_{0} - \hat{\omega}_z{}^b{}_d\, \hat{\omega}_\mu{}^{dc} - \hat{\omega}_z{}^b{}_{\underline{z}}\, \hat{\omega}_\mu{}^{\underline{z}c}$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi}\left[2(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \partial_\mu F^{bc}\right]$$
$$-\left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_d A_\mu\right]\frac{1}{2}e^{2(\beta-\alpha)\phi} F^{dc}$$
$$- e^{(\beta-\alpha)\phi}\left[\beta\partial^b\phi A_\mu + \frac{1}{2}F^b{}_\mu\right]\beta e^{(\beta-\alpha)\phi}\partial^c\phi$$
$$+\frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d\left[\omega_\mu{}^{dc} + \alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\mu\right]$$
$$+\beta e^{(\beta-\alpha)\phi}\partial^b\phi\ e^{(\beta-\alpha)\phi}\left[\beta\partial^c\phi A_\mu + \frac{1}{2}F^c{}_\mu\right]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \frac{1}{2}\partial_\mu F^{bc} + \frac{1}{2}\omega_\mu{}^b{}_d F^{dc}$$
$$+\frac{1}{2}\alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d})F^{dc} + \beta^2\partial^b\phi\,\partial^c\phi\, A_\mu + \frac{1}{2}\beta F^b{}_\mu\partial^c\phi$$
$$-\frac{1}{2}\omega_\mu{}^{dc}F^b{}_d - \frac{1}{2}\alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c)F^b{}_d - \beta^2\partial^b\phi\,\partial^c\phi\, A_\mu$$
$$-\frac{1}{2}\beta\partial^b\phi\, F^c{}_\mu\Big]$$
$$+ e^{4(\beta-\alpha)\phi}\left[\frac{1}{4}F^b{}_d F^{dc} A_\mu - \frac{1}{4}F^b{}_d F^{dc} A_\mu\right]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} - \frac{1}{2}\alpha\partial^d\phi\, F^c{}_d e_\mu{}^b + \frac{1}{2}\alpha\partial^b\phi\, F^c{}_\mu$$
$$-\frac{1}{2}\alpha\partial^c\phi\, F^b{}_\mu + \frac{1}{2}\alpha\partial^d\phi\, F^b{}_d e_\mu{}^c + \frac{1}{2}\beta\partial^c\phi\, F^b{}_\mu - \frac{1}{2}\beta\partial^b\phi\, F^c{}_\mu$$
$$-\frac{1}{2}\omega_\mu{}^b{}_d F^{cd} + \frac{1}{2}\omega_\mu{}^c{}_d F^{bd} + \frac{1}{2}\partial_\mu F^{bc}\Big]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \alpha\,\partial^{[b}\phi\, F^{c]}{}_\mu + \alpha\,\partial^d\phi\, F^{[b}{}_d\, e_\mu{}^{c]}$$

$$+ \beta\, F^{[b}{}_\mu\, \partial^{c]}\phi - \omega_\mu{}^{[b}{}_d\, F^{c]d} + \frac{1}{2}\partial_\mu F^{bc}\Big]$$

$$= -\hat{R}_{z\mu}{}^{bc}$$

- $\hat{R}_{\mu z}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{D\underline{z}} - (\mu \leftrightarrow z)$

$$= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \overbrace{\hat{\omega}_z{}^{\underline{z}\underline{z}}}^{0}$$

$$- \underbrace{\partial_z \hat{\omega}_\mu{}^{b\underline{z}}}_{0} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}} - \omega_z{}^b{}_{\underline{z}}\, \underbrace{\hat{\omega}_\mu{}^{\underline{z}\underline{z}}}_{0}$$

$$= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}}$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\big[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi\big]$$

$$- \Big[\omega_\mu{}^b{}_c + \alpha(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}) - \frac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\Big]\beta\, e^{(\beta-\alpha)\phi}\partial^c\phi$$

$$- \frac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_c \cdot e^{(\beta-\alpha)\phi}\Big[\beta\,\partial^c\phi\, A_\mu + \frac{1}{2}F^c{}_\mu\Big]$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \omega_\mu{}^b{}_c\,\partial^c\phi$$

$$+ \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b - \alpha\,\partial^b\phi\,\partial_\mu\phi\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\Big[\underline{\frac{1}{2}\beta\, F^b{}_c\,\partial^c\phi\, A_\mu} - \underline{\frac{1}{2}\beta\, F^b{}_c\,\partial^c\phi\, A_\mu}$$

$$- \frac{1}{4}F^b{}_c\, F^c{}_\mu\Big]$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-2\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b + \omega_\mu{}^b{}_c\,\partial^c\phi\Big]$$

$$- \frac{1}{4}e^{3(\beta-\alpha)\phi} F^b{}_c\, F^c{}_\mu = -\hat{R}_{\mu z}{}^{\underline{z}b} = -\hat{R}_{z\mu}{}^{b\underline{z}} = \hat{R}_{z\mu}{}^{\underline{z}b}$$

▲ With the Riemann tensor we compute now the curved/flat Ricci tensor

$$\hat{R}_{Nc} = \hat{R}_{MN}{}^{B}{}_{c}\, \hat{e}_{B}{}^{M}$$

• $\hat{R}_{\nu c} = \hat{R}_{M\nu}{}^{B}{}_{c}\, \hat{e}_{B}{}^{M}$

$$= \hat{R}_{\mu\nu}{}^{b}{}_{c}\, \hat{e}_{b}{}^{\mu} + \hat{R}_{\mu\nu}{}^{z}{}_{c}\, \overbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0} + \hat{R}_{z\nu}{}^{b}{}_{c}\, e_{b}{}^{z} + \hat{R}_{z\nu}{}^{z}{}_{c}\, \hat{e}_{\underline{z}}{}^{z}$$

$$= e^{-\alpha\phi} e_{b}{}^{\mu} \hat{R}_{\mu\nu}{}^{b}{}_{c} - e^{-\alpha\phi} A_{b} \hat{R}_{z\nu}{}^{b}{}_{c} + e^{-\beta\phi} R_{z\nu}{}^{z}{}_{c}$$

$$= e^{-\alpha\phi} R_{\nu c} + e^{-\alpha\phi}\alpha \big(\partial_b \partial_c \phi\, e_\nu{}^b - \partial_\nu \partial_c \phi\, D + \partial_c \phi\, \partial_b e_\nu{}^b - \partial_c \phi\, \partial_\nu e_\mu{}^b e_b{}^\mu$$
$$- \partial^2 \phi\, e_{\nu c} + \partial_\nu \partial^b \phi\, \eta_{bc} - \partial^b \phi\, \partial_b e_{\nu c}$$
$$+ \partial^b \phi\, \partial_\nu e_{\mu c}\, e_b{}^\mu \big)$$

$$- e^{(2\beta-3\alpha)\phi} \Big[(\beta-\alpha)(\partial_b \phi\, F^b{}_c A_\nu - \partial_\nu \phi\, F^b{}_c A_b)$$
$$+ \tfrac{1}{2}\beta (\partial^b \phi\, F_{c\nu} A_b + \partial_b \phi\, F^b{}_c A_\nu)$$
$$+ \tfrac{1}{2}\beta (\cancelto{0}{\partial_c \phi\, F^b{}_b A_\nu} - \partial_c \phi\, F^b{}_\nu A_b)$$
$$+ \tfrac{1}{2}\alpha\, \partial_b \phi\, F^b{}_c (A_\nu \underbrace{\delta_d{}^d}_{D} - \underbrace{A_d e_\nu{}^d}_{A_\nu})$$
$$- \tfrac{1}{2}\alpha\, \partial_b \phi\, (F^b{}_c A_\nu - F_{\nu c} A^b)$$
$$+ \tfrac{1}{2}\alpha\, \partial_c \phi\, (F^b{}_\nu A_b - \cancelto{0}{F^b{}_b A_\nu})$$
$$+ \tfrac{1}{2}\alpha\, \partial_b \phi\, F^{bd} (A_d e_{\nu c} - A_\nu \eta_{dc})$$
$$+ \tfrac{1}{2}(\partial_b F^b{}_c A_\nu - \partial_\nu F^b{}_c A_b) + \tfrac{1}{2} F^b{}_c F_{b\nu}$$
$$+ \tfrac{1}{4}(\cancelto{0}{F^b{}_b F_{c\nu}} - F^b{}_\nu F_{cb})$$
$$+ \tfrac{1}{2} F^d{}_c (\omega_b{}^b{}_d A_\nu - \omega_\nu{}^b{}_d A_b)$$
$$+ \tfrac{1}{2} F^b{}_d (\omega_\nu{}^d{}_c A_b - \omega_b{}^d{}_c A_\nu) \Big]$$

SO(1,D-1) generators are antisymmetric

$$+\,e^{-\alpha\phi}\Big[\alpha\,\omega_b{}^b{}_d\left(\partial_c\phi\, e_a{}^d-\partial^d\phi\, e_{ac}\right)-\alpha\,\omega_a{}^b{}_d\left(\cancelto{0}{\partial_c\phi}\,\delta_b{}^d-\partial^d\phi\,\eta_{bc}\right)$$

$$+\alpha\,\omega_a{}^d{}_c\Big(\partial_d\phi\,\overbrace{\delta_b{}^b}^{D}-\partial^b\phi\,\eta_{bd}\Big)-\alpha\,\omega_b{}^d{}_c\left(\partial_d\phi\, e_a{}^b-\partial^b\phi\, e_{ad}\right)$$

$$+\alpha^2\Big(\partial_a\phi\,\partial_c\phi\,\overbrace{\delta_b{}^b}^{D}-\partial_b\phi\,\partial_c\phi\, e_a{}^b+\partial^b\phi\,\partial_b\phi\, e_{ac}-\partial^b\phi\,\partial_a\phi\,\eta_{bc}$$

$$-\partial_d\phi\,\partial^d\phi\,\underbrace{\delta_b{}^b}_{D}\, e_{ac}+\partial_d\phi\,\partial^d\phi\, e_a{}^b\,\eta_{bc}\Big)\Big]$$

$$-\,A_b\, e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\,\partial_a\phi\, F^b{}_c-\tfrac{1}{2}\alpha\,\partial^d\phi\, F_{cd}\, e_a{}^b+\tfrac{1}{2}\alpha\,\partial^b\phi\, F_{ca}$$

$$-\tfrac{1}{2}\alpha\,\partial_c\phi\, F^b{}_a+\tfrac{1}{2}\alpha\,\partial^d\phi\, F^b{}_d\, e_{ac}+\tfrac{1}{2}\beta\,\partial_c\phi\, F^b{}_a$$

$$-\tfrac{1}{2}\beta\,\partial^b\phi\, F_{ca}-\tfrac{1}{2}\omega_a{}^b{}_d\, F_c{}^d+\tfrac{1}{2}\,\omega_{acd}\, F^{bd}$$

$$+\tfrac{1}{2}\,\partial_a F^b{}_c\Big]$$

$$-\,\beta\, e^{-\alpha\phi}\Big[(\beta-2\alpha)\,\partial_a\phi\,\partial_c\phi+\partial_a\partial_c\phi+\alpha\,\partial_d\phi\,\partial^d\phi\, e_{ac}+\omega_{acd}\,\partial^d\phi\Big]$$

$$-\,\tfrac{1}{4}\, e^{(2\beta-3\alpha)\phi}\, F_{cd}\, F^d{}_a$$

$$=e^{-\alpha\phi}\Big[R_{ac}+\alpha\Big(\partial_b\partial_c\phi\, e_a{}^b-\partial^2\phi\, e_{ac}-(D-1)\,\partial_a\partial_c\phi$$

$$+\partial_c\phi\,\partial_b e_a{}^b-\partial^b\phi\,\partial_b e_{ac}-\partial_c\phi\,\partial_a e_\mu{}^b\, e_b{}^\mu+\partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu\Big)$$

$$+\omega_b{}^b{}_a\,\partial_c\phi-\omega_b{}^{bd}\,\partial_d\phi\, e_{ac}+(3-D)\,\omega_{ac}{}^d\,\partial_d\phi$$

$$+\omega^d{}_{ac}\,\partial_d\phi\Big)+\alpha^2(D-2)\left(\partial_a\phi\,\partial_c\phi-\partial^d\phi\,\partial_d\phi\, e_{ac}\right)$$

$$-\beta^2\,\partial_a\phi\,\partial_c\phi+\alpha\beta\left(2\,\partial_a\phi\,\partial_c\phi-\partial_d\phi\,\partial^d\phi\, e_{ac}\right)$$

$$-\beta\left(\partial_a\partial_c\phi+\omega_{ac}{}^d\,\partial_d\phi\right)\Big]$$

$$-\,e^{(2\beta-3\alpha)\phi}\Big[\alpha\,\frac{D-4}{2}\,\partial_b\phi\, F^b{}_c\, A_a+\beta\,\frac{3}{2}\,\partial_b\phi\, F^b{}_c\, A_a+\tfrac{1}{2}\,\partial_b F^b{}_c\, A_a+\tfrac{1}{2}\, F_{ba}\, F^b{}_c$$

$$+\tfrac{1}{2}\,\omega_b{}^b{}_d\, F^d{}_c\, A_a-\tfrac{1}{2}\,\omega_b{}^d{}_c\, F^b{}_d\, A_a\Big]$$

$$\bullet\quad \hat R_{zc} = \hat R_{Mz}{}^{B}{}_{c}\,\hat e_B{}^M$$

$$= \hat R_{\mu z}{}^{b}{}_{c}\,\hat e_b{}^\mu + \overbrace{\hat R_{\mu z}{}^{\underline z}{}_{c}\,\hat e_{\underline z}{}^\mu}^{0} + \overbrace{\hat R_{zz}{}^{b}{}_{c}\,\hat e_b{}^z}^{0} + \overbrace{\hat R_{zz}{}^{\underline z}{}_{c}\,\hat e_{\underline z}{}^z}^{0}$$

$$= e^{-\alpha\phi}\, e_b{}^\mu\, \hat R_{\mu z}{}^{b}{}_{c}$$

$$= -e^{(2\beta-3\alpha)\phi}\Big[\alpha\Big(-\partial_b\phi\, F^b{}_c + \frac{D}{2}\partial^d\phi\, F_{dc} - \partial^b\phi\, F_{bc}\Big) + \frac{1}{2}\partial_b F^b{}_c$$

$$+ \frac{3}{2}\partial_b\phi\, F^b{}_c + \frac{1}{2}\omega_b{}^b{}_d\, F^d{}_c - \frac{1}{2}\omega_{bcd}\, F^{db}\Big]$$

$$\bullet\quad \hat R_{\nu\underline z} = \hat R_{M\nu}{}^{B}{}_{\underline z}\,\hat e_B{}^M$$

$$= \hat R_{\mu\nu}{}^{b}{}_{\underline z}\,\hat e_b{}^\mu + \overbrace{\hat R_{\mu\nu}{}^{\underline z}{}_{\underline z}\,\hat e_{\underline z}{}^\mu}^{0} + \hat R_{z\nu}{}^{b}{}_{\underline z}\,\hat e_b{}^z + \overbrace{\hat R_{z\nu}{}^{\underline z}{}_{\underline z}\,\hat e_{\underline z}{}^z}^{0}$$

$$= e^{-\alpha\phi}\, e_b{}^\mu\, \hat R_{\mu\nu}{}^{b}{}_{\underline z} - e^{-\alpha\phi} A_b\, \hat R_{z\nu}{}^{b}{}_{\underline z}$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\big(\partial_b\phi\,\partial^b\phi\, A_\nu - \partial_\nu\phi\,\partial^b\phi\, A_b\big)$$

$$+ \alpha\beta\big(-2\partial_b\phi\,\partial^b\phi\, A_\nu + 2\partial_\nu\phi\,\partial^b\phi\, A_b + (D-1)\partial_c\phi\,\partial^c\phi\, A_\nu\big)$$

$$+ \alpha\Big(\frac{1}{2}\partial_c\phi\,(D-1)F^c{}_\nu - \frac{1}{2}\partial_b\phi\, F^b{}_\nu - \partial^b\phi\, F_{b\nu}\Big)$$

$$+ \beta\Big(\partial^2\phi\, A_\nu - \partial_\nu\partial^b\phi\, A_b + \frac{1}{2}\partial_b\phi\, F^b{}_\nu + \partial^b\phi\, F_{b\nu}$$

$$+ \omega_b{}^b{}_c\,\partial^c\phi\, A_\nu - \omega_\nu{}^b{}_c\,\partial^c\phi\, A_b\Big) + \frac{1}{2}\partial_b F^b{}_\nu$$

$$-\frac{1}{2}\partial_\nu F^b{}_\mu\, e_b{}^\mu + \frac{1}{2}\omega_b{}^b{}_c\, F^c{}_\nu - \frac{1}{2}\omega_\nu{}^b{}_c\, F^c{}_b\Big]$$

$$+ e^{(3\beta-4\alpha)\phi}\Big[\frac{1}{4}F^b{}_c F^c{}_\nu A_b - \frac{1}{4}F^b{}_c F^c{}_b A_\nu\Big]$$

$$-\beta e^{(\beta-2\alpha)\phi} A_b\Big[\alpha\big(-2\partial_\nu\phi\,\partial^b\phi + \partial_c\phi\,\partial^c\phi\, e_\nu{}^b\big) + \beta\,\partial_\nu\phi\,\partial^b\phi$$

$$+ \partial_\nu\partial^b\phi + \omega_\nu{}^b{}_c\,\partial^c\phi\Big] - \frac{1}{4}e^{(3\beta-4\alpha)\phi} A_b F^b{}_c F^c{}_\nu$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\,\partial_b\phi\,\partial^b\phi\,A_\nu + \alpha\beta\,(D-2)\,\partial_b\phi\,\partial^b\phi\,A_\nu$$

$$+\alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_\nu + \beta\left(\partial^2\phi\,A_\nu + \frac{3}{2}\partial_b\phi\,F^b{}_\nu + \omega_b{}^b{}_c\,\partial^c\phi\,A_\nu\right)$$

$$+\frac{1}{2}\partial_b F^b{}_\nu - \frac{1}{2}\partial_\nu F^b{}_\mu\,e_b{}^\mu + \frac{1}{2}\omega_b{}^b{}_c F^c{}_\nu - \frac{1}{2}\omega_\nu{}^b{}_c F^c{}_b\Big]$$

$$-\frac{1}{4}e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b\,A_\nu$$

- $$\hat{R}_{\underline{z}\underline{z}} = \hat{R}_{Mz}{}^B{}_{\underline{z}}\,\hat{e}_B{}^M$$

$$= \hat{R}_{\mu z}{}^b{}_{\underline{z}}\,\hat{e}_b{}^\mu + \hat{R}_{\mu z}{}^{\underline{z}}{}_{\underline{z}}\,\underbrace{\hat{e}_{\underline{z}}{}^\mu}_{0} + \underbrace{\hat{R}_{zz}{}^b{}_{\underline{z}}}_{0}\,\hat{e}_b{}^z + \underbrace{\hat{R}_{zz}{}^{\underline{z}}{}_{\underline{z}}}_{0}\,e_{\underline{z}}{}^z$$

$$= e^{-\alpha\phi}\,e_b{}^\mu\,\hat{R}_{\mu z}{}^b{}_{\underline{z}}$$

$$= -\beta\,e^{(\beta-2\alpha)\phi}\left[(\beta-2\alpha)\,\partial_b\phi\,\partial^b\phi + \partial^2\phi + \alpha D\,\partial_c\phi\,\partial^c\phi + \omega_b{}^b{}_c\,\partial^c\phi\right]$$

$$-\frac{1}{4}e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b$$

$$= -e^{(\beta-2\alpha)\phi}\left[\beta^2\,\partial_b\phi\,\partial^b\phi + \alpha\beta\,(D-2)\,\partial_b\phi\,\partial^b\phi + \beta\left(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\right)\right]$$

$$-\frac{1}{4}e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b$$

Now we compute the vielbein components of the Ricci tensor

$$\hat{R}_{AC} = \hat{e}_A{}^N\,\hat{R}_{NC}$$

- $$\hat{R}_{ac} = \hat{e}_a{}^N\,\hat{R}_{Nc} = \hat{e}_a{}^\nu\,\hat{R}_{\nu c} + \hat{e}_a{}^z\,\hat{R}_{zc}$$

$$= e^{-\alpha\phi}\,e_a{}^\nu\,\hat{R}_{\nu c} - e^{-\alpha\phi}\,A_a\,\hat{R}_{zc}$$

$$
= e^{-2\alpha\phi}\Big[R_{ac} + \alpha\Big(\partial_a\partial_c\phi - \partial^2\phi\,\eta_{ac} - (D-1)\,\partial_a\partial_c\phi
$$

$$
+ \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu
$$

$$
+ \omega_b{}^b{}_a\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\,\eta_{ca} + (3-D)\,\omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi \Big)
$$

$$
+ \alpha^2 (D-2)\left(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\right) - \beta^2\,\partial_a\phi\,\partial_c\phi
$$

$$
+ \alpha\beta\left(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\right) - \beta\left(\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi\right)\Big]
$$

$$
- e^{(2\beta-4\alpha)\phi}\Big[\alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_c A_a + \beta\,\frac{3}{2}\,\partial_b\phi\,F^b{}_c A_a + \frac{1}{2}\,\partial_b F^b{}_c A_a
$$

$$
+ \frac{1}{2} F_{ba} F^b{}_c + \frac{1}{2}\,\omega_b{}^b{}_d\,F^d{}_c A_a - \frac{1}{2}\,\omega_b{}^d{}_c\,F^b{}_d\,A_a\Big]
$$

$$
+ e^{(2\beta-4\alpha)\phi}\Big[\alpha\left(-\partial_b\phi\,F^b{}_c A_a + \frac{D}{2}\,\partial^d\phi\,F_{dc} A_a - \partial^b\phi\,F_{bc} A_a\right)
$$

$$
+ \frac{1}{2}\,\partial_b F^b{}_c A_a + \beta\,\frac{3}{2}\,\partial_b\phi\,F^b{}_c A_a
$$

$$
+ \frac{1}{2}\,\omega_b{}^b{}_d\,F^d{}_c A_a - \frac{1}{2}\,\omega_{bcd}\,F^{db} A_a\Big]
$$

$$
= e^{-2\alpha\phi}\Big[R_{ac} + \alpha\Big(-(D-2)\,\partial_a\partial_c\phi - \partial^2\phi\,\eta_{ac}
$$

$$
+ \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu
$$

$$
+ \omega_b{}^b{}_a\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\,\eta_{ca} + (3-D)\,\omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi \Big)
$$

$$
+ \alpha^2 (D-2)\left(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\right) - \beta^2\,\partial_a\phi\,\partial_c\phi
$$

$$
+ \alpha\beta\left(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\right) - \beta\left(\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi\right)\Big]
$$

$$
- e^{(2\beta-4\alpha)\phi}\Big[\partial_b\phi\,F^b{}_c A_a\,\frac{(D-4)\alpha + 3\beta}{2} + \frac{1}{2}\,\partial_b F^b{}_c A_a + \frac{1}{2} F_{ba} F^b{}_c
$$

$$
+ \frac{1}{2}\,\omega_b{}^b{}_d\,F^d{}_c A_a - \frac{1}{2}\,\omega_b{}^d{}_c\,F^b{}_d\,A_a
$$

$$
- \alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_c A_a - \frac{1}{2}\,\partial_b F^b{}_c A_a
$$

$$-\beta\,\frac{3}{2}\,\partial_b\phi\,F^b{}_c\,A_a$$

$$-\frac{1}{2}\,\omega_b{}^b{}_d\,F^d{}_c\,A_a+\frac{1}{2}\,\omega_b{}^d{}_c\,F^b{}_d\,A_a\Big]$$

$$=e^{-2\alpha\phi}\Big[R_{ac}+\alpha\Big(-(D-2)\,\partial_a\partial_c\phi\;\underbrace{-\partial^2\phi\,\eta_{ac}}_{\Box\phi}$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu-\partial^b\phi\,\partial_b e_{\nu c}e_a{}^\nu-\partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu+\partial^b\phi\,\partial_a e_{\mu c}\,e_b{}^\mu$$

$$+\omega_b{}^b{}_a\,\partial_c\phi-\omega_b{}^{bd}\,\partial_d\phi\,\eta_{ac}\underbrace{-(D-3)}_{-(D-2)+1}\omega_{ac}{}^d\,\partial_d\phi+\omega^d{}_{ac}\,\partial_d\phi\Big)$$

$$+\alpha^2(D-2)\Big(\partial_a\phi\,\partial_c\phi-\partial^d\phi\,\partial_d\phi\,\eta_{ac}\Big)-\beta^2\,\partial_a\phi\,\partial_c\phi$$

$$+\alpha\beta\Big(2\,\partial_a\phi\,\partial_c\phi-\partial_d\phi\,\partial^d\phi\,\eta_{ac}\Big)-\beta\Big(\underbrace{\partial_a\partial_c\phi+\omega_{ac}{}^d\,\partial_d\phi}_{\nabla_a\nabla_c\phi}\Big)\Big]$$

$$-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb}=(*)$$

NOTE 4: We will see later that one must set $\beta=-(D-2)\alpha$

$$(*)=-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb}+e^{-2\alpha\phi}\Big[R_{ac}+\overbrace{(D-2)\alpha}^{-\beta}\,\nabla_a\nabla_c\phi$$

$$+\partial_a\phi\,\partial_c\phi\Big(\underbrace{\alpha^2(D-2)-\beta^2+2\alpha\beta}_{\alpha\beta-\beta^2=-(D-2)(D-1)\alpha^2}\Big)-\partial^b\phi\,\partial_b\phi\,\eta_{ac}\Big(\underbrace{\alpha^2(D-2)+\alpha\beta}_{0}\Big)$$

$$+\alpha\Big(-\Box\phi\,\eta_{ac}-(D-2)\,\nabla_a\nabla_c\phi+\omega_{ac}{}^d\,\partial_d\phi+\omega^d{}_{ac}\,\partial_d\phi$$

$$+\omega_b{}^b{}_a\,\partial_c\phi+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu-\partial_d\phi\,\partial^d e_{\nu c}\,e_a{}^\nu$$

$$-\partial_c\phi\,\partial_a e_\nu{}^b e_b{}^\nu+\partial_d\phi\,\partial_a e_{\nu c}\,e^{d\nu}\Big)\Big]$$

NOTE 5: $\alpha^2=\dfrac{1}{2(D-2)(D-1)}$ [We will see later]

$$=-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb}+e^{-2\alpha\phi}\Big[R_{ac}-\frac{1}{2}\,\partial_a\phi\,\partial_c\phi-\alpha\,\Box\phi\,\eta_{ac}$$

$$+\partial_d\phi\Big(\underbrace{\omega_{ac}{}^d+\omega^d{}_{ac}-\partial^d e_{\nu c}\,e_a{}^\nu+\partial_a e_{\nu c}\,e^{d\nu}}_{0}\Big)+\partial_c\phi\Big(\underbrace{\omega_b{}^b{}_a+\partial_b e_\nu{}^b e_a{}^\nu-\partial_a e_\nu{}^b e_b{}^\nu}_{0}\Big)=(*)$$

Remark 1

$$\omega_b{}^{ba} = e_b{}^\mu \omega_\mu{}^{ba} = -e_b{}^\mu \omega_\mu{}^{ab}(e)$$

$$= -e_b{}^\mu \tfrac{1}{2}\big[e^{\nu a}\partial_\mu e_\nu{}^b - e^{\nu b}\partial_\mu e_\nu{}^a - e^{\nu a}\partial_\nu e_\mu{}^b + e^{\nu b}\partial_\nu e_\mu{}^a$$
$$- e^{\nu a} e^{b\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c + e^{\nu b} e^{a\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c\big]$$

$$= -\tfrac{1}{2}\big[\partial_b e_\nu{}^b e^{\nu a} - \partial_b e_\nu{}^a e^{\nu b} - \partial^a e_\mu{}^b e_b{}^\mu + \partial^b e_\mu{}^a e_b{}^\mu$$
$$- e^{\nu a} e^{b\sigma}\partial_\nu e_{\sigma b} + e^{\nu b} e^{a\sigma}\partial_\nu e_{\sigma b}\big]$$

$$= -\tfrac{1}{2}\big[\partial_b e_\nu{}^b e^{\nu a} - \partial^\nu e_\nu{}^a - \partial^a e_\mu{}^b e_b{}^\mu + \partial^\mu e_\mu{}^a$$
$$- \partial^a e_{\sigma b} e^{b\sigma} + \partial^b e_{\sigma b} e^{a\sigma}\big]$$

$$= -\tfrac{1}{2}\big[2\partial_b e_\nu{}^b e^{\nu a} - \partial^a e_{\sigma b} e^{b\sigma}\big] = \partial^a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e^{\nu a}$$

$$\Rightarrow \omega_b{}^b{}_a = \partial_a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e_a{}^\nu$$

Remark 2 [We can compute faster by using anholonomy coefficients]

$$\omega_{acd} + \omega_{dac} = \tfrac{1}{2}\big[\underline{\Omega_{[ac]d}} - \underline{\Omega_{[cd]a}} + \Omega_{[da]c}$$
$$+ \Omega_{[da]c} - \underline{\Omega_{[ac]d}} + \underline{\Omega_{[cd]a}}\big]$$
$$= \Omega_{[da]c}$$

see note below

$$\Rightarrow \omega_{ac}{}^d + \omega^d{}_{ac} = \Omega_{[ba]c}\,\eta^{bd} = -(\partial_b e_a{}^\rho - \partial_a e_b{}^\rho)\, e_{\rho c}\,\eta^{bd}$$
$$= -\partial^d e_a{}^\nu e_{\nu c} + \partial_a e^{d\nu} e_{\nu c}$$
$$= \partial^d e_{\nu c} e_a{}^\nu - \partial_a e_{\nu c} e^{d\nu}$$

NOTE: $\Omega_{[\mu\nu]\rho} = (\partial_\mu e_\nu{}^d - \partial_\nu e_\mu{}^d)\, e_{d\rho}$

$$\Omega_{[ab]c} = e_a{}^\mu e_b{}^\nu e_c{}^\rho\, \Omega_{[\mu\nu]\rho} = (\partial_a e_\nu{}^d e_b{}^\nu - \partial_b e_\mu{}^d e_a{}^\mu)\,\eta_{cd}$$

Important $= -(\partial_a e_b{}^\rho - \partial_b e_a{}^\rho)\, e_{\rho c}$

$$(*) = e^{-2\alpha\phi}\left[R_{ac} - \frac{1}{2}\partial_a\phi\partial_c\phi - \alpha\Box\phi\,\eta_{ac}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^b F_{cb}$$

- $$\hat{R}_{\underline{z}\underline{z}} = \hat{e}_{\underline{z}}{}^N \hat{R}_{N\underline{z}} = \underbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0} \hat{R}_{\mu\underline{z}} + \hat{e}_{\underline{z}}{}^{z}\hat{R}_{z\underline{z}}$$

$$= e^{-\beta\phi}\hat{R}_{z\underline{z}}$$

$$\overbrace{\eta^{ab}\nabla_a\nabla_b\phi = \Box\phi}$$

$$= -e^{-2\alpha\phi}\left[\partial_b\phi\partial^b\phi\left(\beta^2 + (D-2)\alpha\beta\right) + \beta\left(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\right)\right]$$

$$+ \frac{1}{4}e^{(2\beta-4\alpha)\phi}F_c{}^b F^c{}_b$$

$$= e^{-2\alpha\phi}\left[-\underbrace{(\beta^2 + (D-2)\alpha\beta)}_{0 \text{ (see note 4)}}\partial_b\phi\partial^b\phi - \beta\Box\phi\right] + \frac{1}{4}e^{(2\beta-4\alpha)\phi}F^2$$

▲ Using these two vielbein components of the Ricci tensor it's finally straightforward to compute the Ricci scalar

$$\hat{R} = \hat{R}_{AB}\,\eta^{AB}$$

We find:

$$\hat{R} = \hat{R}_{ac}\eta^{ac} + \hat{R}_{\underline{z}\underline{z}} = e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \overbrace{(D\alpha+\beta)}^{D\alpha-(D-2)\alpha = 2\alpha}\Box\phi\right]$$

$$-\frac{1}{2}e^{-2D\alpha\phi}\left(F^2 - \frac{1}{2}F^2\right) =$$

$$= e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \underbrace{2\alpha\Box\phi}_{\text{boundary term}} - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]$$

▶ The full (D+1)-dimensional action then reduces to

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2}\int d^{D+1}x\ \hat{e}\ \hat{R}$$

$$= \frac{1}{2\kappa_{D+1}^2}\int_0^{2\pi L} dz \int d^Dx\ e^{(\alpha D+\beta)\phi}\, e\ \hat{R}$$

$$= \frac{1}{2\underbrace{\frac{\kappa_{D+1}^2}{2\pi L}}_{\kappa_D^2 = \frac{\kappa_{D+1}^2}{2\pi L}}}\int d^Dx\ \underbrace{e^{[(D-2)\alpha+\beta]\phi}}_{\substack{\text{Canonical E-H if} \\ \beta = -(D-2)\alpha \\ \text{(see note 4)}}}\, e\left[R \underbrace{-\frac{1}{2}(\partial\phi)^2}_{\substack{\text{Proper normalisation if} \\ \alpha^2 = \frac{1}{2(D-2)(D-1)}}} - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]$$

(see note 5)

$$= \frac{1}{2\kappa_D^2}\int d^Dx\, e\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right] = S_D$$

Therefore we recover an Einstein – Maxwell – Dilaton theory !!

$$\boxed{S_{D+1} = \frac{1}{2\kappa_D^2}\int d^Dx\ e\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]}$$

with $\kappa_D^2 = \dfrac{\kappa_{D+1}^2}{2\pi L}$

Example : If $D=4 \Rightarrow e^{-2(D-1)\alpha\phi} = e^{-\sqrt{3}\,\phi}$

Exercise: Compute the $\hat{R}_{b\underline{z}}$ component of the Ricci tensor

$$\hat{R}_{b\underline{z}} = \hat{e}_b{}^N \hat{R}_{N\underline{z}} = \hat{e}_b{}^0 \hat{R}_{0\underline{z}} + \hat{e}_b{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\alpha\phi} e_b{}^0 \hat{R}_{0\underline{z}} - e^{-\alpha\phi} A_b \hat{R}_{z\underline{z}}$$

$$= -e^{(\beta-3\alpha)\phi}\Big[\beta^2 \partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \alpha\frac{D-4}{2}\partial_c\phi F^c{}_b$$
$$+\beta\left(\partial^2\phi A_b + \tfrac{3}{2}\partial_c\phi F^c{}_b + \omega_c{}^c{}_d\partial^d\phi A_b\right) + \tfrac{1}{2}\partial_c F^c{}_0 e_b{}^0$$
$$-\tfrac{1}{2}\partial_b F^c{}_0 e_c{}^0 + \tfrac{1}{2}\omega_c{}^c{}_d F^d{}_b - \tfrac{1}{2}\omega_b{}^c{}_d F^d{}_c\Big]$$

$$+\tfrac{1}{4} e^{(3\beta-5\alpha)\phi} F^2 A_b$$

$$+ e^{(\beta-3\alpha)\phi}\left[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \beta\,\Box\phi A_b\right]$$

$$-\tfrac{1}{4} e^{(3\beta-5\alpha)\phi} F^2 A_b$$

$$= -\frac{1}{2}\underbrace{e^{(\beta-3\alpha)\phi}}_{e^{-(D+1)\alpha}}\Big[-\overbrace{\big((D-4)\alpha+3\beta\big)}^{-2(D-1)\alpha}\partial_c\phi F_b{}^c$$
$$+\partial_c F^c{}_0 e_b{}^0 - \partial_b F^c{}_0 e_c{}^0$$
$$+\omega_c{}^c{}_d F^d{}_b + \omega_b{}^c{}_d F_c{}^d\Big]$$

NOTE: $-(D+1)\alpha = (D-3)\alpha - 2(D-1)$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi}\Big[-2(D-1)\alpha\,\partial_c\phi F_b{}^c$$
$$\underbrace{-\partial_c F^c{}_0 e_b{}^0 + \partial_b F^c{}_0 e_c{}^0}_{\partial_c F_b{}^c - F_0{}^c\partial_c e_b{}^0 + F_0{}^c\partial_b e_c{}^0} + \omega_c{}^c{}_d F_b{}^d + \omega_b{}^c{}_d F^d{}_c\Big]$$

NOTE: $\nabla_c F_b{}^c = \partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi} \Big[-2(D-1)\alpha \overbrace{\partial_c \phi}^{\nabla_c \phi} F_b{}^c$$

$$+ \underbrace{\partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c}_{\nabla_c F_b{}^c} + \omega_{cdb} F^{dc} - F_\mu{}^c \partial_c e_b{}^\mu + F_\mu{}^c \partial_b e_c{}^\mu$$

$$+ \omega_{bcd} F^{dc} \Big] = (*)$$

Remark 3

$$\omega_{cdb} + \omega_{bcd} = \frac{1}{2} \Big[\underline{\Omega_{[cd]b}} - \underline{\Omega_{[db]c}} + \Omega_{[cb]d}$$

$$+ \Omega_{[cb]d} - \underline{\Omega_{[cd]b}} + \underline{\Omega_{[db]c}} \Big]$$

$$= \Omega_{[cb]d} = -(\partial_b e_c{}^\mu - \partial_c e_b{}^\mu) e_{\mu d}$$

$$\Rightarrow (\omega_{cdb} + \omega_{bcd}) F^{dc} = -\partial_b e_c{}^\mu e_{\mu d} F^{dc} + \partial_c e_b{}^\mu e_{\mu d} F^{dc}$$

$$= F_\mu{}^c \partial_c e_b{}^\mu - F_\mu{}^c \partial_b e_c{}^\mu$$

$$(*) = \frac{1}{2} e^{(D-3)\alpha\phi} \nabla_c \Big[e^{-2(D-1)\alpha\phi} F_b{}^c \Big] = \hat{R}_{\underline{z} b}$$

II. (D+1)-dimensional vs D-dimensional EOMs and symmetries

In this section we discuss the equations of motion (EOMs) that result from S_{D+1} and S_D.

i) (D+1)-dimensional EOMs

$$S_{D+1} = \frac{1}{2K_{D+1}^2} \int d\overset{D+1}{x}\ \hat{e}\ \hat{R} \implies \hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = 0$$

NOTE: $\hat{g}^{MN}\hat{G}_{MN} = \hat{R} - \frac{1}{2}(D+1)\hat{R} = \left(1 - \frac{1}{2}(D+1)\right)\hat{R} = 0$

$$\implies \hat{R} = 0 \implies \hat{R}_{MN} = 0$$

$\hookrightarrow D \neq 1$

$$\implies \hat{R}_{AB} = 0$$

$$\hat{R}_{AB} = 0 \quad \begin{cases} \hat{R}_{ab} = e^{-2\alpha\phi}\left[R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\,\Box\phi\,\eta_{ab}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^cF_{bc} = 0 \\ \hat{R}_{a\underline{z}} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla_c\left[e^{-2(D-1)\alpha\phi}F_b{}^c\right] = 0 \\ \hat{R}_{\underline{z}\underline{z}} = (D-2)\alpha\, e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}F^2 = 0 \end{cases}$$

- It is important to notice that

$$\phi = 0 \implies \hat{R}_{\underline{z}\underline{z}} = \frac{1}{4}F^2 = 0 \implies \underline{F = 0}$$

$\hookrightarrow$ Trivial Maxwell !!

ii) D-dimensional EOMs

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx\, e\left[R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}e^{-2(D-1)\alpha\phi} F_{\mu\nu}F^{\mu\nu}\right]$$

The EOMs that follow from the above action are:

- $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) + \frac{1}{2}e^{-2(D-1)\alpha\phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^2 g_{\mu\nu}\right)$$

- $$\nabla^\mu\left(e^{-2(D-1)\alpha\phi}F_{\mu\nu}\right) = 0$$

- $$\Box\phi = -\frac{1}{2}(D-1)\alpha\, e^{-2(D-1)\alpha\phi} F^2$$

▲ It is important to notice again that

$$\phi = 0 \implies F^2 = 0 \implies \underline{F=0}$$

↳ Trivial Maxwell !!

IMPORTANT: Having set $\phi = 0$ in the Ansatz for the (D+1)-dimensional metric would have been inconsistent !! [common mistake]
[Einstein - Maxwell - DILATON]

iii) (D+1)-dimensional symmetries

The symmetry group is (D+1)-dimensional general coordinate transformations. At the infinitesimal level we have

$$\delta_{\hat\xi}\, \hat g_{MN} = \hat\xi^P \partial_P \hat g_{MN} + \hat g_{PN} \partial_M \hat\xi^P + \hat g_{MP} \partial_N \hat\xi^P$$

with

$$\hat\xi^M(x,z) = \left(\hat\xi^\mu(x,z) \,,\, \hat\xi^z(x,z) \right)$$

- However, in order to preserve the KK Ansatz of the (D+1)-dimensional metric, there are the restrictions:

Diffeom: $\hat\xi^\mu = \xi^\mu(x)$, $\hat\xi^z = \lambda(x) + \underbrace{c\, z}$

linear dependence on S^1

- On the other hand, the (D+1)-dimensional EOMs have a scaling symmetry [not the E-H action!!] of the form:

$$\left.\begin{aligned} \hat g_{MN} &\to a^2\, \hat g_{MN} \\ \hat e &\to a^{(D+1)}\, \hat e \\ \hat R &\to a^{-2}\, \hat R \end{aligned}\right\} \quad \hat e\, \hat R \to \underbrace{a^{(D-1)}}_{a\in\mathbb{R}}\, \hat e\, \hat R \implies \delta_a \hat g_{MN} = 2a\, \hat g_{MN}$$

infinitesim.

iv) D-dimensional symmetries

Starting from (D+1)-dimensional diffeomorphisms we will obtain D-dimensional diff + U(1) gauge symmetry + Global symmetries.

Ex : Using $\left\{ \begin{aligned} \hat{g}_{\mu\nu} &= e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu \\ \hat{g}_{\mu z} &= \hat{g}_{z\mu} = e^{2\beta\phi} A_\mu \\ \hat{g}_{zz} &= e^{2\beta\phi} \end{aligned} \right\}$ with $\beta = -(D-2)\alpha$

show that $\delta \hat{g}_{MN} = (\delta\xi + \delta a)\, \hat{g}_{MN}$ gives rise to :

$$\delta\phi = \xi^\rho \partial_\rho \phi - \frac{1}{(D-2)\alpha}(c+a)$$

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda - c A_\mu$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \frac{2}{(D-2)} \left[c + a(D-1) \right] g_{\mu\nu}$$

- Setting $a = -\dfrac{c}{(D-1)}$ one finds :

$$\delta\phi = \underline{\delta_\xi \phi} - \overbrace{\frac{c}{(D-1)\alpha}}^{\text{shift} \leftrightarrow \text{Non-linear action}}$$

$$\delta A_\mu = \underline{\delta_\xi A_\mu} + \underline{\partial_\mu \lambda} - \overbrace{c A_\mu}^{\text{scaling} \leftrightarrow \text{linear action}}$$

$$\delta g_{\mu\nu} = \underline{\delta_\xi g_{\mu\nu}}$$

→ Global symmetry $\equiv \mathbb{R}$ (real parameter)

→ $U(1)$ gauge symmetry

→ D-dimensional diffeomorphisms

- Setting $a=-c$ one finds : $\underbrace{n\text{-legs} \Rightarrow nc}$

$$\delta\phi = \delta_\xi \phi \qquad (0\text{-legs})$$

$$\delta A_\mu = \delta_\xi A_\mu + \partial_\mu \lambda - \underbrace{c A_\mu} \qquad (1\text{-leg})$$

$$\delta g_{\mu\nu} = \delta_\xi g_{\mu\nu} - \underbrace{2c\, g_{\mu\nu}} \qquad (2\text{-legs})$$

$\longrightarrow$ Real scaling $\mathbb{R}$ symmetry of the D-dimensional EOMs known as "trombone" scaling symmetry.

Important : There are two inequivalent $\mathbb{R}$ global symmetries. One is an actual symmetry of the D-dimens action whereas the other is only of the EOMs.

Important : In modern language, the global symmetries of the action upon dimensional reduction are referred to as "dualities". In this case the duality group is just $G_{global} = \mathbb{R}$ symmetry and affects scalar and vector fields in the reduced theory.

III. Kaluza-Klein reduction of Maxwell and scalar on S^1

In this section we look at other reductions on S^1. The starting point is a $(D+1)$-dimensional Maxwell field $\hat{B}_M$ with field strength $\hat{F}_{MN} = \partial_M \hat{B}_N - \partial_N \hat{B}_M$.

- The K-K Ansatz for $\hat{B}_M$ reads:

$$\boxed{\hat{B}_M = \left(\hat{B}_\mu , \hat{B}_z \right) = \left(B_\mu(x) , \chi(x) \right)}$$

which yields a field strength of the form:

$$\hat{F}_{MN} = \begin{bmatrix} \hat{F}_{\mu\nu} & \hat{F}_{\mu z} \\ \hat{F}_{z\nu} & 0 \end{bmatrix} = \begin{bmatrix} F_{\mu\nu} & \partial_\mu \chi \\ -\partial_\nu \chi & 0 \end{bmatrix}$$

with:

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$F_{\mu z} = \partial_\mu \chi$$

$$F_{z\nu} = -\partial_\nu \chi$$

- The Maxwell's action in $(D+1)$-dimensions then reduces to:

$$S_{\hat{B}} = -\frac{1}{4}\int d^{D+1}x \, \underbrace{\sqrt{|\hat{g}|}}_{\hat{e}} \, \hat{F}_{MN} \hat{F}^{MN} = -\frac{1}{4}\int d^{D+1}x \, \hat{e} \, \hat{F}_{AB} \hat{F}^{AB} = (*)$$

NOTE 1: $\hat{F}_{AB} = \hat{e}_A{}^M \hat{e}_B{}^N \hat{F}_{MN}$

- $\hat{F}_{ab} = \hat{e}_a{}^M \hat{e}_b{}^N \hat{F}_{MN}$

$$= \hat{e}_a{}^\mu \hat{e}_b{}^\nu \hat{F}_{\mu\nu} + \hat{e}_a{}^z \hat{e}_b{}^\nu \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_b{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_b{}^z \underbrace{\hat{F}_{zz}}_{0}$$

$$= e^{-2\alpha\phi} F_{ab} + e^{-2\alpha\phi} A_a \partial_b \chi - e^{-2\alpha\phi} A_b \partial_a \chi$$

$$= e^{-2\alpha\phi} \Big[\underbrace{F_{ab} - (\partial_a\chi A_b - \partial_b \chi A_a)}_{\mathcal{F}_{ab} \equiv F_{ab} - 2\partial_{[a}\chi A_{b]}} \Big] = e^{-2\alpha\phi} \mathcal{F}_{ab}$$

- $\hat{F}_{a\underline{z}} = \hat{e}_a{}^M \hat{e}_{\underline{z}}{}^N \hat{F}_{MN}$

$$= \hat{e}_a{}^\mu \underbrace{\hat{e}_{\underline{z}}{}^\nu}_{0} \hat{F}_{\mu\nu} + \hat{e}_a{}^z \underbrace{\hat{e}_{\underline{z}}{}^\nu}_{0} \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_{\underline{z}}{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_{\underline{z}}{}^z \underbrace{\hat{F}_{zz}}_{0}$$

$$= e^{-(\alpha+\beta)\phi} \partial_a \chi = - \hat{F}_{\underline{z}a}$$

- $\hat{F}_{\underline{z}\underline{z}} = 0$

NOTE 2: $\hat{e} = e^{(\alpha D + \beta)\phi} e$

$$(*) = -\frac{1}{4} e^{(\alpha D+\beta)\phi} (2\pi L) \int d^D x \, e \Big[\hat{F}_{ab} \hat{F}^{ab} + \underbrace{\hat{F}_{a\underline{z}} \hat{F}^{a\underline{z}} + \hat{F}_{\underline{z}b} \hat{F}^{\underline{z}b}}_{2 \hat{F}_{a\underline{z}} \hat{F}^{a\underline{z}}} \Big]$$

$$= -\frac{1}{4} (2\pi L) e^{(\alpha D+\beta)\phi} \int d^D x \, e \Big[e^{-4\alpha\phi} \mathcal{F}_{ab} \mathcal{F}^{ab} + 2 e^{-2(\alpha+\beta)\phi} \partial_a \chi \partial^a \chi \Big]$$

$$\boxed{S_{\hat{B}} = (2\pi L) \int d^D x \, e \Big[-\frac{1}{4} e^{-2\alpha\phi} \mathcal{F}^2 - \frac{1}{2} e^{2(D-2)\alpha\phi} (\partial\chi)^2 \Big]}$$

- The reduction of the scalar field is straightforward. The KK Ansatz reads:

$$\boxed{\hat{\Phi} = \Phi(x)}$$

So that the action reduces as:

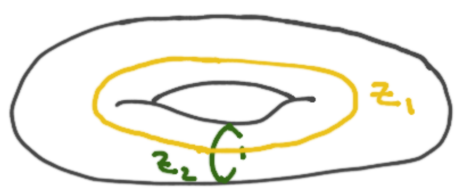
$$\boxed{S_{\hat{\Phi}} = -\frac{1}{2}\int d^{D+1}x \; \hat{e} \; \partial_M \hat{\Phi} \, \partial^M \hat{\Phi} = (2\pi L) \int d^D x \, e \left[-\frac{1}{2} \partial_\mu \Phi \, \partial^\mu \Phi \right]}$$

NOTE 1: $\hat{e} = e^{(D\alpha+\beta)\phi} \, e \overset{\beta = -(D-2)\alpha}{=} e^{2\alpha\phi} e$

NOTE 2: $\partial_A \hat{\Phi} = (\hat{e}_a{}^\mu \partial_\mu \Phi, 0) = e^{-\alpha\phi} (\partial_a \Phi, 0)$

IV. Kaluza-Klein reduction on T^2 and $SL(2)$ duality

In this section we are combining the previous results above and will present the KK reduction of gravity in $(D+2)$ dimensions:



$T^2 \equiv$ 2-torus

coordinates (z_1, z_2)

$$\hat{\hat{g}}_{\mathcal{M}\mathcal{N}} \underset{\text{step 1}}{\Longrightarrow} \hat{g}_{MN} + \hat{A}_{M1} + \hat{\phi}_1 \underset{\text{step 2}}{\Longrightarrow} g_{\mu\nu} + A_{\mu 2} + \phi_2 + A_{\mu 1} + \chi + \phi_1$$

step 1: $\mathcal{M} = M, z_1$

step 2: $M = \mu, z_2$

- Reduction along z_1 :

$$S_{D+2} = \frac{1}{2\kappa_{D+2}^2} \int d^Dx \, dz_2 \, dz_1 \; \hat{\hat{e}} \; \hat{\hat{R}}$$

$$= \frac{1}{2\kappa_{D+1}^2} \int d^Dx \, dz_2 \; \hat{e} \left[\hat{R} - \frac{1}{2} (\partial \hat{\phi}_1)^2 - \frac{1}{4} e^{-2D\alpha_1 \hat{\phi}_1} \hat{F}_1^2 \right] \equiv S_{D+1}$$

with $\kappa_{D+1}^2 = \frac{\kappa_{D+2}^2}{2\pi L_1}$ and $\alpha_1^2 = \frac{1}{2(D-1)D}$

- Reduction along z_2 :

$$S_D = \frac{1}{2\kappa_D^2} \int d^Dx \; e \left[R - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^2 \right.$$

$$- \frac{1}{2} (\partial \phi_1)^2$$

$$\left. + e^{-2D\alpha_1 \phi_1} \left(-\frac{1}{4} e^{-2\alpha_2 \phi_2} \mathcal{F}_1^2 - \frac{1}{2} e^{2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right) \right]$$

$$= \frac{1}{2\kappa_D^2} \int d^Dx \; e \left[R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{2} e^{-2D\alpha_1 \phi_1 + 2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right.$$

$$\left. - \frac{1}{4} e^{-2D\alpha_1 \phi_1 - 2\alpha_2 \phi_2} \mathcal{F}_1^2 - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^2 \right]$$

with $\alpha_2^2 = \frac{1}{2(D-2)(D-1)}$ and $\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2 \partial_{[\mu} \chi A_{\nu]2}$

The action S_D can be enlightening rewritten as

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx\, e\left[R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}e^{\vec{c}\vec{\phi}}(\partial\chi)^2 - \frac{1}{4}e^{\vec{c}_1\vec{\phi}}\mathcal{F}_1^{\,2} - \frac{1}{4}e^{\vec{c}_2\vec{\phi}}\mathcal{F}_2^{\,2}\right]$$

with

$$\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2\,\partial_{[\mu}\chi\, A_{\nu]2}$$

$$\mathcal{F}_{\mu\nu 2} = F_{\mu\nu 2}$$

and

$$\kappa_D^{\,2} = \frac{\kappa_{D+1}^2}{2\pi L_2} = \frac{\kappa_{D+2}^2}{(2\pi L_1)(2\pi L_2)} = \frac{\kappa_{D+2}^2}{\mathrm{Vol}(T^2)}$$

$$\vec{c} = \left[-\sqrt{\frac{2D}{D-1}}\,,\ \sqrt{\frac{2(D-2)}{D-1}}\right]$$

$$\vec{c}_1 = \left[-\sqrt{\frac{2D}{D-1}}\,,\ -\sqrt{\frac{2}{(D-1)(D-2)}}\right]$$

$$\vec{c}_2 = \left[\ 0\ ,\ -\sqrt{\frac{2(D-1)}{D-2}}\right]$$

- Let us rotate the two dilatons $\vec{\phi} = (\phi_1, \phi_2)$ to new ones:

$$\phi = -\frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_1 + \frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_2$$

$$\varphi = -\frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_1 - \frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_2$$

so the D-dimensional action becomes

$$S_D = \frac{1}{2K_D^2}\int d^Dx\, e\left[R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 \right.$$
$$\left. - \frac{1}{4}e^{q\varphi+\phi}\mathcal{F}_1^2 - \frac{1}{4}e^{q\varphi-\phi}\mathcal{F}_2^2\right]$$

with $q^2 = \frac{D}{D-2}$ and the $(D+2)$-dimensional metric reads

$$ds^2_{D+2} = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}\, ds_D^2 + e^{\sqrt{\frac{D-2}{D}}\varphi}\, ds_2^2$$

with

$$ds_2^2 = e^{\phi}(dz_1 + A_{\mu 1}dx^\mu + \chi dz_2)^2 + e^{-\phi}(dz_2 + A_{\mu 2}dx^\mu)^2$$

$$\Rightarrow ds_2^2\Big|_{\phi=\chi=A_{\mu 1,2}=0} = dz_1^2 + dz_2^2$$

Moduli space: (scalars ≡ "moduli")

- The scalar φ parameterises the volume of volume of T^2 as it appears as a factor in front of ds_2^2.

- The scalar ϕ and χ play different roles. The scalar ϕ parameterises a shape-changing of the torus. It scales the z_1-cycle and the z_2-cycle in opposite manners. The scalar χ varies the angle between the z_1-cycle and the z_2-cycle.

Scalar sector and SL(2) duality

Let's focus on the scalar sector of the above 8D action:

$$\mathcal{L}_{scalar} = -\frac{1}{2}(\partial\varphi)^2 \underbrace{- \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi^2)}_{\mathcal{L}(\phi,\chi)}$$

Global symmetries (or dualities)

i) The scalar φ decouples from the others. It has a global $\mathbb{R}$ shift symmetry

$$\varphi \longrightarrow \varphi + K \qquad \text{with} \quad K \in \mathbb{R}$$

↳ Non-linear action

ii) The symmetry analysis for $\mathcal{L}(\phi,\chi)$ is more interesting. To make the symmetry manifest we define a complex modulus field on T^2 as

$$\tau = \chi + i\, e^{-\phi}$$

in terms of which

$$\mathcal{L}(\phi,\chi) = -\frac{1}{2}\left[(\partial\phi)^2 + e^{2\phi}(\partial\chi)^2\right] = -\frac{\partial_\mu\tau\,\partial^\mu\bar{\tau}}{2\,\mathrm{Im}^2(\tau)}$$

Ex: Show that $\mathcal{L}(\phi, \chi)$ is invariant under the global fractional linear transformation:

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

with $ad - bc = 1$. Show that this transformation acts on (ϕ, χ) as:

$$\left.\begin{aligned} e^{\phi} &\longrightarrow e^{\phi'} = (c\chi + d)^2 e^{\phi} + c^2 e^{-\phi} \\ \chi e^{\phi} &\longrightarrow \chi' e^{\phi'} = (a\chi + b)(c\chi + d) e^{\phi} + ac\, e^{-\phi} \end{aligned}\right\} \text{Non-linear } SL(2) \text{ action}$$

iii) As scalars couple to vectors, these must also transform. Let us write a constant 2×2 matrix Λ of the form

$$\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with} \qquad |\Lambda| = ad - bc = 1$$

so that $\Lambda \in SL(2)$. Using this matrix Λ, the transformation

$$\begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \longrightarrow (\Lambda^t)^{-1} \begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \quad \Rightarrow \text{Linear } SL(2) \text{ action}$$

precisely cancells against the transformation of scalars and leaves the action invariant.

As a result, the global symmetry (duality) of the KK reduction of gravity on T^2 turns out to be:

$$G_{global} = \mathbb{R} \times SL(2) \equiv GL(2)$$

Some final remarks:

- If gravity in $(D+n)$ dimensions is reduced on T^n then the duality group becomes $G_{global} = \mathbb{R} \times SL(n)$

- If we start from the Type II supergravity theories in 10 D and reduce it on T^n then the duality group gets enhanced to the exceptional $G_{global} = E_{n(n)}$

$$S_{10D}^{SUGRA} = \frac{1}{2\kappa_{10D}^2} \int d^{10}x \, \hat{e} \left[\underset{\downarrow}{\hat{R}} - \frac{1}{2 \times n!} \underset{\downarrow}{\hat{F}_{(n)}^2} - \frac{1}{2} \partial_M \hat{\Phi} \partial^M \hat{\Phi} + \ldots \right]$$

$GL(n) = \mathbb{R} \times SL(n)$ enhancement to $E_{n(n)}$

where $\hat{F}_{(n)}^2 = \hat{F}_{M_1 \cdots M_n} \hat{F}^{M_1 \cdots M_n}$ with $\hat{F}_{M_1 \cdots M_n} = \partial_{[M_1} \underbrace{\hat{A}_{M_2 \cdots M_n]}}_{(n-1)\text{-form}}$

- Duality transformations allow us to explore different regimes of the theory. For example large vs small extra dimensions or weak vs strong coupling.

* Scalar kinetic terms and "coset" spaces

The scalar kinetic terms can be understood geometrically from a "fictitious" (or auxiliary) scalar space perspective where scalar fields $\phi_i \in \mathbb{R}$ $(i=1,\dots,N)$ play the role of coordinates:

$$S_\phi = \int d^4x \sqrt{-|g|} \left[- \underbrace{K_{ij}(\phi)}_{\text{"metric" in field space}} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi) \right]$$

- One canonically normalised scalar:

$$K_{\phi\phi} = \frac{1}{2}$$

- N canonically normalised scalars:

$$K_{ij} = \frac{1}{2} \delta_{ij}$$

The geometrical interpretation becomes obvious when writing the kinetic terms as:

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = - K_{ij}(\phi) \underbrace{\partial_\mu \phi^i}_{d\phi^i} \underbrace{\partial^\mu \phi^j}_{d\phi^j} \quad \Rightarrow \quad \text{"Scalar geometry"} \ [\sigma\text{-model}]$$

$\Rightarrow$ Line element in field space !!

Important: In supergravity the scalar geometries are of a specific type called coset spaces.

Coset space $\mathcal{M} = \frac{G}{H}$: Coordinates on $\mathcal{M}$ (fields ϕ^i) correspond to an element of G not being an element of its maximal compact subgroup $H \subset G$:

- generators of G : $\{ \underbrace{h_1, \dots, h_{\dim H}}_{\text{generators of } H} ; t_1, \dots, t_{\dim G - \dim H} \}$

  [in a given representation]

- Coset representative : $\mathcal{V}(\phi) = e^{\sum_{i=1}^{N = \dim G - \dim H} \phi^i t_i} \in \frac{G}{H}$

  scalars = algebra parameters

- Scalar matrix : $M(\phi) = \mathcal{V}^t \mathcal{V} \in G$

- Scalar kinetic terms :

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = - K_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right]$$

Important: Coset representatives transform as

$$\mathcal{V}' = h(x)\, \mathcal{V}\, g \qquad \text{with} \quad g \in G \,,\; h(x) \in H$$

g ↳ global, $h(x)$ ↳ local

$$\Rightarrow \quad M' = v'^t v' = g^t \overbrace{v^t v}^{M} g = g^t M g$$

As a result, $\mathcal{L}_{kin}$ is invariant under the action of $g \in G$

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}' = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M' \partial^\mu M'^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g^t \partial_\mu M \underbrace{g\, g^{-1}}_{\mathbb{I}} \partial^\mu M^{-1} g^{-t}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g^t \partial_\mu M \partial^\mu M^{-1} g^{-t}\right]$$

cyclicity

$$= \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \partial^\mu M^{-1}\right] = \frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}$$

Example : $\mathcal{M} = \dfrac{SL(2)}{SO(2)} \Rightarrow G = SL(2)\,, \; H = SO(2) \subset SL(2)$

- Generators of G : $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
  [fundamental represent]

$\Rightarrow$ Commutators: $[T, E_\pm] = \pm 2 E_\pm$
$[E_+, E_-] = T$

- Some examples of group elements of $G = SL(2)$

$$g_T = e^{\frac{1}{2}\phi T} = \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix}, \quad g_{E_+} = e^{x E_+} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$g_h = e^{\theta \underbrace{(E_+ - E_-)}_{h \text{ generator}}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2) = H$$

- When constructing $\mathcal{V} \in \frac{SL(2)}{SO(2)}$ one must be careful for not to exponentiate $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One choice is

$$\mathcal{V}(\phi, \chi) = g_T \, g_{E_+} = \begin{bmatrix} e^{\frac{\phi}{2}} & e^{\frac{\phi}{2}} \chi \\ 0 & e^{-\frac{\phi}{2}} \end{bmatrix} \in \frac{SL(2)}{SO(2)}$$

so that

$$M(\phi, \chi) = \mathcal{V}^t \, \mathcal{V} = \begin{bmatrix} e^{\phi} & e^{\phi} \chi \\ e^{\phi} \chi & e^{-\phi} + \chi e^{\phi} \chi \end{bmatrix}$$

and

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right]$$

$$= -\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \chi \, \partial^\mu \chi$$

$$\Rightarrow \quad K_{ij}(\phi, \chi) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\phi} \end{bmatrix}$$

NOTE$_3$: Coset spaces of the form $\frac{G}{H}$ with H being the maximal compact subgroup of G (like $\frac{SL(2)}{SO(2)}$) are important when describing the scalar geometries arising from Kaluza-Klein reductions.

V. Prelude to superstrings and D=10, 11 supergravity

* From strings to $\mathcal{N}=2$, D=10 Supergravity

Particle evolution in D-dimensions

$\bullet \approx> X^M(\underbrace{\tau}_{\text{proper time}})$

String evolution in D-dimensions

$\approx> X^M(\tau,\sigma)$

$l_s^2 \sim 2\alpha'$

$+\text{susy} \Rightarrow \left.\begin{matrix}\theta^1(\tau,\sigma)\\ \theta^2(\tau,\sigma)\end{matrix}\right\}$ Grassman variables

Set D = 10 and $\theta^{1,2}$ being $\underbrace{\text{M-W}}_{\text{Majorana-Weyl}}$ fermions

→ 2D conformal field theory : $X^M(\tau,\sigma)$, $\theta^{1,2}(\tau,\sigma)$

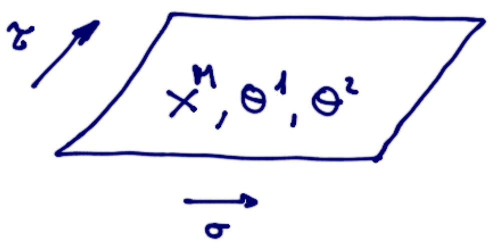
$$S_{2D} = -\frac{1}{4\pi\alpha'}\int d\sigma^2\, \underbrace{\eta^{\alpha\beta}}\, \partial_\alpha X^M \partial_\beta X^M + \text{fermion terms}$$

with $\sigma^\alpha = (\tau,\sigma)$

$\eta^{\alpha\beta} = (-1,1)$

↳ gauge fixing : diff + Weyl in 2D

→ Mode expansion and states



$\Rightarrow$

$$X_M \supset \sum_n \left(a_M^{(n)} e^{-2in(\tau-\sigma)} + \tilde{a}_M^{(n)} e^{-2in(\tau+\sigma)}\right)$$

$$\theta^1 \supset \sum_n b^{(n)} e^{-2in(\tau-\sigma)} \,;\; \theta^2 \supset \sum_n \tilde{b}^{(n)} e^{-2in(\tau+\sigma)}$$

Promote a's, $\tilde{a}$'s, b's, $\tilde{b}$'s to operators with $[\,,\,]$ or $\{\,,\,\}$ relations : "dilaton"

$$|\text{state}\rangle = a^{\dagger}_M a^{\dagger}_N |0\rangle \Rightarrow \underbrace{G_{MN}}_{\text{metric}} \oplus \underbrace{B_{MN}}_{\text{antisym}} \oplus \underbrace{\Phi}_{\text{scalar trace}}$$

$D=10$:

→ Mass of a state :

$$M^2 = \frac{1}{l_s^2}\left[N(a,b) + \tilde{N}(\tilde{a},\tilde{b})\right] \Rightarrow \begin{matrix} l_s \to 0 \\ \bigcirc \to \bullet \\ M^2 \to \infty \end{matrix} \Rightarrow \text{Keep only massless states !!}$$

ocupation numbers

"low energy"

→ $\mathcal{N}=2$, $D=10$ massless spectrum : $\underbrace{G_{MN}, B_{(2)}, \Phi, C_{(p)}}_{\text{Bosons}}$; $\underbrace{\chi^{1,2}_{\alpha}, \Psi^{1,2}_{M\alpha}}_{\text{Fermions}}$

$(\text{ch}\,\Psi^1 \neq \text{ch}\,\Psi^2)$ IIA : $p=1,3 \Rightarrow C_M, C_{MNP}$

$(\text{ch}\,\Psi^1 = \text{ch}\,\Psi^2)$ IIB : $p=0,2,4 \Rightarrow C_{(0)}, C_{MN}, C_{MNPQ}$

NOTE: A p-form $C_{(p)}$ has p antisymmetric indices $C_{(p)} = C_{[M_1 \cdots M_p]}$

- Lagrangian : a candidate

$$\mathcal{L}_{10D} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \sqrt{G}\left[R - \frac{1}{2}\partial_M\Phi\,\partial^M\Phi - \frac{1}{2\times 3!}e^{-\Phi}\underbrace{H_{MNP}}H^{MNP} + \ldots + \text{fermi}\right.$$

with $2\kappa_{10}^2 = \frac{1}{2\pi}(2\pi l_s)^8$ $\qquad H_{(3)} \equiv H_{MNP} = \partial_{[M}B_{NP]} = dB_{(2)}$

→ We can also study a probe string propagating in a background $\{ G_{MN} , B_{MN} , \Phi , C_{(p)} \}$ generated by other strings around :

$$S_{\substack{\text{probe}\\\text{string}}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[(\partial_\alpha X^M)(\partial^\alpha X^N)\, \underbrace{G_{MN}(X)} + \epsilon^{\alpha\beta}\, \partial_\alpha X^M \partial_\beta X^N\, \underbrace{B_{MN}(X)} + \dots \right.$$

G_{MN}, $B_{(2)}$, etc can be viewed as couplings in the 2D field theory !!

$$\text{Conformal invariance} \Rightarrow \underbrace{\beta_G^{MN} = \beta_B^{MN} = \dots = 0}$$

At lowest order in $\frac{\sqrt{\alpha'}}{L}$ ~ system scale

$\Rightarrow$ E.O.M of an action !!

→ $\mathcal{N}=2$, $D=10$ Supergravity action :

$$S_{\substack{SUGRA\\ \mathcal{N}=2,\, D=10}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left[R - \frac{1}{2} \partial^M \Phi\, \partial_M \Phi - \frac{1}{2\times 3!} e^{-\Phi} H_{MNP} H^{MNP} \right.$$

$$- \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{G} \begin{cases} \text{IIA}: \ \frac{1}{2!} e^{3/2\,\Phi}\, \tilde{F}_{MN} \tilde{F}^{MN} + \frac{1}{4!} e^{1/2\,\Phi}\, \tilde{F}_{M_1 \dots M_4} \tilde{F}^{M_1 \dots M_4} \\ \text{IIB}: \ e^{2\Phi}\, \partial_M C_{(0)} \partial^M C_{(0)} + \frac{1}{3!} e^{\Phi}\, \tilde{F}_{MNP} \tilde{F}^{MNP} + \frac{1}{5!} \tilde{F}_{M_1 \dots M_5} \tilde{F}^{M_1 \dots M_5} \end{cases}$$

$$B_{(2)} \wedge F_{(4)} \wedge F_{(4)} \Rightarrow \text{wedge products}$$

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x \begin{cases} \text{IIA: } \underbrace{\epsilon^{\mu_1\ldots\mu_{10}} \, B_{\mu_1\mu_2} \, F_{\mu_3\ldots\mu_6} \, F_{\mu_7\ldots\mu_{10}}} \\ \text{IIB: } \underbrace{\epsilon^{\mu_1\ldots\mu_{10}} \, C_{\mu_1\ldots\mu_4} \, H_{\mu_5\mu_6\mu_7} \, F_{\mu_8\mu_9\mu_{10}}}_{C_{(4)} \wedge H_{(3)} \wedge F_{(3)}} \end{cases}$$

$$+ \, S_{\text{Fermi}}(\chi^{1,2}, \Psi^{1,2})$$

where the gauge invariant field strengths are given by:

IIA: $\tilde{F}_{(2)} = F_{(2)} = dC_{(1)}$

$\tilde{F}_{(4)} = \underbrace{F_{(4)}}_{dC_{(3)}} + C_{(1)} \wedge H_{(3)}$

NOTE: There is a massive IIA theory with $F_{(0)} = \text{cte}$

IIB: $\tilde{F}_{(3)} = \underbrace{F_{(3)}}_{dC_{(2)}} - H_{(3)} \wedge C_{(0)}$

$\tilde{F}_{(5)} = \underbrace{F_{(5)}}_{dC_{(4)}} + \frac{1}{2}\left[B_{(2)} \wedge F_{(3)} - C_{(2)} \wedge H_{(3)}\right]$

$\Rightarrow$ Self-dual : $\boxed{\tilde{F}_{(5)} = * \tilde{F}_{(5)}}$

Math : $F_{(n)} = F_{\mu_1\cdots\mu_n} = \partial_{[\mu_1} C_{\mu_2\cdots\mu_n]} \equiv d\,C_{(n)}$

$\Rightarrow$ Starting from closed superstrings we have obtained $\mathcal{N}=2$, $D=10$ Supergravities as the low-energy limit !!

$\Rightarrow$ Superstrings live in a ten-dimensional space-time ...

... so what about $10-4=6$ extra dimensions ?

→ The type IIA supergravity can be connected to the one and unique $\mathcal{N}=1$, $D=11$ Supergravity conjectured to be the low-energy limit of a mysterious theory of membranes called "M-theory"

$$\boxed{\begin{aligned} S_{\substack{SUGRA\\ \mathcal{N}=1,\,D=11}} &= \frac{1}{2K_{11}^2}\int d^{11}x\,\sqrt{G}\left[R-\frac{1}{2\times 4!}F_{\hat{M}_1\ldots\hat{M}_4}F^{\hat{M}_1\ldots\hat{M}_4}\right] \\ &\quad -\frac{1}{12K_{11}^2}\int d^{11}x\,\underbrace{\epsilon^{\hat{M}_1\ldots\hat{M}_{11}}A_{\hat{M}_1\hat{M}_2\hat{M}_3}F_{\hat{M}_4\ldots\hat{M}_7}F_{\hat{M}_8\ldots\hat{M}_{11}}}_{A_{(3)}\wedge F_{(4)}\wedge F_{(4)}} \\ &\quad + S_{Fermi}(\Psi)\end{aligned}}$$

with $2K_{11}^2 = \frac{1}{2\pi}(2\pi\,\ell_p)^9$

↳ Planck scale

* The field content of the theory is $G_{\hat{M}\hat{N}}\oplus A_{\hat{M}\hat{N}\hat{P}}\oplus\Psi_{\hat{M}\alpha}$

with $F_{(4)}\equiv F_{\hat{M}_1\ldots\hat{M}_4}=\partial_{[\hat{M}_1}A_{\hat{M}_2\hat{M}_3\hat{M}_4]}\equiv dA_{(3)}$. It is invariant under local supersymmetry transformations

$$\delta_\epsilon e_{\hat{M}}{}^{\hat{A}} = \bar{\epsilon}\,\Gamma^{\hat{A}}\,\Psi_{\hat{M}}$$

$$\delta_\epsilon A_{\hat{M}\hat{N}\hat{P}} = -3\,\bar{\epsilon}\,\Gamma_{[\hat{M}\hat{N}}\Psi_{\hat{P}]}$$

$$\delta_\epsilon \Psi_{\hat{M}} = D_{\hat{M}}\epsilon + \frac{1}{12}\left[\Gamma_{\hat{M}}\frac{1}{4!}F_{\hat{Q}\hat{R}\hat{S}\hat{T}}\Gamma^{\hat{Q}\hat{R}\hat{S}\hat{T}} - 3\,\frac{1}{3!}F_{\hat{M}\hat{N}\hat{P}\hat{Q}}\Gamma^{\hat{N}\hat{P}\hat{Q}}\right]\epsilon$$

Important : Note that there is no coupling to be tuned !!

* M-theory $\Rightarrow$ IIA $\begin{cases} G_{\hat{M}\hat{N}} \Rightarrow G_{MN} \oplus G_{M\,10} \equiv C_M \oplus G_{10\,10} \equiv \Phi \\ A_{\hat{M}\hat{N}\hat{P}} \Rightarrow A_{MNP} \equiv C_{MNP} \oplus A_{MN\,10} \equiv B_{MN} \end{cases}$

$\hat{M} = (M, 10)$

$\hookrightarrow M = 0, \dots, 9$

$\hookrightarrow \hat{M} = 0, \dots, 10$

Then one finds that

$$\underbrace{G_{\hat{M}\hat{N}} \;,\; A_{\hat{M}\hat{N}\hat{P}}}_{\mathcal{N}=1,\ D=11 \text{ SUPERGRAVITY}} \Longrightarrow \underbrace{G_{MN} \,,\, B_{MN} \,,\, \Phi \,,\, C_M \,,\, C_{MNP}}_{\mathcal{N}=2,\ D=10 \text{ Type IIA SUPERGRAVITY}}$$

Important : The 11D action also reduces consistently to the type IIA action (not only the field content)

VI. Type II reductions on T^6

Let us consider type II SUGRAS in 10D and perform a KK decomposition :

NOTE : 10D index splitting $M = (\mu, m)$ (μ: 4D, m: 6D)

- G_{MN} : $G_{\mu\nu}$, $\underline{G_{\mu m}}$ (6) , $\underline{G_{mn}}$ (21)
- B_{MN} : $\underline{B_{\mu\nu}}$ (1) , $\underline{B_{\mu m}}$ (6) , $\underline{B_{mn}}$ (15)
- Φ : $\underline{\Phi}$ (1)

$\Rightarrow$ Universal sector

scalars = 38

vectors = 12

metric = 1

IIA : odd p-forms $p = 1, 3$

- C_M : C_μ (1) , $\underline{C_m}$ (6)
- C_{MNP} : $\underbrace{C_{\mu\nu\rho}}_{\substack{\text{not-independent} \\ \text{(dual to V)}}}$, $\underline{C_{\mu\nu m}}$ (6) , $\underline{C_{\mu mn}}$ (15) , $\underline{C_{mnp}}$ (20)

$\Rightarrow$ scalars = 32

vectors = 16

IIB : even p-forms $p = 0, 2, 4$ (self-dual)

- $C_{(0)}$: $\underline{C_{(0)}}$ (1)
- $C_{(2)}$: $\underbrace{\text{non-dyn}}$ $\overbrace{\text{(dual to V)}}^{\text{not-indep}}$, $\underline{C_{\mu\nu}}$ (1) , $\underline{C_{\mu m}}$ (6) , $\underline{C_{mn}}$ (15)
- $C_{(4)}$: $C_{\mu\nu\rho\sigma}$, $C_{\mu\nu\rho m}$, $C_{\mu\nu mn}$ (15) , $C_{\mu mnp}$ (20) , C_{mnpq} (15)

$$\approx\!> \quad \text{scalars} = 32$$
$$\text{vectors} = 16$$

Important: Upon suitable dualisations (2-form ↔ scalars) and Kaluza-Klein inspired field redefinitions, the dimensionally reduced theory in 4D is:

Field content:

* scalars : $38 + 32 = \underbrace{70}_{\phi^{i=1,\dots,70}}$ $\approx\!> M_{MN}(\phi) \in \underbrace{\frac{E_{7(7)}}{SU(8)}}_{\text{coset space } \frac{G}{H}}$ (56×56 matrix) [like $\frac{SL(2)}{SO(2)}$]

NOTE: $E_{7(7)}$ irreps : $\underbrace{56}_{\text{fundam } \mathbb{M}}$, $\underbrace{133}_{\text{adjoint } \alpha}$, ...

* vectors : $12 + 16 = \underbrace{28}_{A_\mu^{\Lambda=1,\dots,28}}$ $\approx\!>$ Abelian vector fields "ungauged theory"

* metric : $g_{\mu\nu}$

$\approx\!>$ Bosonic sector of $\mathcal{N}=8$ SUGRA !!

NOTE: Reducing 10D fermions $\Rightarrow$ $\mathcal{N}=8$ SUGRA

<u>Action</u> : ungauged theory $\Rightarrow$ $D_\mu = \partial_\mu$

$F_{\mu\nu}{}^{\Lambda} = \partial_\mu A_\nu{}^{\Lambda} - \partial_\nu A_\mu{}^{\Lambda}$

$$
\begin{aligned}
S^{\text{ungauged}}_{\mathcal{N}=8} = \int d^4x \sqrt{-|g|} \Big\{ & \frac{R}{2} \\
& + \frac{1}{96} \operatorname{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right] \\
& + \frac{1}{4} \overbrace{\mathcal{I}_{\Lambda\Sigma}(\phi)}^{\text{"} g^{-2} \delta_{\Lambda\Sigma} \text{ like"}} F_{\mu\nu}{}^{\Lambda} F^{\mu\nu\,\Sigma} \\
& + \frac{1}{4} \frac{1}{2\sqrt{-|g|}} \overbrace{\mathcal{R}_{\Lambda\Sigma}(\phi)}^{\text{"} \frac{1}{8\pi^2} \theta \delta_{\Lambda\Sigma} \text{ like"}} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma} \\
& + \text{fermi-terms} \Big\} = \int d^4x \sqrt{-|g|} \underbrace{\mathcal{L}}_{\text{Lagrangian}}
\end{aligned}
$$

$\approx>$ KK reduction of Type II SUGRA in 10D yields ungauged $\mathcal{N} = 8$ (maximal) supergravity in 4D

Symmetries :

* Global $G = E_{7(7)}$ of the scalar sector $M \in \frac{G}{H}$
* Local $H = SU(8)$ R-symmetry acting on fermions
* $U(1)^{28}$ gauge theory with $\underbrace{\text{uncharged matter}}_{D_\mu M_{MN} = \partial_\mu M_{MN}}$

Electric-magnetic Sp(56) duality

As in classical electromagnetism we can associate with the electric $F_{\mu\nu}{}^{\Lambda}$ their magnetic duals $G_{\mu\nu\Lambda}$

$$G_{\mu\nu\Lambda} \equiv -\varepsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}{}^{\Lambda}} = \mathcal{R}_{\Lambda\Sigma}(\phi)\, F_{\mu\nu}{}^{\Sigma} - \mathcal{I}_{\Lambda\Sigma}(\phi) \underbrace{* F_{\mu\nu}{}^{\Sigma}}_{\text{4D Hodge dual}}$$

with

$$* F_{\mu\nu}{}^{\Sigma} \equiv \frac{\sqrt{-|g|}}{2!} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma\,\Sigma}$$

$$** = -1$$

NOTE: In ordinary Maxwell theory without scalars $(\phi^{i} = 0)$ one has $\mathcal{I}_{\Lambda\Sigma} = -\delta_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma} = 0$.

In terms of $(F_{\mu\nu}{}^{\Lambda}, G_{\mu\nu\Lambda})$ the $\underbrace{\text{vacuum}}_{\text{no charged matter}}$ Maxwell equations are

$$\nabla^{\mu} (* F_{\mu\nu}{}^{\Lambda}) = 0 \quad , \quad \nabla^{\mu} (* G_{\mu\nu\Lambda}) = 0$$

which can be expressed as

$$d\, \mathcal{G}_{\mu\nu}{}^{M} = 0 \qquad \text{with} \qquad \mathcal{G}_{\mu\nu}{}^{\overbrace{M}^{M=1,\dots,56}} = \begin{pmatrix} F_{\mu\nu}{}^{\Lambda} \\ G_{\mu\nu\Lambda} \end{pmatrix}$$

Using $G_{\mu\nu}{}^{\mathbb{M}}$ the vector sector of the Lagrangian can be expressed as

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}\sqrt{-|g|}\; M_{\mathbb{M}\mathbb{N}}(\phi)\; G_{\mu\nu}{}^{\mathbb{M}}\, G^{\mu\nu\,\mathbb{N}}$$

with

$$\underbrace{M_{\mathbb{M}\mathbb{N}}}_{\text{symmetric}} = \begin{bmatrix} M_{\Lambda\Sigma} & M_{\Lambda}{}^{\Sigma} \\ M^{\Lambda}{}_{\Sigma} & M^{\Lambda\Sigma} \end{bmatrix} = \begin{bmatrix} -(\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma} \\ (\mathcal{I}^{-1}\mathcal{R})^{\Lambda}{}_{\Sigma} & -(\mathcal{I}^{-1})^{\Lambda\Sigma} \end{bmatrix}$$

Importantly the electric $F_{\mu\nu}{}^{\Lambda}$ and magnetic $G_{\mu\nu\,\Lambda}$ field strengths do NOT carry independent dynamics as they obey (by construction) twisted self-duality conditions

$$*G^{\mathbb{M}} = -\underbrace{\Omega^{\mathbb{M}\mathbb{N}}}_{\text{Symplectic } Sp(56)\text{-inv matrix}} M_{\mathbb{N}\mathbb{P}}(\phi)\; G^{\mathbb{P}}$$

$$\Omega_{\mathbb{M}\mathbb{N}} = \begin{pmatrix} 0 & \mathbb{I}_{28} \\ -\mathbb{I}_{28} & 0 \end{pmatrix}$$

NOTE: The scalar matrix satisfies $M(\phi)\,\Omega\,M(\phi) = \Omega$

The reformulation of the vector sector in terms of $\mathcal{G}_{\mu\nu}{}^{M}$ allows to elevate the $G = E_{7(7)}$ global symmetry of the scalar sector to global symmetries of field equations and Bianchi identities. [on-shell]

More concretely

$$g \in G = E_{7(7)} \begin{cases} \phi \longrightarrow \overbrace{g \circ \phi}^{\text{action on scalars}} & \text{(non-linear action)} \\ \mathcal{G}^{M} \longrightarrow [R(g)]^{M}{}_{N}\, \mathcal{G}^{N} & \text{(linear action)} \end{cases}$$

and invariance of $d\mathcal{G} = 0$ and $*\mathcal{G} = -\Omega M \mathcal{G}$ impose (sufficient conditions)

i) $R(g) \in Sp(56) \Rightarrow R(g)^{t}\, \Omega\, R(g) = \Omega$

ii) $M(\underbrace{g \circ \phi}_{\text{non-linear action}}) = R(g)^{-t}\, \underbrace{M(\phi)}_{\text{linear action}}\, R(g)^{-1}$

These two conditions are verified by virtue of supersymmetry.

Symplectic frame : It is a choice of embedding of $R(g) \subset Sp(56)$

$\Rightarrow$ NOT UNIQUE $\Rightarrow$ Important consequences when having a "gauging"

VII. Type II reductions on T^6 with background fluxes

We will now consider reductions in the presence of fluxes and sources : $F_{(p)}$, D-branes , ...
The charges of these sources are quantised in string theory (not in supergravity) and so fluxes :
[quantum]

$$\frac{1}{(2\pi\sqrt{\alpha'})^{p-1}} \int_{\underbrace{\Sigma_p}_{p\text{-cycle within } T^6}} F_{(p)} \in \mathbb{Z} \quad \Rightarrow \quad \text{Dirac quantisation}$$

Ex : Type IIB on T^6 : $F_{(3)} = dC_{(2)} + F^{(bg)}_{(3)}$; $H_{(3)} = dB_{(2)} + H^{(bg)}_{(3)}$

* $F_{(3)}$: $\quad F^{(bg)}_{mnp} \rightsquigarrow \binom{6}{3} = 20$ indep. flux parameters
* $H_{(3)}$: $\quad \underbrace{H^{(bg)}_{mnp}}_{\text{along } T^6} \rightsquigarrow \binom{6}{3} = 20$ indep. flux parameters

$\rightsquigarrow$ Type IIB action :

$$S_{IIB} \supset \int_{\mathcal{M}_4 \times T^6} H_{(3)} \wedge F_{(3)} \wedge C_{(4)}$$

$$\Rightarrow \quad \int_{T^6} H_{(3)}^{(bg)} \wedge F_{(3)}^{(bg)} = \underbrace{N_3}_{\Downarrow}$$

Net charge of
D3-branes / O3-planes
(pos charge) (neg charge)

$\underbrace{\hphantom{xxxxxxxxxxxxxxxx}}$
"Tadpole cancellation conditions"

[Gauss law]

Upon dimensional reduction in the presence of background fluxes one obtains general gauged supergravities in 4D. The action is given by:

$$S_{\mathcal{N}=8}^{\text{gauged}} = \int d^4x \sqrt{-|g|} \Big\{ \frac{R}{2}$$

$$+ \frac{1}{96} \operatorname{Tr}\Big[\underbrace{D_\mu M \; D^\mu M^{-1}}_{\text{charged fields}}\Big] - \overbrace{V(\phi, \underbrace{\text{fluxes}}_{\text{couplings}})}^{\text{Scalar potential}}$$

$$+ \frac{1}{4} \mathcal{I}_{\Lambda\Sigma}(\phi) \; H_{\mu\nu}{}^{\Lambda} H^{\mu\nu\,\Sigma}$$

↳ non-abelian vectors

$$+ \frac{1}{4} \frac{1}{2\sqrt{-|g|}} \mathcal{R}_{\Lambda\Sigma}(\phi) \; \varepsilon^{\mu\nu\rho\sigma} H_{\mu\nu}{}^{\Lambda} H_{\rho\sigma}{}^{\Sigma}$$

$$+ \mathcal{L}_{\text{top}} + \text{fermi-terms} + \text{fermi masses} \Big\}$$

↳ topological terms

↳ to restore susy

Strategy : Use gauged supergravities as an effective 4D description of flux compactifications.

$\cong>$ $G = E_{7(7)}$ symmetry as a guiding principle !!

Gauging : Promote a subgroup $G_0 \subset G = E_{7(7)}$ from global to local (gauge)

$$D_\mu = \partial_\mu - \overset{\text{gauge coupling}}{g}\, \underbrace{A_\mu^{\mathbb{P}}}_{\substack{\text{gauge}\\\text{fields}}}\ \underbrace{\Theta_{\mathbb{P}}{}^{\alpha}}_{\substack{\text{embedding}\\\text{tensor}}}\ \underbrace{t_\alpha}_{E_{7(7)} \text{ generators } t_{\alpha=1,\dots,133}}$$

Both electric/magnetic [dyonic gaugings]

"Selector"

Ex : $D_\mu M_{\mathbb{M}\mathbb{N}} = \partial_\mu M_{\mathbb{M}\mathbb{N}} - g\, A^{\mathbb{P}} \underbrace{\Theta_{\mathbb{P}}{}^{\alpha} [t_\alpha]_{(\mathbb{M}}{}^{\mathbb{Q}}}_{X_{\mathbb{P}\mathbb{M}}{}^{\mathbb{Q}} \iff \text{"Charges"}} M_{\mathbb{N})\mathbb{Q}}$

Important : The constant embedding tensor charges $X_{\mathbb{M}\mathbb{N}}{}^{\mathbb{P}}$ encodes all the information about the 4D theory !!

* Consistency conditions on $X_{MN}{}^{P}$

i) Linear constraint ≡ Representation constraint

$\mathcal{N}=8$ susy
[tensor hierarchy] $\Rightarrow$ $X_{MN}{}^{P} \in$ 912 of $E_{7(7)}$

ii) Quadratic constraints

Closure of the
gauge group G_0 $\Rightarrow$ $\Omega^{MN} X_{MP}{}^{Q} X_{NR}{}^{S} = 0$

Then i) and ii) imply

$$[X_M, X_N] = - X_{MN}{}^{P} X_P$$

↳ close gauge algebra
in the 4D theory

* Vector-tensor sector : Vectors fields $A_\mu{}^M$ span now a non-abelian gauge group $G_0 \subset G = E_{7(7)}$

$$\mathcal{G}_{\mu\nu}{}^{M} \longrightarrow H_{\mu\nu}{}^{M} = 2\partial_{[\mu} A_{\nu]}{}^{M} + g\, X_{[PQ]}{}^{M} A_\mu{}^{P} A_\nu{}^{Q}$$
$$+ g\, \frac{1}{2} \Omega^{MN} \Theta_N{}^{\alpha} B_{\mu\nu\,\alpha}$$

Auxiliary two-forms dual to scalars
=> Non-dynamical

[They also enter $\mathcal{L}_{top}$]

relevant when el. magnetic charges are present

* Scalar potential : This is probably the most distinctive feature of a gauged supergravity. It takes the form :

$$V(M, X) = \frac{g^2}{672} \Big[X_{MN}{}^{R} X_{PQ}{}^{S} M^{MP} M^{NQ} M_{RS} + 7 X_{MN}{}^{Q} X_{PQ}{}^{N} M^{MP} \Big]$$

$\approx\!>$ $\boxed{V(M, X) \text{ vs } V(M, \text{fluxes})}$

$\Downarrow$

$\boxed{\text{Embedding Tensor} \Longleftrightarrow \text{Type II fluxes}}$

$\Downarrow$

"CHARTING THE LANDSCAPE OF TYPE II FLUX COMPACTIFICATIONS"

VIII. Gauged supergravities from Type IIB fluxes

As we saw before, background fluxes $H_{mnp}^{(bg)}$, etc carry internal space-time indices $m,n=1,\dots,6$ that transform under $SL(6)$ internal diffeomorphisms.

In order to establish a neat dictionary between fluxes and components of the embedding tensor it will prove useful to perform a group-theoretical branching of $X_{MN}{}^{P} \in \mathbf{912}$ of $E_{7(7)}$:

$$E_{7(7)} \supset SL(2)\times SO(6,6) \supset SL(2)\times SL(6)$$

$$\mathbf{912} \equiv X_{MN}{}^{P} \rightarrow (\mathbf{2},\mathbf{12}) \equiv \xi_{\alpha M} \longrightarrow (\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{6'})$$

$$(\mathbf{2},\mathbf{220}) \equiv f_{\alpha MNP} \quad \left\{ \begin{array}{l} \underbrace{(\mathbf{2},\mathbf{20})}_{(H_{mnp},\,F_{mnp})} + (\mathbf{2},\mathbf{6}+\mathbf{84}) \\ + (\mathbf{2},\mathbf{20}) + (\mathbf{2},\mathbf{6'}+\mathbf{84'}) \end{array}\right.$$

$$(\mathbf{3},\mathbf{32}) \equiv \Xi_{\alpha\beta\mu} \rightarrow \underbrace{(\mathbf{3},\mathbf{6'})}_{F_m} + (\mathbf{3},\mathbf{20}) + (\mathbf{3},\mathbf{6})$$

$$(\mathbf{1},\mathbf{352'}) \equiv F_{M\dot{\mu}} \quad \left\{ \begin{array}{l} \underbrace{(\mathbf{1},\mathbf{6})}_{F_{mnpqr}} + \underbrace{(\mathbf{1},\mathbf{6'}+\mathbf{84'})}_{\omega_{mn}{}^{p}} \\ + (\mathbf{1},\mathbf{70}+\mathbf{20}+\mathbf{70'}) \\ + \underbrace{(\mathbf{1},\mathbf{6})}_{H_m} + (\mathbf{1},\mathbf{6}+\mathbf{84}) \end{array}\right.$$

NOTE : $\alpha=1,2$ of $SL(2)$
$M=1,\dots,12$ of $SO(6,6)$
$\mu,\dot{\mu}=1,\dots,32$ of $SO(6,6)$ [M-W]

In this manner we encounter the following Type IIB background fluxes :

- $(F_{(3)}, H_{(3)}) \in (2, 20)$
- $F_{(1)} = d\,C_{(0)} \in (3, 6')$
- $H_{(1)} = d\,\Phi \in (1, 6)$
- $F_{(5)} \in (1, 6)$

"Gauge background fluxes"

- $\omega_{mn}{}^{p} \in (1, 84')$ $\Longrightarrow$ " Metric fluxes "

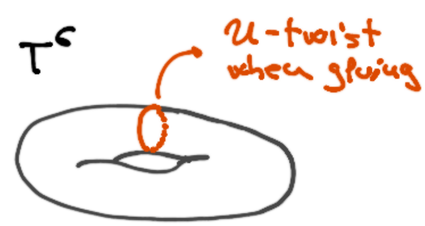
as components of the embedding tensor $X_{MN}{}^{P} \in 912$ of $E_{7(7)}$.

$*$ Metric fluxes and twisted tori : Introduce a twist on the T^6 one-form basis

$$[\mathcal{U}(y)]^{m}{}_{n} \in G_T$$

so that

$$e^{m} = [\mathcal{U}(y)]^{m}{}_{n}\, dy^{n}$$



and

$$ds_6^2 = \delta_{mn}\, e^m\, e^n$$

The twist is based on a twist group G_T with algebra

structure constants of $\mathfrak{g}_T$

$$[E_m, E_n] = \underbrace{\omega_{mn}{}^p}\, E_p$$

The left-invariant one-forms $e^p(y)$ satisfy the "Maurer-Cartan" equation

vanishing curvature

$$de^p + \frac{1}{2}\,\omega_{mn}{}^p\, e^m \wedge e^n = 0$$

with $\omega_{mn}{}^p \equiv$ torsion on the original torus given by

$$\omega_{mn}{}^p = [\mathcal{U}^{-1}]_m{}^{m'}\, [\mathcal{U}^{-1}]_n{}^{n'} \left(\partial_{m'} [\mathcal{U}]^p{}_{n'} - \partial_{n'} [\mathcal{U}]^p{}_{m'}\right)$$

Introducing a twisted exterior derivative $D \equiv d + \omega$ and demanding $D^2 = 0$ one gets

$$\omega_{[mn}{}^p\, \omega_{r]s}{}^q = 0 \quad \Rightarrow$$ Jacobi identity the algebra $\mathfrak{g}_T$

$\Downarrow$

Quadratic constraints on the metric fluxes

NOTE: Twisted torus $\equiv$ Group manifold (locally)

* **Quadratic constraints and sources** : Let us start from the Quadratic Constraints (QC) on the embedding tensor $X_{MN}{}^{P}$ with non-zero components

$$\underbrace{H_{mnp}\ ,\ F_{mnp}\ ,\ \omega_{mn}{}^{p}}_{\text{Background fluxes}} \subset \underbrace{X_{MN}{}^{P}}_{\text{Embedding tensor}}$$

- $QC_{\mathcal{N}=8}$ in 4D :

$$\Omega^{MN}\, X_{MP}{}^{Q}\, X_{NR}{}^{S} = 0 \quad \Rightarrow$$

i) $H_{[mnp}\, F_{qrs]} = 0$

ii) $\omega_{[mn}{}^{p}\, \omega_{q]p}{}^{r} = 0$

iii) $\omega_{[mn}{}^{p}\, H_{qr]p} = 0$

iv) $\omega_{[mn}{}^{p}\, F_{qr]p} = 0$

- Sources in 10D : [Twisted derivative $D \equiv d + \omega$]

[$N_3 = N_{O3} - N_{D3}$]

i) $H_3 \wedge F_3 = N_3 = 0 \Rightarrow$ Absence of D3/O3 sources

ii) $D^2 = 0 \Rightarrow$ Absence of KK5-branes

iii) $DH_{(3)} = 0 \Rightarrow$ Absence of NS5-branes

iv) $\underbrace{\omega\, F_{(3)}} = 0 \Rightarrow$ Absence of D5-branes

4-form $\wedge\, C_{(6)} \Rightarrow N_5 = 0$

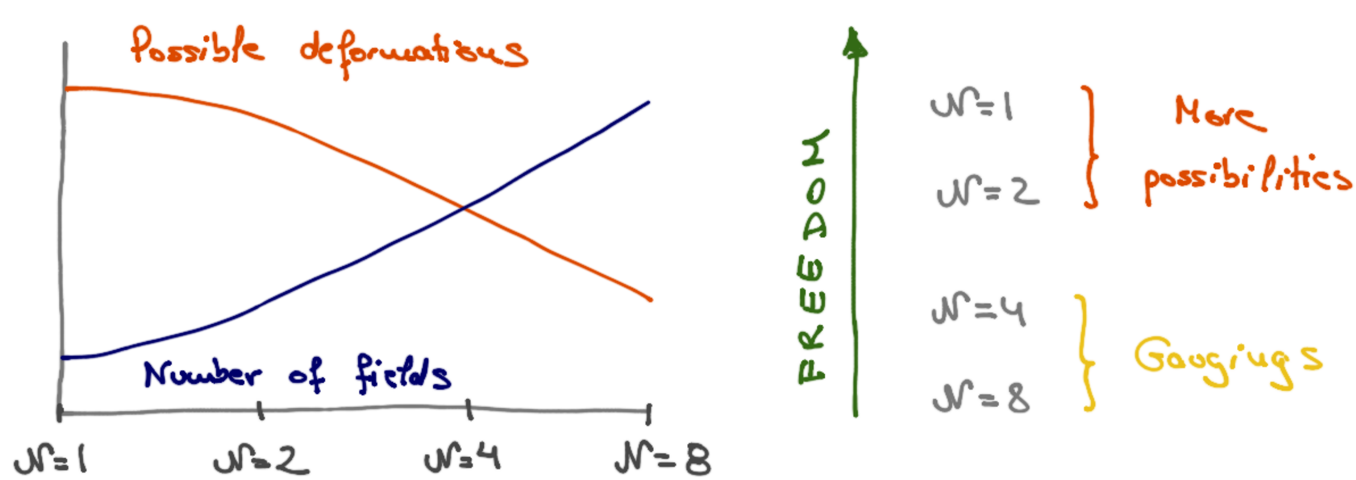
Message : No net charge is allowed for any type of sources in order to preserve $\mathcal{N}=8$ supersymmetry in the 4D compactified theory.

If we start adding sources in 10D we will break (some) supersymmetries

$QC_{\mathcal{N}=1}$	$\subset$	$QC_{\mathcal{N}=2}$	$\subset$	$QC_{\mathcal{N}=4}$	$\subset$	$QC_{\mathcal{N}=8}$
[String Pheno]		[Black holes]		[DFT]		[AdS/CFT]

- Moduli stabilisation
- String Cosmology

$\approx>$



IX. Type IIB moduli stabilisation

Type IIB dimensional reduction from 10D to 4D produces a large set of scalar fields $\phi^{i=1,\dots,70}$ spanning the coset space

$$M(\phi) \in \frac{E_{7(7)}}{SU(8)} \quad \Rightarrow \quad \underbrace{133}_{E_{7(7)}} - \underbrace{63}_{SU(8)} = 70 \text{ scalars}$$

$\Downarrow$

$\boxed{\text{Moduli fields}}$

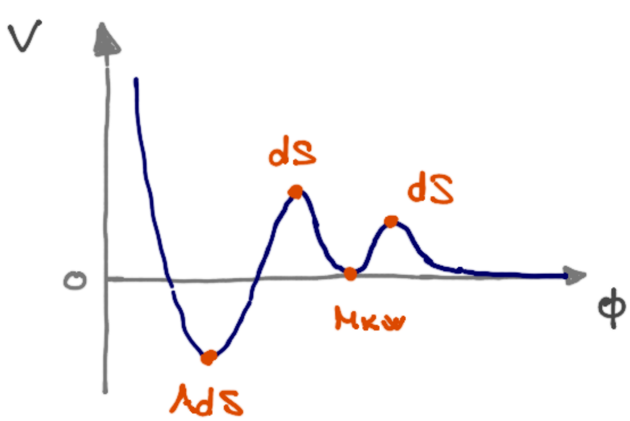
Problem: Massless scalars = Long range interactions

$\Downarrow$

(Precision tests of GR)

"Moduli problem"

Solution (?): Fluxes $\Rightarrow$ $V(\phi, \text{fluxes}) = \underbrace{m^2_{ij}}_{\text{masses = fluxes}} \phi_i \phi_j + \dots$



$\Rightarrow$ E.O.M : $\Box \phi = \dfrac{\partial V}{\partial \phi}$

$\langle \phi \rangle = \phi_0 \Rightarrow \left.\dfrac{\partial V}{\partial \phi}\right|_{\phi=\phi_0} = 0$

$\Rightarrow \boxed{\Lambda_{cc} \equiv V(\phi_0)}$

Question : Do fluxes suffice to $\overbrace{\text{stabilise}}^{m_{ij}^2>0}$ moduli in a de Sitter $\underbrace{(dS)}_{\Lambda_{cc}>0}$ [quasi Minkowski (Mkw)] vacuum ?

Technical difficulty : 70 scalars are way too many to extremise $V(\phi)$

$\Rrightarrow$ Set most of them to zero consistently by virtue of symmetry argument :

$$\underbrace{\mathbb{Z}_2^*}_{\substack{\text{Orientifold}\\\text{involution}\\ \Omega_p(-1)^{F_L}\sigma^*}} \times \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\frac{T^6}{\mathbb{Z}_2\times\mathbb{Z}_2}\ \text{orbifold}} \subset E_{7(7)} \quad \approx> \quad \underbrace{\frac{E_{7(7)}}{SU(8)}}_{70\ \text{scalars}} \supset \underbrace{\left[\frac{SL(2)}{SO(2)}\right]^7}_{14\ \text{scalars}}$$

$$T^6 = \underbrace{T^2 \times T^2 \times T^2}_{\substack{\text{exchange}\\\text{symmetry}}}$$

$$\Downarrow$$

$$\underbrace{\left[\frac{SL(2)}{SO(2)}\right]^7}_{14\ \text{scalars}} \supset \underbrace{\left[\frac{SL(2)}{SO(2)}\right]^3}_{\boxed{6\ \text{scalars}}}\ \checkmark$$

Sources : We will allow for sources preserving at least $\mathcal{N}=1$ supersymmetry.

Equivalently

$$QC_{\mathcal{N}=1} = 0 \quad \text{but} \quad QC_{\mathcal{N}>1} = N_{\text{sources}} \neq 0$$

✓

and the resulting 4D supergravity is then $\mathcal{N}=1$ supersymmetric. The potential $V(\phi, \text{fluxes})$ takes the schematic form

$$V(\phi) = V_{\text{fluxes}}(\phi) + \underbrace{V_{\text{sources}}(\phi)}_{\substack{\text{Terms proportional} \\ \text{to } QC_{\mathcal{N}>1} \neq 0}}$$

* **The $\mathcal{N}=1$ SUGRA model** : We have reduced the model to an $\mathcal{N}=1$ SUGRA coupled to 3 chiral superfields whose (complex) scalar component we denote :

- $S = \chi_S + i\, e^{-\phi_S} \in \frac{SL(2)}{SO(2)}$

- $T = \chi_T + i\, e^{-\phi_T} \in \frac{SL(2)}{SO(2)}$ $\Rightarrow$ "STU models"

  [6 (real) scalars]

- $U = \chi_U + i\, e^{-\phi_U} \in \frac{SL(2)}{SO(2)}$

The 10D type IIB origin of these scalar fields is given by

- $S = C_{(0)} + i\, e^{-\Phi}$ $\Rightarrow$ Axion-dilaton

- U $\Rightarrow$ Complex structure modulus [shape of T^6]

$$G_{mn} = \frac{\text{Im}\, T}{\text{Im}\, U} \begin{bmatrix} |U|^2 & -\text{Re}\, U \\ -\text{Re}\, U & 1 \end{bmatrix} \otimes \mathbb{I}_3$$

$\hookrightarrow T^6 = T^2 \times T^2 \times T^2$

- $$T = \frac{1}{\text{Vol}_6} \int_{T^6} \overbrace{C_{(4)}}^{\text{purely internal}} \wedge \underbrace{\omega}_{\text{2-cycle on } T^6} + i\, e^{-\Phi} A^2_{T^2}$$

  $\hookrightarrow A_{T^2} \equiv \text{Vol}_{T^2}$

$\Rightarrow$ Kähler modulus [size of T^6]

In order to generalise the results here to more general SU(3)-structure manifolds (like CY_3 manifolds), let us introduce a set of SU(3)-structure forms:

$$J \equiv \text{2-form} \in \mathbb{R} \quad , \quad \Omega \equiv \text{Holomorphic 3-form} \in \mathbb{C}$$

in terms of which

$$\Omega(U) = (e^1+Ue^2)\wedge(e^3+Ue^4)\wedge(e^5+Ue^6)$$

$$T = \frac{1}{\text{vol}_6} \int_{M_6} \underbrace{\Big(C_{(4)} + \frac{i}{2} e^{-\Phi} \overbrace{J\wedge J}^{A_{T^2}} \Big)}_{\mathcal{J} \equiv \text{complexified Kähler 4-form}} \wedge \underbrace{\omega}_{\text{2-cycle}} \quad , \quad \text{vol}_6 = \int_{M_6} \Omega \wedge \bar{\Omega}$$

As an $\mathcal{N}=1$ theory, the full Lagrangian is encoded in a Kähler potential $K(S,T,U) \in \mathbb{R}$ and a holomorphic superpotential $W(S,T,U) \in \mathbb{C}$.

The Kähler potential for this model is given by

$$K(S,T,U) = -\text{Log}\left[-i(S-\bar{S})\right] - 3\,\text{Log}\left[-i(T-\bar{T})\right] - 3\,\text{Log}\left[-i(U-\bar{U})\right]$$

The superpotential depends on the IIB fluxes F_{mnp}, H_{mnp}, etc. that are being considered. Including only gauge background fluxes (F_{mnp}, H_{mnp}) one gets

$$W(S,U) = \int_{M_6 = T^6} \left(F_{(3)} - S\, H_{(3)} \right) \wedge \Omega(U)$$

$$\hookrightarrow T^6 = T^2 \times T^2 \times T^2$$

NOTE:

$$F_{(3)} = a_0\,\beta^0 + a_1\,\beta^1 + a_2\,\alpha_1 + a_3\,\alpha_0$$
$$H_{(3)} = b_0\,\beta^0 + b_1\,\beta^1 + b_2\,\alpha_1 + b_3\,\alpha_0$$

$\underbrace{(\alpha_0, \alpha_1, \beta^1, \beta^0)}_{\text{3-cycles on } T^6}$

$$= P_F(U) - S\,P_H(U)$$

with

$$P_F(U) = a_0 - 3\,a_1\,U + 3\,a_2\,U^2 - a_3\,U^3$$
$$P_H(U) = b_0 - 3\,b_1\,U + 3\,b_2\,U^2 - b_3\,U^3$$

The $\mathcal{N}=1$ scalar potential for $\Phi^i = \{S, T, U\}$

$$V = e^K \left[K^{i\bar{J}}\, D_i W\, D_{\bar{J}} \overline{W} - 3\,W\,\overline{W} \right]$$

with $K_{i\bar{J}} = \dfrac{\partial K}{\partial \Phi^i\, \partial \overline{\Phi}^{\bar{J}}}$ and $D_i W = \dfrac{\partial W}{\partial \Phi^i} + \dfrac{\partial K}{\partial \Phi^i}\, W$ being the "Kähler derivative".

Important: This scalar potential V suffices to stabilise (S, U) by solving $D_S W = 0$, $D_U W = 0$

... but leaves T unstabilised !!

$\Rightarrow$ How to stabilise T ??

Important : Stabilising (S,U) requires background fluxes for which

$$H_{(3)} \wedge F_{(3)} = N_3 = \underbrace{N_{O3}}_{} - N_{D3} > 0$$

O3-planes must be present !!

$\approx\!> \quad QC_{\mathcal{N}=4} = 0 \quad \subset \quad QC_{\mathcal{N}=8} \neq 0$

✓ ✗

$\approx\!>$ SUSY reduced to $\mathcal{N}=4$ due to the presence of sources.

X. Stabilising the Kähler modulus T

We will present now two possible mechanisms to stabilise the Kähler modulus

a) Non-perturbative effects : They introduce exponentials in the superpotential

$$W(S,T,U) = W_{fluxes}(S,U) + W_{np}(T)$$

with

$$W_{np} = A\, e^{-aT}$$

and $(A, a) \equiv$ Model dependent quantities

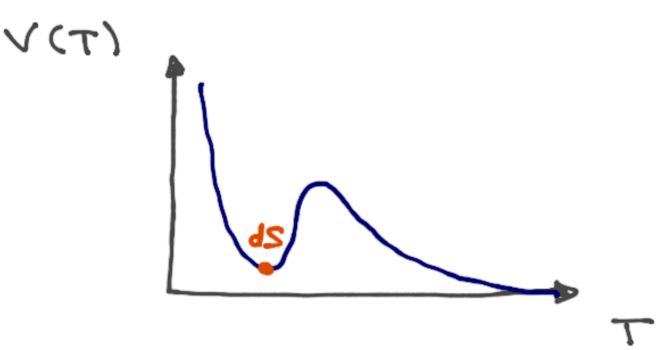
- Gaugino condensation on D7-branes
- D-brane instantons

! (anti D3)

* Application : dS vacua
  - Two-step procedure [KKLT]
  - Single-step procedure

$A = A(S,U)$; $A = A(M)$

squarks condensates $\equiv$ open string sector



b) Non-geometric fluxes : Conjectured on the basis of $E_{7(7)}$ covariance (stringy dualities)

Recalling the embedding tensor group theoretical decompositions

$$E_{7(7)} \supset SL(2) \times SO(6,6) \supset SL(2) \times SL(6)$$

$$912 \equiv X_{MN}{}^{P} \rightarrow (2,220) \equiv f_{\alpha MNP} \left\{ \begin{array}{l} \underbrace{(2,20)}_{(F_{mnp},\, H_{mnp})} + \underbrace{(2,6+84)}_{(Q^{mn}{}_{p},\, P^{mn}{}_{p})} \\ + (2,20) + (2,6'+84') \end{array} \right.$$

with $(Q^{mn}{}_{p}\,,\, P^{mn}{}_{p}) \equiv$ " Non-geometric fluxes "

$D^2 = 0$ with $D \equiv d + Q\cdot + P\cdot$

$$\approx> \quad QC_{\mathcal{N}=8} \text{ in } D=4 \quad => \left\{ \begin{array}{l} \cdot\; Q\,Q = P\,P = Q\,P + P\,Q = 0 \\ \cdot\; \underbrace{Q\cdot F_{(3)}}_{2\text{-form}} = P\cdot H_{(3)} = Q\cdot H_{(3)} + P\cdot F_{(3)} = 0 \end{array} \right.$$

2-form $\wedge\, C_{(8)} \Rightarrow N_7 = 0$

The superpotential is given by

specific index contraction

$$W(S,U,T) = \int_{M_6 = T^6} \Big[\big(F_{(3)} - S\, H_{(3)}\big) + \big(Q - S\, P\big)\cdot \mathcal{J}(T) \Big] \wedge \Omega(U)$$

$$= P_F(U) - S\, P_H(U) + \underbrace{\big(P_Q(U) - S\, P_P(U) \big)\, T}_{\text{New flux couplings from } E_{7(7)}\text{-covariance !!}}$$

with

$$P_Q(U) = c_0 + 3\,c_1\,U - 3\,c_2\,U^2 - c_3\,U^3$$

$$P_P(U) = d_0 + 3\,d_1\,U - 3\,d_2\,U^2 - d_3\,U^3$$

* Application : dS vacua with $N_3 \neq 0$ & $N_7 = 0$ ✓

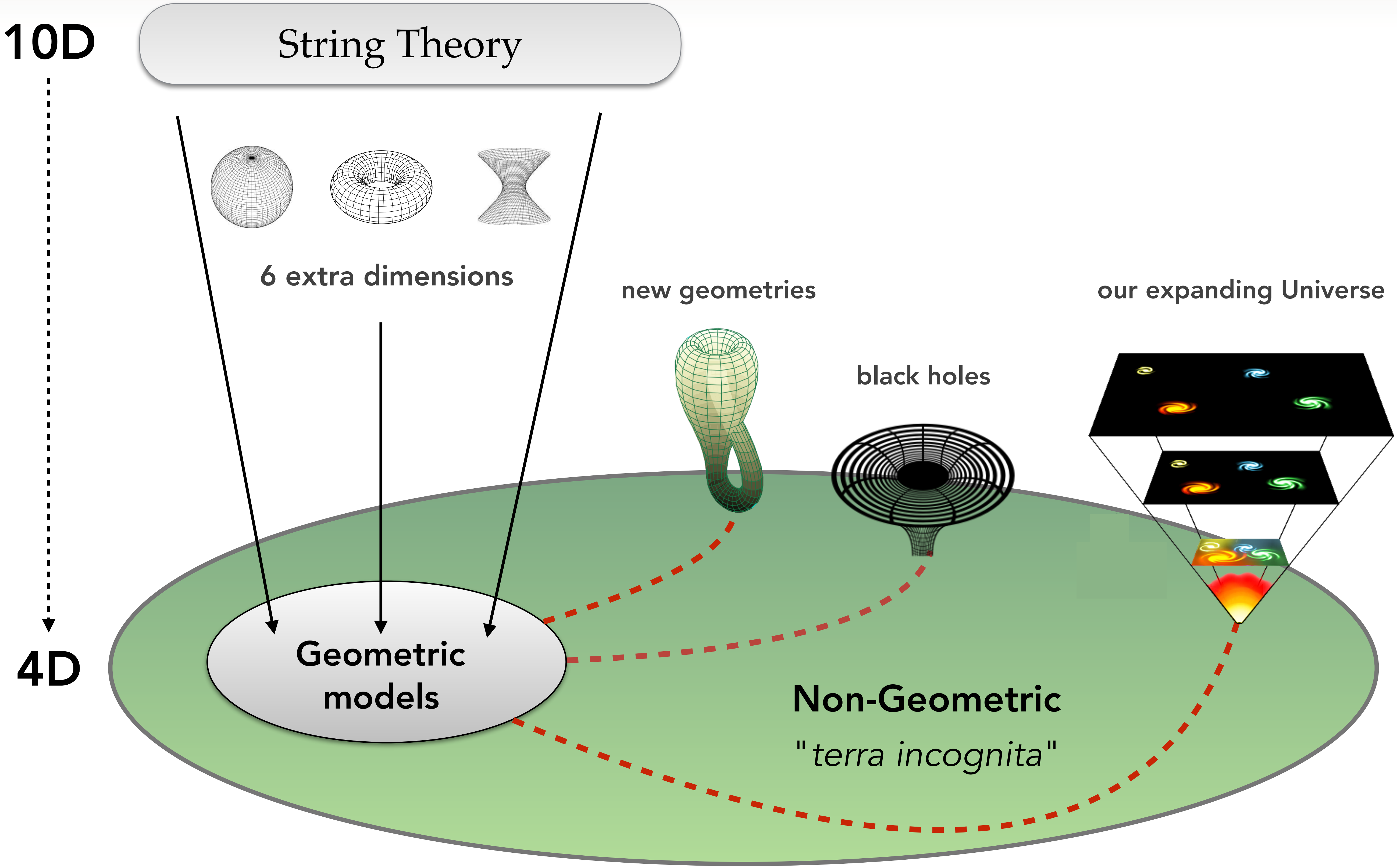
... but ... higher-dimensional origin ?

XI. Some final considerations

- Poincaré vs duality covariance
- Generalised geometries :

$$D = d + \omega + Q\cdot + \ldots$$

- DFT and EFT : Duality-covariant reformulation of Type IIB and 11D supergravities
- Phenomenological implications : moduli stabilisation, string cosmology, ...
- Holography



10D
String Theory
6 extra dimensions
new geometries
our expanding Universe
black holes
4D
Geometric models
Non-Geometric
"terra incognita"