I. The general static and isotropic metric

In this section we are discussing a broad class of metrics describing various types of gravitational systems in GR.

- static : There must be a "time" coordinate $x^0 \equiv t$ such that ds^2 is independent of it
- isotropic : There must be some "space coordinates" $\vec{x} = (x^1, x^2, x^3)$ such that ds^2 only depends on rotation invariant (isotropic) combinations $d\vec{x}^2$, $\vec{x}\, d\vec{x}$ and $\vec{x}^2$

Then

$$ds^2 = -F(r)\, dt^2 + 2\, E(r)\, dt\, (\vec{x}\, d\vec{x}) + D(r)\, (\vec{x}\, d\vec{x})^2 + C(r)\, d\vec{x}^2$$

where $F(r)$, $E(r)$, $D(r)$ and $C(r)$ are functions of $r^2 = \vec{x}^2$

Changing to spherical coordinates

$$x^1 = r \sin\theta \cos\varphi \;,\quad x^2 = r \sin\theta \sin\varphi \;,\quad x^3 = r \cos\theta$$

one gets

$$ds^2 = -F(r)\, dt^2 + \underbrace{2\, r\, E(r)\, dt\, dr}_{\text{this piece can be eliminated by a change of "time"}} + r^2 D(r)\, dr^2 + C(r) \underbrace{\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \right]}_{\text{spherical coordinates}}$$

We can change to a new "time" coordinate of the form

$$t' = t + \Phi(r) \quad \text{with} \quad \frac{d\Phi}{dr} = -r\,\frac{E(r)}{F(r)}$$

so that

$$ds^2 = -F(r)\,dt'^2 + G(r)\,dr^2 + \underbrace{C(r)}_{\text{This can also be eliminated}}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right]$$

where

$$G(r) \equiv r^2\left[D(r) + \frac{E^2(r)}{F(r)}\right]$$

We can also redefine the "radial coordinate" as

$$r'^2 = C(r)\,r^2$$

so that

$$\boxed{ds^2 = -B(r')\,dt'^2 + A(r')\,dr'^2 + r'^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]}$$

with

$$B(r') \equiv F(r)$$

$$A(r') \equiv \left(1 + \frac{G(r)}{C(r)}\right)\left(1 + \frac{r}{2C(r)}\frac{dC(r)}{dr}\right)^{-2}$$

The various components of $R_{\mu\nu}$ take the form [removing primes !!]

$$R_{tt} = \frac{B''(r)}{2A(r)} - \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left[\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right] + \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right)$$

$$R_{rr} = -\frac{B''(r)}{2B(r)} + \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left[\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right] + \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right)$$

$$R_{\theta\theta} = 1 - \frac{1}{A(r)} + \frac{r}{2A(r)}\left[\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}\right]$$

$$R_{\varphi\varphi} = \sin^2\theta \, R_{\theta\theta}$$

We can plug this $R_{\mu\nu}$ into Einstein equation and search for the $A(r)$ and $B(r)$ functions compatible with a given energy-momentum tensor $T_{\mu\nu}$.

II. The Schwarzchild black hole

It is a non-trivial solution (1.916) of the vacuum Einstein's equations $G_{\mu\nu} = 0$. The Schwarzchild metric is static and isotropic with a free parameter M (BH mass) and

$$B(r) = 1 - \frac{2MG}{r} \qquad \text{and} \qquad A(r) = \frac{1}{B(r)}$$

so that

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

* Singularities :

$\hookrightarrow r_s \equiv$ event horizon $\equiv$ "Schwarzschild radius"

- $g_{tt} = 0$: $B(r) = 0 \Rightarrow r = r_s = 2MG$

- $g_{rr} = 0$: $A(r) = 0 \Rightarrow r = 0$

<u>NOTE:</u> For the Sun : $R_\odot = 700\,000$ km and $r_s = 3$ km

<u>Important:</u> The singularity at $r = r_s$ is a coordinate singularity and can be eliminated using different coordinates.

Ex: Lemaître coordinates

$$d\tau = dt + \sqrt{\frac{r_s}{r}}\left(1 - \frac{r_s}{r}\right)^{-1} dr$$

$$d\rho = dt + \sqrt{\frac{r}{r_s}}\left(1 - \frac{r_s}{r}\right)^{-1} dr$$

$$\Rightarrow \quad ds^2 = -d\tau^2 + \frac{r_s}{r}\, d\rho^2 + r^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

with $r \equiv \left[\frac{3}{2}\underbrace{(\rho - \tau)}\right]^{\frac{2}{3}} r_s^{1/3}$

- singularity at $r = r_s$ dissapears
- singularity at $r = 0$ remains

<u>Important:</u> The computation of the invariant quantity

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{12\, r_s^2}{r^6}$$ "Kretschmann scalar"

shows that $r = 0$ is a physical singularity

An explicit computation for the Schwarzschild black hole shows that

$$R^{t}{}_{rrt} = 2\,R^{\theta}{}_{r\theta r} = 2\,R^{\varphi}{}_{r\varphi r} = \frac{r_s}{r^2\,(r_s - r)}$$

$$2\,R^{t}{}_{\theta\theta t} = 2\,R^{r}{}_{\theta\theta r} = R^{\varphi}{}_{\theta\varphi\theta} = \frac{r_s}{r}$$

$$2\,R^{t}{}_{\varphi\varphi t} = 2\,R^{r}{}_{\varphi\varphi r} = -R^{\theta}{}_{\varphi\varphi\theta} = \frac{r_s}{r}\sin^2\theta$$

$$R^{r}{}_{trt} = -2\,R^{\theta}{}_{t\theta t} = -2\,R^{\varphi}{}_{t\varphi t} = \frac{r_s\,(r_s - r)}{r^4}$$

III. The Reissner-Nordström black hole

It is a non-trivial solution (1916-1918) of the Einstein-Maxwell equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}(F)$. The Reissner-Nordström metric is static and isotropic with two free parameters (M, Q) (BH mass and charge) and

$$B(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \qquad \text{and} \qquad A(r) = \frac{1}{B(r)}$$

so that

$$ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)dt^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

with $r_s = 2GM$ and $r_Q^2 = \dfrac{Q^2 G}{4\pi\varepsilon_0}$.

This $F_{\mu\nu}$ also satisfies Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0\,, \quad \nabla_{[\rho} F_{\mu\nu]} = 0$$

The electromagnetic field is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{with} \quad A_\mu = \frac{\sqrt{2}}{\kappa}\left(\frac{r_Q}{r}, \vec{0}\right) \Rightarrow F_{tr} = -F_{rt} = \frac{\sqrt{2}}{\kappa}\,\frac{r_Q}{r^2}$$

$$\Rightarrow \quad T_\mu{}^\nu = \frac{r_Q^2}{\kappa^2 r^4}\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix} = \begin{bmatrix} -\rho & & & \\ & -p_r & & \\ & & p_\theta & \\ & & & p_\varphi \end{bmatrix}$$

* Singularities :

- $g_{tt} = 0$: $B(r) = 0 \Rightarrow r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right)$
- $g_{rr} = 0$: $A(r) = 0 \Rightarrow r = 0$

Important : The two horizons $r_\pm$ get degenerated when

$$r_s = 2r_Q \Rightarrow B(r) = \left(1 - \frac{r_Q}{r}\right)^2 \qquad \text{“Extremal BH”}$$

Important : If $2r_Q > r_s$ then there is a naked singularity at $r = 0$ $\Rightarrow$ Cosmic censorship hypothesis

$\Rightarrow$ Singularities must be hidden behind a horizon.

IV. Kerr-Newman black hole

It is a non-trivial solution (1963-1965) of the Einstein-Maxwell equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$. This metric is neither static (rotation) [though stationary] nor spherically symmetric

↱ $dt\,d\varphi$ term (rotation)

$$ds^2 = \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) - \left(dt - a\sin^2\theta\, d\varphi\right)^2 \frac{\Delta}{\rho^2}$$

“Boyer-Lindquist coordinates”

$$+ \left[(r^2 + a^2)\, d\varphi - a\, dt\right]^2 \frac{\sin^2\theta}{\rho^2}$$

↳ $dt\,d\varphi$ term (rotation)

where $J \equiv$ angular momentum

$$a = \frac{J}{M} \quad , \quad \rho^2 = r^2 + a^2 \cos^2\theta \quad , \quad \Delta = r^2 - r_s r + a^2 + r_Q^2$$

The electromagnetic field is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{with} \quad A_\mu = \frac{\sqrt{2}}{\kappa}\left(\frac{r_Q r}{\rho^2}, 0, 0, -\frac{a\, r_Q\, r \sin^2\theta}{\rho^2} \right)$$

* Singularities :

- $g_{rr} = \infty$: $r_\pm = \frac{r_s}{2} \pm \sqrt{\frac{r_s^2}{4} - a^2 - r_Q^2}$ $\equiv$ inner/outer horizons
- $g_{tt} = \infty$: $r_\pm^E = \frac{r_s}{2} \pm \sqrt{\frac{r_s^2}{4} - a^2 \cos^2\theta - r_Q^2}$ $\equiv$ inner / outer ergosphere

NOTE: Note that $r_\pm^E = r_\pm$ at $\theta = 0, \pi$

NOTE: The Kerr black hole (1.963) corresponds to the case $Q = 0$ and it is again a solution of the vacuum Einstein equation $G_{\mu\nu} = 0$

The Kerr-Newman black hole then encompasses all the types of black holes :

	Non-rotating ($a=0$)	Rotating ($a \neq 0$)
Uncharged ($Q=0$)	Schwarzchild	Kerr
Charged ($Q \neq 0$)	Reissner-Nordström	Kerr-Newman

V. Wormhole (Einstein-Rosen bridge)

It is a non-trivial solution (1935) of Einstein or Einstein-Maxwell equations $G_{\mu\nu} = 0$ or $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$. Wormholes describe space-times with two asymptotic regions.

1) Neutral bridge : Start from Schwarzschild BH and remove the region inside the horizon [r=0 singularity]

$$u^2 = r - r_s \quad \text{with} \quad r \in [r_s, \infty) \implies u \in (-\infty, \infty)$$

↳ two-folded !! ↳ interior removed !!

One obtains the WH metric

$$ds^2 = -\left(\frac{u^2}{u^2 + r_s}\right) dt^2 + 4(u^2 + r_s)\, du^2 + (u^2 + r_s)^2 \overbrace{\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]}^{d\Omega}$$

Important : $g_{tt} = 0$ at $u = 0$ ⟹ Singularity

This geometry describes two sheets ($u>0$ & $u<0$) connected by a "bridge".

We can visualise this by embedding a slice of the WH in Euclidean $\mathbb{R}^3$:

$$ds^2_{\mathbb{R}^3} = dx^2 + dy^2 + dz^2 = \underbrace{dr^2 + r^2 d\varphi^2 + dz^2}_{\text{cylindrical coordinates}}$$

$$ds^2_{WH} = f(r)\, dr^2 + r^2\, d\varphi^2$$

at $t = \text{fixed}$, $\theta = \frac{\pi}{2}$

Since the WH is spherically symmetric, one has that

$$z = z(r) \Rightarrow \quad ds^2_{\mathbb{R}^3} = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2\, d\varphi^2$$

$$ds^2_{WH} = f(r)\, dr^2 + r^2\, d\varphi^2$$

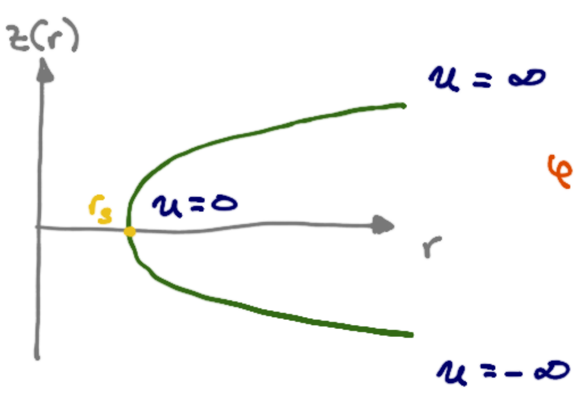
$$\Rightarrow \frac{dz}{dr} = \pm \sqrt{f(r) - 1}$$

$$\Rightarrow z(r) = \pm \int dr \sqrt{f(r) - 1}$$

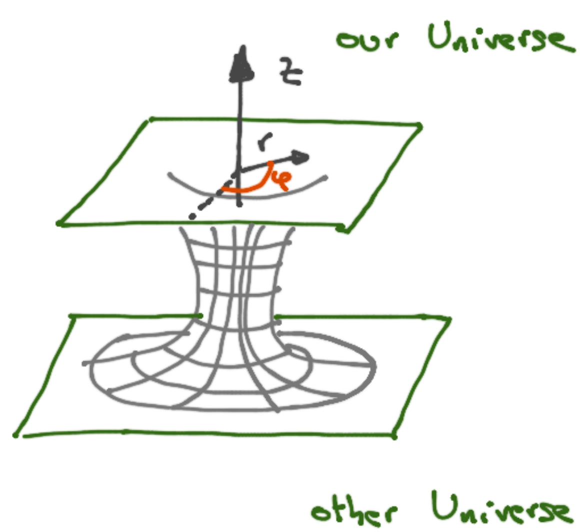
Example: $f(r) = \left(1 - \frac{r_s}{r}\right)^{-1}$ (Schwarzchild)

$$\Rightarrow z(r) = \pm \int dr \sqrt{\left(1 - \frac{r_s}{r}\right)^{-1} - 1}$$

$$= \pm\, 2 r_s \sqrt{\frac{r - r_s}{r_s}}$$



φ rotating $\Rightarrow$



Important : It is a vacuum solution : $G_{\mu\nu} = 0$

2) Charged bridge : Start from Reissner-Nordström BH set the mass $M = 0$ ($r_s = 0$) and replace $r_Q^2 \to -r_Q^2$

$$ds^2_{BH} = -\left(1 - \frac{r_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{r_Q^2}{r^2}\right) dr^2 + r^2 d\Omega$$

Removing the interior [r=0 singularity]

$$u^2 = r^2 - r_Q^2 \quad \text{with} \quad r \in [r_Q, \infty) \quad \Rightarrow \quad u \in (-\infty, \infty)$$

↳ two-folded !! ↳ interior removed !!

One obtains the WH metric

$$ds^2 = -\left(\frac{u^2}{u^2 + r_Q^2}\right) dt^2 + du^2 + (u^2 + r_Q^2)\overbrace{\left[d\theta^2 + \sin^2\theta \, d\varphi^2\right]}^{d\Omega}$$

Important : $g_{tt} = 0$ at $u = 0$ $\Rightarrow$ Singularity

Computing the Einstein tensor yields

$$G^\mu{}_\nu = \frac{r_Q^2}{(r_Q^2 + u^2)^2} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} = \kappa^2 \begin{bmatrix} -\rho & & & \\ & -p_u & & \\ & & p_\theta & \\ & & & p_\varphi \end{bmatrix}$$

$\Rightarrow \rho < 0$ "Exotic matter"

VI. Friedmann - Lemaître - Robertson - Walker metric

This metric describes a homogeneous and isotropic spatial geometry that is expanding or contracting. This metric is the one used for cosmological purposes.

$$ds^2 = -dt^2 + a^2(t) \underbrace{\left[d\vec{x}^2 + K \frac{(\vec{x}\cdot d\vec{x})^2}{1-K\vec{x}^2} \right]}_{\text{spacial sections}}$$

with

- $a(t) \equiv$ scale factor
- $K = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperbolic} \\ 0 & \text{flat} \end{cases}$

Theorem: This metric is unique (up to a g.c.t)

Changing to spherical coordinates so that

$$d\vec{x}^2 = dr^2 + r^2 \left[\underbrace{d\theta^2 + \sin^2\theta\, d\varphi^2}_{d\Omega_2} \right] = dr^2 + r^2 d\Omega_2$$

one arrives at

$$ds^2 = \overbrace{-dt^2}^{g_{tt}=-1} + \underbrace{a^2(t)}_{\text{scale factor}} \overbrace{\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega_2 \right]}^{\tilde{g}_{ij}(\vec{x}) \equiv \text{spatial metric}}$$

scale factor $\Rightarrow g_{ij} = a^2(t)\, \tilde{g}_{ij}(\vec{x})$

* **Proper distance** : At a given time t^*, the proper distance from the origin at $r=0$ to a point at location r is given by

$$d(t^*, r) \equiv \int ds = \int_0^r \sqrt{g_{rr}}\, dr = \int_0^r a(t^*) \frac{dr}{\sqrt{1-Kr^2}}$$

$$= \begin{cases} \underbrace{a(t^*)}_{} \sin^{-1}(r) & \text{if} \quad K=+1 \\ a(t^*) \sinh^{-1}(r) & \text{if} \quad K=-1 \\ a(t^*)\, r & \text{if} \quad K=0 \end{cases}$$

distances are proportional to $a(t^*)$
[scale factor]

* **Perfect fluid in FLRW metric** : Let us consider a perfect fluid in the inertial frame $[u^a = (1, \vec{0})]$

$$T^{\mu\nu} = P g^{\mu\nu} + (P+\rho) u^\mu u^\nu =$$

$$= \begin{bmatrix} \rho & \\ & P g^{ij} \end{bmatrix} = \begin{bmatrix} \rho & \\ & P a^{-2}(t)\, \tilde{g}^{ij}(\vec{x}) \end{bmatrix}$$

Since $T^{\mu\nu}$ is conserved :

- $\nabla_\nu T^{i\nu} = 0$ [automatically satisfied]
- $\nabla_\nu T^{0\nu} = \dfrac{d\rho}{dt} + 3\dfrac{\dot{a}}{a}(P+\rho) = 0$

Assuming an equation of state of the form

$$f(P, \rho) = 0 \quad : \quad P = \underbrace{w}_{\text{cte}} \rho$$

then

$$\dot{\rho} + 3\frac{\dot{a}}{a}(P + \rho) = \dot{\rho} + 3\frac{\dot{a}}{a}\rho(w+1) = 0$$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = -3(w+1)\frac{\dot{a}}{a}$$

$$\Rightarrow \frac{d}{dt}(\ln \rho) = -3(w+1)\frac{d}{dt}(\ln a)$$

$$\Rightarrow \frac{d}{dt}(\ln \rho) = \frac{d}{dt}\ln\left(a^{-3(w+1)}\right)$$

$$\Rightarrow \rho \propto a^{-3(w+1)}$$

- Cold matter (e.g. dust) : $P = 0$ $(w = 0)$ $\Rightarrow \rho \propto a^{-3}$
- Hot matter (e.g. radiation) : $P = \frac{1}{3}\rho$ $(w = \frac{1}{3})$ $\Rightarrow \rho \propto a^{-4}$
- Vacuum energy (e.g. cosmological constant) : $P = -\rho$ $(w = -1)$
  $\Rightarrow \rho \propto 1 \equiv$ indep of $a(t)$

* Einstein's equations : From the FLRW metric one can derive the relevant (geometrical) quantities

- $R_{tt} = -3\dfrac{\ddot{a}}{a}$

- $R_{ti} = 0$

- $R_{ij} = \tilde{R}_{ij} + (a\ddot{a} + 2\dot{a}^2)\,\tilde{g}_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2K)\,\tilde{g}_{ij}$

  ↳ spatial Ricci tensor

$$\tilde{R}_{ij} = 2K\,\tilde{g}_{ij}$$

Using the energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid one gets the two equations : $R_{\mu\nu} = \kappa^2\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]$

(tt)-comp : $-3\ddot{a} = \dfrac{\kappa^2}{2}(\rho + 3P)\,a$

(ij)-comp : $a\ddot{a} + 2\dot{a}^2 + 2K = \dfrac{\kappa^2}{2}(\rho - P)\,a^2$

Solving for $\ddot{a}$ in the (tt)-component and substituting into the (ij)-component one gets

$$\ddot{a} = -\frac{\kappa^2}{6}(\rho + 3P)\,a$$

$$\Rightarrow -a^2\frac{\kappa^2}{6}(\rho + 3P) + 2\dot{a}^2 + 2K = \frac{\kappa^2}{2}(\rho - P)\,a^2$$

$$\Rightarrow a^2\left[-\frac{\kappa^2}{6}\rho - \frac{\kappa^2}{2}P + \frac{\kappa^2}{2}P - \frac{\kappa^2}{2}\rho\right] + 2\dot{a}^2 + 2K = 0$$

$$\Rightarrow -\frac{1}{3}\kappa^2\rho a^2 + \dot{a}^2 + K = 0 \Rightarrow \underbrace{\dot{a}^2 + K = \frac{\kappa^2}{3}\rho a^2}_{\text{First-order equation for } a(t)}$$

* Friedmann equations : We have arrived at a set of equations

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\frac{\kappa^2}{2}(\rho + 3P)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho - \frac{K}{a^2}$$

$\Longrightarrow$ Friedmann equations in Cosmology

It is customary in Cosmology to define the following quantities :

- $H \equiv \dfrac{\dot{a}}{a}$ "Hubble parameter"

- $\Omega \equiv \underbrace{\dfrac{\kappa^2}{3H^2}}_{=\frac{1}{\rho_{crit}}}\rho \equiv \dfrac{\rho}{\rho_{crit}}$

And then the second Friedmann equation becomes

$$H^2 = H^2\Omega - \frac{K}{a^2} \implies \Omega - 1 = \frac{K}{H^2 a^2}$$

Important : The sign of K is determined by

- $\rho < \rho_{crit} \implies \Omega < 1 \implies K = -1$ (open)
- $\rho = \rho_{crit} \implies \Omega = 1 \implies K = 0$ (flat)
- $\rho > \rho_{crit} \implies \Omega > 1 \implies K = +1$ (closed)

VII. Maximally symmetric spaces

Maximally symmetric spaces are homogeneous and isotropic and then have the same number of isometries as Minkowski space-time

- Homogeneous : Invariance under translations
- Isotropic : Invariant under Lorentz $SO(1,3)$ rotations

$$\Rightarrow \quad \#\text{ isometries} = D + \frac{D(D-1)}{2} = \frac{D(D+1)}{2}$$

$$\Rightarrow \quad \frac{D(D+1)}{2} \text{ linearly independent Killing vectors}$$

* Riemann tensor : $R_{\rho\sigma\mu\nu} = \dfrac{\overbrace{R}^{R \,\equiv\, \text{cte}}}{D(D-1)} \left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \right)$

* Ricci tensor : $R_{\mu\nu} = \dfrac{R}{D} g_{\mu\nu}$

* Classification : Maximally symmetric spaces are then classified in terms of the signature (Euclidean vs Lorentzian), their constant Ricci scalar R and discrete information related to the global topology. Ignoring questions about the global topology one has that :

- Euclidean signature $\begin{cases} \mathbb{H}^D & \text{if } R<0 & \text{(Hyperboloid)} \\ \mathbb{R}^D & \text{if } R=0 & \text{(Flat)} \\ \mathbb{S}^D & \text{if } R>0 & \text{(Sphere)} \end{cases}$

- Lorentzian signature $\begin{cases} AdS_D & \text{if } R<0 & \text{(Anti de Sitter)} \\ Mkw_D & \text{if } R=0 & \text{(Minkowski)} \\ dS_D & \text{if } R>0 & \text{(de Sitter)} \end{cases}$

* Einstein equations : Substituting $R_{\mu\nu} = \frac{R}{D} g_{\mu\nu}$ into the Einstein equations one finds

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = R \underbrace{\left(\frac{1}{D} - \frac{1}{2}\right)}_{-\frac{D-2}{2D}} g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

$$\Rightarrow \quad g_{\mu\nu} = -\frac{2D}{D-2} \frac{\kappa^2}{R} T_{\mu\nu}$$

Important : If $D=4$ with a Lorentzian metric one has

$$|g| < 0 \quad \Rightarrow \quad \begin{cases} T_{00} < 0 \text{ and } T_{ii} > 0 \\ T_{00} > 0 \text{ and } T_{ii} < 0 \end{cases} \qquad (i = 1,2,3)$$

$$\Rightarrow \quad \text{sign}(\rho) \neq \text{sign}(P) \qquad \text{"exotic matter/energy"}$$

[eg. $P = -\rho$]

Let us consider the case of vacuum energy with $P = -\rho$ so that

$$T^{\mu\nu} \equiv -\frac{\Lambda}{\kappa^2} g^{\mu\nu} \Rightarrow g_{\mu\nu} = \underbrace{-4}_{-\frac{2D}{D-2}\big|_{D=4}} \frac{\kappa^2}{R} T_{\mu\nu} = \frac{4}{R} \Lambda g_{\mu\nu}$$

$$\Rightarrow \boxed{\Lambda = \frac{R}{4}}$$

As a result we have three possible cases :

- $\Lambda = 0 \iff R = 0 \Rightarrow ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ [Minkowski]

- $\Lambda > 0 \iff R > 0 \Rightarrow ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\Omega_2$

  [de Sitter]

  with

  "Static coordinates"

  $$B(r) = \frac{1}{A(r)} = 1 - \frac{r^2}{L^2}$$

  $\hookrightarrow$ de Sitter radius

  and

  $$\Lambda = \frac{3}{L^2}$$

  $\Rightarrow$ Horizon at $r = L$!!

- $\Lambda < 0 \iff R < 0 \Rightarrow ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\Omega_2$

  [anti de Sitter]

  with

  $$B(r) = \frac{1}{A(r)} = 1 + \frac{r^2}{L^2}$$

  "Global coordinates"

  and

  $$\Lambda = -\frac{3}{L^2}$$

  $\Rightarrow$ No horizon !!

* Embedding into (D+1) : Maximally symmetric dS_D and AdS_D in D dimensions can be embedded as hypersurfaces in (D+1) dimensions

- de Sitter dS_D : $ds^2 = -dx_0^2 + \sum_{i=1}^{D} dx_i^2 \quad \langle \equiv \rangle \quad O(1,D)$

  subject to the hyperboloid submanifold

  $$-x_0^2 + \sum_{i=1}^{D} x_i^2 = L^2$$

  → Static coordinates

  $$x_0 = \sqrt{L^2 - r^2}\, \sinh\left(\frac{t}{L}\right)$$

  $$x_1 = \sqrt{L^2 - r^2}\, \cosh\left(\frac{t}{L}\right)$$

  $$x_i = r\, y_i \quad (i = 2, \dots, D) \rightarrow \sum_{i=2}^{D} y_i^2 = 1 \Rightarrow S^{D-2} \text{ with } d\Omega_{D-2}$$

  → Flat slicing (Relevant in Cosmology)

  $$\left.\begin{aligned} x_0 &= L \sinh\left(\frac{t}{L}\right) + \frac{r^2}{2L} e^{\frac{t}{L}} \\ x_1 &= L \cosh\left(\frac{t}{L}\right) - \frac{r^2}{2L} e^{\frac{t}{L}} \end{aligned}\right\} \quad r^2 \equiv \sum_{i=2}^{D} z_i^2$$

  $$x_i = e^{\frac{t}{L}} z_i \quad (i = 2, \dots, D)$$

  $$\Rightarrow \quad ds^2 = -dt^2 + \underbrace{e^{\frac{2t}{L}}}_{\text{scale factor}} \underbrace{\sum_{i=2}^{D} dz_i^2}_{\text{Flat (D-1)-dim space}}$$

  FLRW $\Rightarrow$ $a(t) = e^{\frac{2t}{L}}$

- Anti de Sitter AdS_D : $$ds^2 = -dx_0^2 - dx_1^2 + \sum_{i=2}^{D} dx_i^2 \Leftrightarrow O(2, D-1)$$

subject to the hyperboloid submanifold

$$-x_0^2 - x_1^2 + \sum_{i=2}^{D} x_i^2 = -L^2$$

→ Global coordinates

$$x_0 = L \cosh\rho \cos\tau$$

$$x_1 = L \cosh\rho \sin\tau$$

$$x_i = L \sinh\rho \, y_i \quad (i = 2, \ldots, D) \rightarrow \sum_{i=2}^{D} y_i^2 = 1 \Rightarrow S^{D-2} \text{ with } d\Omega_{D-2}$$

$$ds^2 = L^2 \left[-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega_{D-2} \right]$$

Changing coordinates to $r = L \sinh\rho$ and $t = L\tau$

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 \, d\Omega_{D-2}$$

→ Poincaré coordinates (Relevant in Holography)

$$x_0 = \frac{L^2}{2r} \left[1 + \frac{r^2}{L^4} \left(L^2 + \vec{z}^2 - t^2 \right) \right]$$

$$x_1 = \frac{r}{L} t$$

$$x_i = \frac{r}{L} z_i \quad (i = 2, \ldots, D-1)$$

$$x_D = \frac{L^2}{2r} \left[1 - \frac{r^2}{L^4} \left(L^2 - \vec{z}^2 + t^2 \right) \right]$$

$$ds^2 = -\frac{r^2}{L^2}\,dt^2 + \frac{L^2}{r^2}\,dr^2 + \frac{r^2}{L^2}\,d\vec{z}^{\,2}$$

Changing coordinates to $y = \frac{L^2}{r}$

$$ds^2 = \frac{L^2}{y^2}\left[dy^2 + \underbrace{\left(-dt^2 + d\vec{z}^{\,2}\right)}_{\text{Minkowski} \,\equiv\, \text{MKW}_{D-1} \text{ sections}}\right]$$

static and isotropic

NOTE: Asymptotically AdS_4 charged black holes are important objects within the context of string theory. They are solutions of Einstein - Maxwell - Λ equations of the form

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,d\Omega_2$$

with

$$B(r) = \frac{1}{A(r)} = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} + \frac{r^2}{L^2} \underset{r\to\infty}{\sim} 1 + \frac{r^2}{L^2}$$

so that at $r \to \infty$ they asymptote to AdS_4.