I. Postulates and Lorentz transformations

Principle of special relativity : The laws of Nature are invariant under a particular group of space-time transformations. These are the Lorentz transformations.

Denoting coordinates as $x^a = (\underbrace{x^0}_{ct}, \underbrace{x^1, x^2, x^3}_{\vec{x}})$, the Lorentz group has an action

$$x'^a = \underbrace{\Lambda^a{}_b}_{\text{cte}} \, x^b$$

with

$$\Lambda^a{}_c \, \Lambda^b{}_d \, \eta_{ab} = \eta_{cd} \iff \Lambda^t \eta \Lambda = \eta$$

$$\Downarrow$$

for the Minkowski metric

$$\eta_{ab} = \eta_{ba} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$\Lambda \in O(1,3)$

"Lie group"

* Line element (proper time) : Lorentz transformations leave invariant the element line

$ds^2 = \underbrace{-d\tau^2}$ ≡ "proper time"

$$ds^2 = \eta_{ab} \, dx^a \, dx^b = -c^2 dt^2 + d\vec{x}^2$$

Proof : $ds'^2 = \eta_{ab} \, dx'^a \, dx'^b = \underbrace{\eta_{ab} \, \Lambda^a{}_c \, \Lambda^b{}_d}_{\eta_{cd}} \, dx^c \, dx^d$

$$= \eta_{cd} \, dx^c dx^d = ds^2$$

Important: Light propagation $\Leftrightarrow$ $ds^2 = 0$

$\Rightarrow$ Michelson – Morley experiment: speed of light is the same in all inertial frames related by Lorentz transformations.

* Lorentz transformations are the most general coordinate transformations $x \to x'$ leaving invariant ds^2

$$ds'^2 = \eta_{ab}\, dx'^a dx'^b = \eta_{ab} \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} dx^c dx^d$$

$$\overset{!}{=} \eta_{cd}\, dx^c dx^d \quad \Rightarrow \quad \underbrace{\eta_{cd} = \eta_{ab} \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d}}_{\text{This must hold !!} \ (*)}$$

Differentiating $(*)$ with respect to x^e:

$$0 = \eta_{ab} \frac{\partial^2 x'^a}{\partial x^e \partial x^c} \frac{\partial x'^b}{\partial x^d} + \eta_{ab} \frac{\partial x'^a}{\partial x^c} \frac{\partial^2 x'^b}{\partial x^e \partial x^d}$$

free indices (c,e,d)

$c \leftrightarrow e$ $\qquad$ $d \leftrightarrow e$

Adding $+O$ and substracting $-O$

$$0 = \eta_{ab} \Bigg[\frac{\partial^2 x'^a}{\partial x^e \partial x^c} \frac{\partial x'^b}{\partial x^d} + \frac{\partial x'^a}{\partial x^c} \frac{\partial^2 x'^b}{\partial x^e \partial x^d}$$

$$+ \frac{\partial^2 x'^a}{\partial x^c \partial x^e} \frac{\partial x'^b}{\partial x^d} + \frac{\partial^2 x'^b}{\partial x^c \partial x^d} \frac{\partial x'^a}{\partial x^e}$$

$$- \frac{\partial^2 x'^a}{\partial x^d \partial x^c} \frac{\partial x'^b}{\partial x^e} - \frac{\partial^2 x'^b}{\partial x^d \partial x^e} \frac{\partial x'^a}{\partial x^c} \Bigg]$$

NOTE: ~~~ cancel because $\eta_{ab} = \eta_{ba}$

$$= 2 \frac{\partial^2 x'^a}{\partial x^e \partial x^c} \underbrace{\frac{\partial x'^b}{\partial x^d}}_{\neq 0} \underbrace{\eta_{ab}}_{\neq 0} \Rightarrow \underbrace{\frac{\partial^2 x'^a}{\partial x^e \partial x^c} = 0}_{\text{Solution:}}$$

Lorentz transformation : $\underbrace{x'^a = \underbrace{\Lambda^a{}_b}_{\text{Lorentz}} x^b + \underbrace{\sigma^a}_{\text{Translation (new origin)}}}_{\text{"Poincaré group"}}$

$\Lambda^a{}_b\,,\ \sigma^a = cte$

[6+4=10 parameters]

NOTE: If $ds^2 = 0 \Rightarrow$ 15 parameters !!
[conformal group].

* Recall that $\Lambda^t \eta \Lambda = \eta \Rightarrow |\Lambda|^2 = 1$

* Orthocronous : $\Lambda^0{}_0 \geq 1 \Rightarrow \begin{cases} |\Lambda| = +1 & : \text{Proper Lorentz} \\ |\Lambda| = -1 & : \text{Improper Lorentz} \end{cases}$
  ↳ preserve direction of time

NOTE: $|\Lambda| = +1$ is the proper case connected with the identity $\mathbb{I}$ and thus $\Lambda \in SO(1,3)$.

* Subcases of the proper Lorentz transformations :

a) Rotations : $\Lambda = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R^i{}_j & \\ 0 & & & \end{array}\right]$ with $|R| = 1$ and $R^t \mathbb{I} R = \mathbb{I}$
[Subgroup]

$\Rightarrow$ Rotations + Translations = Galileo group

- Setting $v^a = 0$:

$$\left.\begin{aligned} dx'^i &= R^i{}_j\, dx^j \\ dt' &= dt \end{aligned}\right\} \quad \frac{dx'^i}{dt'} = v_i' = R_i{}^j v_j = R_i{}^j \frac{dx^j}{dt}$$

b) Boosts :
[Not a subgroup]

$$\Lambda^a{}_b = \left[\begin{array}{c|c} \Lambda^0{}_0 & \Lambda^0{}_j \\ \hline \Lambda^i{}_0 & \Lambda^i{}_j \end{array}\right] \quad \text{with}$$

components

"Lorentz factor"

$$\Lambda^0{}_0 = \gamma \equiv \frac{1}{\sqrt{1-\frac{|\vec{v}|^2}{c^2}}} \quad , \quad \Lambda^0{}_j = \gamma \frac{v_j}{c} \quad , \quad \Lambda^i{}_0 = \gamma \frac{v_i}{c}$$

and
$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{(\gamma - 1)}{|\vec{v}|^2}$$

in terms of 3 free parameters v_i .

Important : This parameterisation of the boosts reproduces the change of frame $dx'^a = \Lambda^a{}_b\, dx^b$ from an observer O seeing a particle at rest, to an observer O' seeing the particle moving with velocity $\vec{v} = (v_x, v_y, v_z)$. Notice the $|\vec{v}| \to c$ pathology to put light at rest !!

* Meaning of the Lorentz factor ?

$$\underbrace{d\tau^2}_{\text{"proper time"}} = -ds^2 = c^2 dt^2 - d\vec{x}^2 \Rightarrow d\tau = \sqrt{c^2 dt^2 - d\vec{x}^2}$$

$$= \sqrt{c^2 dt^2 - \frac{d\vec{x}^2}{dt^2} dt^2}$$

Momentarily Inertial Frame (MIF)

$$ds^2 = -d\tau^2 + \underbrace{d\vec{x}^2}_{0} = -d\tau^2$$

$$= \sqrt{c^2 dt^2 - c^2 \frac{|\vec{v}|^2}{c^2} dt^2}$$

$$= \sqrt{1 - \frac{|\vec{v}|^2}{c^2}}\, c\, dt$$

$$= \underbrace{\frac{c}{\gamma}}_{} dt$$

$$\Rightarrow \boxed{\gamma \cdot d\tau = c \cdot dt}$$

Rescaling of the time of an observer with respect to the "proper time"

* Energy and momentum

$$p^a = m\, c \underbrace{\frac{dx^a}{d\tau}}_{} \Rightarrow \begin{cases} p^0 = m \cdot c^2 \underbrace{\frac{dt}{d\tau}}_{\gamma/c} = \gamma m c \equiv \frac{E}{c} \text{ (energy)} \\ \vec{p} = m c \frac{d\vec{x}}{d\tau} = m\gamma \frac{d\vec{x}}{dt} = m\gamma\vec{v} \end{cases}$$

$u^a = \gamma(c, \vec{v}) \Leftrightarrow u^a \equiv$ proper velocity

NOTE: If $\frac{|\vec{v}|}{c} \ll 1 \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{|\vec{v}|^2}{c^2} + \frac{3}{8}\frac{|\vec{v}|^4}{c^4} + \dots$

NOTE: $-d\tau^2 = dx^a dx_a \Rightarrow u^a u_a = -c^2$

and so:

$$E = \gamma m c^2 = \underbrace{mc^2}_{\text{Rest energy}} + \underbrace{\frac{1}{2} m |\vec{v}|^2}_{\text{Kinetic energy}} + \dots$$

$$\vec{p} = m\gamma\vec{v} = m\vec{v} + \dots$$

At lowest order one recovers the non-relativistic results!!

* Dispersion relation $E(|\vec{p}|)$: Starting from

$$E = \gamma m c^2 \quad \text{and} \quad \vec{p} = m \gamma \vec{v}$$

one finds that

$$E^2 = \gamma^2 m^2 c^4 = m^2 c^4 \left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{-1} \Rightarrow \frac{|\vec{v}|^2}{c^2} = 1 - \frac{m^2 c^4}{E^2}$$

and then

$$|\vec{p}|^2 = m^2 \gamma^2 |\vec{v}|^2 = \underbrace{\frac{E^2}{c^4}}_{} \left(1 - \frac{m^2 c^4}{E^2}\right) c^2 = \frac{E^2}{c^2} - m^2 c^2$$

$\frac{E^2}{m^2c^4}$

$$\Rightarrow E^2 = m^2 c^4 + |\vec{p}|^2 c^2$$

$$\Rightarrow E(\vec{p}) = c\sqrt{m^2 c^2 + |\vec{p}|^2}$$

"De Broglie dispersion relation"

* Energy and momentum ($m = 0$ and $|\vec{v}| = c$) : This is the special case of "photons". In this case :

$$\frac{|\vec{v}|}{c} = 1 \Rightarrow \gamma = \infty \quad \text{and} \quad m = 0$$

so

$$\vec{p} = m \cdot \gamma \cdot \vec{v} \quad \text{(indeterminate)}$$

$$E = \gamma m c^2 \quad \text{(indeterminate)}$$

but the dispersion relation is still valid $\Rightarrow E = c|\vec{p}|$

II. Vectors and tensors

Objets in the theory of special relativity transform covariantly under Lorentz transformations.

* Coordinates and partial derivatives :

$$x'^a = \Lambda^a{}_b \, x^b \quad \Longleftrightarrow \quad \frac{\partial}{\partial x'^a} = \underbrace{\frac{\partial x^b}{\partial x'^a}}_{\Lambda_a{}^b} \frac{\partial}{\partial x^b} = \Lambda_a{}^b \frac{\partial}{\partial x^b}$$

NOTE: Recall that $\Lambda^a{}_b = \dfrac{\partial x'^a}{\partial x^b}$

so that

$$\Lambda_a{}^b = \eta_{ac} \, \eta^{bd} \, \Lambda^c{}_d$$

with $\eta_{ab} = \eta^{ab}$ and $\eta_{ab} \, \eta^{ac} = \delta^c_a$.

As a result :

$$\Lambda_a{}^b \, \Lambda^a{}_c = \eta_{ae} \, \eta^{bf} \, \Lambda^e{}_f \, \Lambda^a{}_c = \eta_{cf} \, \eta^{bf} = \delta^b_c$$

η_{cf}

$\Rightarrow$ $\Lambda_x{}^x$ and $\Lambda^x{}_x$ are transpose inverse of each other.

* Covariant and contravariant vectors : Contravariant V^a and covariant vectors V_a are well-defined objects

that relate as

$$V_a = \eta_{ab} V^b \quad \Longleftrightarrow \quad V^a = \eta^{ab} V_b$$

Under a Lorentz transformation one has

$$V_a'(x') = \Lambda_a{}^b V_b(x) \quad \text{and} \quad V^{a'}(x') = \Lambda^a{}_b V^b(x)$$

so that full-contractions are Lorentz-invariant quantities

$$U'^a V'_a = \Lambda^a{}_b U^b \Lambda_a{}^c V_c = U^b V_c \underbrace{\Lambda^a{}_b \Lambda_a{}^c}_{\delta_b^c} = U^b V_b$$

NOTE : $\partial_a V^a$ (divergence) is Lorentz-invariant.

NOTE : d'Alembertian is a Lorentz-invariant operator

$$\Box = \eta^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} = \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

* Mixed tensors : General tensors can be constructed transforming multilinearly under Lorentz transformations

$$T^{c'}{}_{ab}(x') = \Lambda^c{}_d \Lambda_a{}^e \Lambda_b{}^f T^d{}_{ef}(x)$$

General mixed tensors satisfy :

- Linearity : $T^a{}_b \equiv \alpha \underbrace{R^a{}_b}_{\text{tensor}} + \beta \underbrace{S^a{}_b}_{\text{tensor}}$ is a tensor.

- Direct product : $T^{a}{}_{b}{}^{c} \equiv \underbrace{A^{a}{}_{b}}_{\text{tensor}} \underbrace{B^{c}}_{\text{tensor}}$ is a tensor.

- Contraction : $T^{ab} = \underbrace{T^{a}{}_{c}{}^{bc}}_{\text{tensor}}$ is a tensor.

- Differentiation : $T_{a}{}^{bc} \equiv \underbrace{\frac{\partial}{\partial x^{a}}}_{\text{tensor}} \underbrace{T^{bc}}_{\text{tensor}}$ is a tensor. [direct product]

* Lorentz-invariant tensors : There are three invariant tensors under Lorentz transformations

- The metric tensor : $\eta'_{ab} = \underbrace{\Lambda_{a}{}^{c} \Lambda_{b}{}^{d} \eta_{cd}}_{\Lambda \eta \Lambda^{t} = \eta} = \eta_{ab}$

- The Levi-Civita tensor : $\varepsilon^{abcd} = \begin{cases} +1 & : \text{even permut of } 0123 \\ -1 & : \text{odd permut of } 0123 \end{cases}$

$\Lambda^{a}{}_{e} \Lambda^{b}{}_{f} \Lambda^{c}{}_{g} \Lambda^{d}{}_{h} \varepsilon^{efgh} \propto \varepsilon^{abcd}$

↳ must be fully antisym
⇒ proportional to ε itself !!

Looking at the component:

$$\varepsilon'^{0123} = \Lambda^{0}{}_{e} \Lambda^{1}{}_{f} \Lambda^{2}{}_{g} \Lambda^{3}{}_{h} \varepsilon^{efgh}$$

$$= \det(\Lambda)\, \varepsilon^{0123} = \varepsilon^{0123}$$

↳ if Λ is a "proper" Lorentz transf.

- The zero tensor with arbitrary pattern of indices.

NOTE: $\varepsilon_{abcd} = \eta_{ae}\,\eta_{bf}\,\eta_{cg}\,\eta_{dh}\,\varepsilon^{efgh} = -\varepsilon^{abcd}$

$|\eta| = -1$

Important: If two tensors have the same index structure and are the same for an observer in a coordinate system $\Rightarrow$ They are the same for any other inertial observer.

$\Rightarrow$ The statement that a tensor vanishes is a Lorentz-invariant statement !!

III. Classical electrodynamics

Maxwell's equations for electromagnetism can be written in a covariant form

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[Gauss]} \qquad \text{and} \qquad \vec{\nabla}\cdot\vec{B} = 0 \quad \text{[Gauss]}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{[Ampère]} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{[Faraday]}$$

$(*)$ $\qquad (**)$

where $\rho \equiv$ elec. charge density, $\vec{J} \equiv$ elec. current density and $\begin{cases}\varepsilon_0 \equiv \text{permitivity} \\ \mu_0 \equiv \text{permeability}\end{cases}$

Question: How do $\vec{E}$ and $\vec{B}$ transform under Lorentz ?

Let us introduce the electromagnetic field strength

$$F^{ab} \equiv \begin{bmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & B_3 & -B_2 \\ -\frac{E_2}{c} & -B_3 & 0 & B_1 \\ -\frac{E_3}{c} & B_2 & -B_1 & 0 \end{bmatrix} = -F^{ba}$$

Then $F^{0i} = \frac{E_i}{c}$ and $F^{ij} = \epsilon_{ijk} B^k$ so that

$(*)$: $\frac{\partial}{\partial x^a} F^{ab} = \partial_a F^{ab} = -\mu_0 J^b$ ↱ Prediction !!

$c^2 = \frac{1}{\varepsilon_0 \mu_0}$

[Gauss] $b=0$: $\frac{\partial}{\partial x^i} F^{i0} = -\mu_0 J^0 \Rightarrow \vec{\nabla} \cdot \vec{E} = c\mu_0 J^0 = \mu_0 \underbrace{c^2}_{} \cdot \rho = \frac{\rho}{\varepsilon_0}$

[Ampère] $b=j$: $\frac{1}{c} \frac{\partial}{\partial t} F^{0j} + \frac{\partial}{\partial x^i} F^{ij} = -\mu_0 J^j$

$$\Rightarrow \frac{1}{c^2} \frac{\partial}{\partial t} E^j + \epsilon^{ijk} \frac{\partial}{\partial x^i} B^k = -\mu_0 J^j$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial}{\partial t} E^j - \epsilon^{jik} \frac{\partial}{\partial x^i} B^k = -\mu_0 J^j$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

where $J^a \equiv (c \cdot \rho, \vec{J})$

$(**)$ $\varepsilon^{abcd} \frac{\partial}{\partial x_b} F_{cd} = 0$ with $F_{cd} = \eta_{ca} \eta_{db} F^{ab}$

[Gauss] $a=0$: $\varepsilon^{0ijk} \frac{\partial}{\partial x^i} F_{jk} = 0 \Rightarrow \underbrace{\epsilon^{ijk} \epsilon_{jkl}}_{2\,\delta^i{}_l} \frac{\partial}{\partial x^i} B^l = 0$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

[Faraday] $a=i$: $\varepsilon^{i0jk}\frac{1}{c}\frac{\partial}{\partial t}F_{jk} + \epsilon^{ij0k}\frac{\partial}{\partial x^j}F_{0k} + \epsilon^{ijk0}\frac{\partial}{\partial x^j}F_{k0} = 0$

$$-\underbrace{\epsilon^{ijk}\epsilon_{jkl}}_{2\delta^i_l}\frac{1}{c}\frac{\partial}{\partial t}B^l + 2\epsilon^{ijk}\frac{\partial}{\partial x^j}\left(-\frac{E^k}{c}\right) = 0$$

$\hookrightarrow F_{0i} = -F^{0i}$

$$\Rightarrow \frac{\partial}{\partial t}\vec{B} + \vec{\nabla}\times\vec{E} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

"dual" $\to$ field strength

Important: $\varepsilon^{abcd}\partial_b F_{cd} = 0 \iff \partial_b\tilde{F}^{ab} = 0$ with $\tilde{F}^{ab} \equiv \varepsilon^{abcd}F_{cd}$

* Bianchi form of Gauss-Faraday law : It will be useful to consider the alternative expression

$$\underbrace{\varepsilon_{aefg}\times\varepsilon^{abcd}}_{\propto\,\delta^{bcd}_{efg}}\frac{\partial}{\partial x^b}F_{cd} = 0 \Rightarrow \partial_{[e}F_{fg]} = 0$$

$\hookrightarrow dF = 0$ (differential form)

* The vector potential A_a : The Gauss-Faraday eqs imply

$$\epsilon^{abcd}\partial_b F_{cd} = 0 \Rightarrow F_{cd} = \partial_c A_d - \partial_d A_c = 2\,\partial_{[c}A_{d]}$$

so we can introduce a 4-vector A_a. Note also the ambiguity $A_a \to A_a + \partial_a\phi$ that yields the same F_{ab} and so the same $(\vec{E},\vec{B})$ $\Rightarrow$ Gauge symmetry of electromagnetism !!

IV. Lie groups and the Lorentz group

Definition: A group G is a set of elements $g \in G$ with a composition law $\cdot$ such that:
↳ multiplication

* closure : If $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$
* associativity : $\forall\, g_1, g_2, g_3 \in G \Rightarrow g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$
* identity element (e) : $\exists\, e \in G$ such that $e \cdot g = g \cdot e = g \quad \forall g \in G$
* inverse : $\forall g \in G,\ \exists\, g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

Exponential map: Lets G be a Lie group (also a differentiable manifold) and $\mathcal{A}$ its associated Lie algebra. Then there is an exponential map connecting both

$$\mathrm{Exp}: \mathcal{A} \longrightarrow G$$

$$X \longrightarrow g = e^{X} = \mathbb{I} + X + \frac{1}{2!} X^2 + \dots$$

$$= \sum_{K=0}^{\infty} \frac{X^K}{K!}$$

Lie algebra $\mathcal{A}$: The Lie algebra $\mathcal{A}$ is a vector space so that

$$X = \sum_{A=1}^{\dim(G)} \underbrace{\theta^A}_{\text{Real parameters} \,\equiv\, \text{Local coordinates}} \ \underbrace{t_A}_{\text{basis elements} \,\equiv\, \text{"generators"}} , \quad A = 1, \dots, \underbrace{\dim(G)}_{\text{dim of } G \,=\, \#\,\text{generators}}$$

and therefore

$$g = e^{X} = e^{\sum_{A=1}^{\dim(G)} \theta^A t_A} \in G$$

Important : The composition law $\cdot$ in the group becomes

$$e^{X} \cdot e^{Y} = e^{Z} \quad \text{with} \quad \underset{\substack{\downarrow \\ \text{Baker-Campbell-Hausdorff}}}{Z} = X + Y + \frac{1}{2} \underbrace{[X,Y]}_{\text{"commutator"}} + \underbrace{\ldots}_{\text{extra commutator terms}}$$

Commutators : There is a composition law in the algebra

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

$$[X, Y] \longrightarrow Z$$

that is antisymmetric and bilinear

$$[X,Y] = -[Y,X] \quad , \quad [aX+bY, Z] = a[X,Z] + b[Y,Z]$$

The algebra $\mathcal{A}$ is totally specified by structure constants defining the commutator between the generators

$$[t_A, t_B] = f_{AB}{}^{C} \, t_C$$

and satisfying the Jacobi identity

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

which, taking $X = t_A$, $Y = t_B$ and $Z = t_C$, translates into

$$f_{AD}{}^{E} f_{BC}{}^{D} + f_{CD}{}^{E} f_{AB}{}^{D} + f_{BD}{}^{E} f_{CA}{}^{D} = 0 \iff f_{[AB}{}^{D} f_{C]D}{}^{E} = 0$$

Representations : A representation D of G is a mapping of G into linear operators (i.e. matrix operators) so that

- $D(e) = \mathbb{I}$ (element with $\theta^A = 0$)
- $D(g_1 \cdot g_2) = D(g_1)\, D(g_2)$

Important : The dimension of a representation is the dimension of the vector space with elements σ_i with $i = 1, \dots, N$ on which it acts

$$\sigma'^i = [D(g)]^i{}_j \, \sigma^j$$

$\Rightarrow$ $\dim(G)$ should not be confused with the dimension of the representation N !!

Important : A group G has various representations D_1, D_2, etc of different dimension N_1, N_2, etc.

$\ast$ **The Lorentz group** : Let us present the Lorentz group as a Lie group.

- $\dim(G) = 6$ $\Rightarrow$ # of independent rotations in $\mathbb{R}^{1,3}$
- $\Lambda = e^{\frac{1}{2}\theta^{ab} M_{ab}}$ with $\begin{cases} \theta^A \equiv \theta^{ab} = -\theta^{ba} \\ t_A \equiv M_{ab} = -M_{ba} \end{cases}$

- 3 rotations M_{ij} + 3 boosts M_{0i}
- Commutation relations

$$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}$$

- Representations:

→ Fundamental (vector) : $\dim(D_F) = 4 \in \mathbb{R}$

$$[M_{ab}]^c{}_d = \delta^c_a \eta_{bd} - \delta^c_b \eta_{ad} = 2\, \delta^c_{[a} \eta_{b]d}$$

→ Adjoint : $\dim(D_{Adj}) = 6 = \dim(G) \in \mathbb{R}$

$$[M_{ab}]^{ef}{}_{cd} = \eta_{ac} \delta^e_b \delta^f_d - \eta_{bc} \delta^e_a \delta^f_d + \eta_{ad} \delta^f_b \delta^e_c - \eta_{bd} \delta^f_a \delta^e_c$$
$$= - f_{ab\, cd}{}^{ef} \quad \text{[in terms of structure constants]}$$

=> Structure constants $f_{AB}{}^C = f_{ab\,cd}{}^{ef}$ with

$$f_{ab\,cd}{}^{ef} = 8\, \eta_{[c[b} \delta^{[e}_{a]} \delta^{f]}_{d]}$$

→ Spinorial :

$$[M_{ab}]^\alpha{}_\beta = \frac{1}{4} [\gamma_{ab}]^\alpha{}_\beta$$

with $\gamma_{ab} = \gamma_a \gamma_b - \gamma_b \gamma_a = [\gamma_a, \gamma_b]$ in terms of γ-matrices.

▲ Spinorial Weyl : $\dim(D_W) = 2^{\frac{4}{2}-1} = 2 \in \mathbb{C}$
$(\alpha = 1, 2)$

$$\gamma_a = (\mathbb{I}_{2\times 2}, \underbrace{\sigma_1, \sigma_2, \sigma_3}_{\text{Pauli matrices}})$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

▲ Spinorial Dirac: $\dim(D_D) = 2^{\frac{4}{2}} = 4 \in \mathbb{C}$
$(\alpha = 1, \dots, 4)$

$$\gamma_0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

▲ Spinorial Majorana : $\dim(D_M) = 2^{\frac{4}{2}} = 4 \in \mathbb{R}$
$(\alpha = 1, \dots, 4)$

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}$$

• Particles in Minkowski space-time : They transform under the above representations of the Lorentz group

- Bosons $\begin{cases} s = 0 : & \Phi \\ s = 1 : & A_a \\ s = 2 : & \eta_{ab} \end{cases}$ $\Rightarrow$ Integer spin $s = 0, 1, 2$

- Fermions $\begin{cases} s = 1/2 : & \psi_\alpha \\ s = 3/2 : & \Psi_{a\alpha} \end{cases}$ $\Rightarrow$ Half-integer spin $s = \frac{1}{2}, \frac{3}{2}$

V. Classical fields in Minkowski space-time

Let us consider a classical field $\phi(x)$ of any of the types discussed before. And let us consider an action

$$S = \int \underbrace{\mathcal{L}(\phi, \partial_a \phi)}_{\text{Lagrangian density}} d^4x$$

* Equation of motion (E.O.M) :

$$\delta S = 0 \equiv> \text{Arbitrary variation } \delta\phi$$

$$\equiv> \quad \frac{\delta S}{\delta \phi} = 0 \iff \frac{\partial \mathcal{L}}{\partial \phi} - \partial_a \left[\frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \right] = 0$$

"Euler-Lagrange equations"

NOTE: Two Lagrangians differing on a divergence $\partial_a T^a$ yields the same E.O.M.

* Conserved current : J^a is a conserved current if

$$\partial_a J^a = 0 \quad \Rightarrow \quad Q = \int J^{(0)} d^3x \equiv \text{Conserved charge}$$

* Symmetry : Let us consider an infinitesimal δ_ϵ transformation on coordinates and/or fields with continuous constant parameter ϵ

$$x'^a = x^a + \delta_\epsilon x^a$$

$$\phi'(x') = \phi(x) + \delta_\epsilon \phi$$

If $\delta_\epsilon S = 0 \Rightarrow$ Symmetry !!

• Acting with a general transformation one finds

linear dependence on ϵ

$$\delta S = \int \partial_a J^a \, d^4x$$

with the so-called "Noether current"

$$J^a = \frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \delta\phi - \left[\frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} (\partial_b \phi) - \delta^a_b \mathcal{L} \right] \delta x^b$$

Then, if the transformation is a symmetry, one has

$$\delta S = 0 \quad \Rightarrow \quad \partial_a J^a = 0$$

* Noether Theorem: If a Lagrangian has a continuous symmetry $\Rightarrow \exists$ a current associated with that symmetry which is conserved provided the EOM holds.

* Types of transformations in SR [Poincaré sym]

i) Internal : $\delta\phi \neq 0$, $\delta x^a = 0$

ii) Translations : $\delta\phi = 0$, $\delta x^a = \epsilon^a$

iii) Lorentz : $\delta\phi \neq 0$, $\delta x^a = \theta^a{}_b \, x^b$

with

$$\Lambda^a{}_b = e^{\frac{1}{2}\theta^{cd}\overbrace{[M_{cd}]^a{}_b}^{\text{algebra } so(1,3)}} \in SO(1,3)$$

$$[M_{cd}]^a{}_b = 2\,\delta^a_{[c}\,\eta_{d]b}$$

Example : Translations $x'^a = x^a + \epsilon^a \Rightarrow \delta x^a = \epsilon^a$

$\delta \phi = 0$

$$\Rightarrow J^a = - T^a{}_b \, \epsilon^b$$

with $T^{ab} = \dfrac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \partial^b \phi - \eta^{ab} \mathcal{L}$

Energy-momentum tensor

Then, invariance under translations requires

$$\partial_a J^a = - (\partial_a T^{ab}) \underbrace{\epsilon_b}_{\text{arbitrary}} = 0$$

$$\Rightarrow \boxed{\partial_a T^{ab} = 0}$$

[conserved energy-momentum tensor]

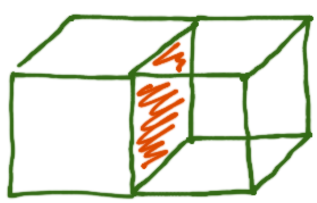
and the conserved quantities are

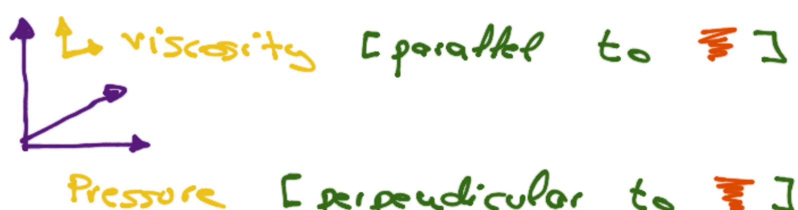
$$P^a = \int T^{0a} \, d^3x$$ [energy and momentum]

VI. Perfect fluid in Minkowski space-time

In many areas of Physics matter is modelled as a perfect fluid. Let us introduce the various concepts involved in its definition.

* Fluid : Continuous medium that can be divided into "fundamental elements" with a contact surface





for which viscosity $\ll$ pressure.

* Energy-momentum tensor : Any continuous medium has its energy and momentum codified into an energy-momentum tensor T^{ab}

T^{00} : Energy flux through a $t = cst$ surface

T^{0j} : Energy flux through a $x^j = cst$ surface

T^{i0} : Momentum flux through a $t = cst$ surface

T^{ij} : Momentum flux through a $x^j = cst$ surface

$$\approx > \quad T^{ii} \equiv \text{Pressure}$$

$$T^{ij} \ (i \neq j) \equiv \text{Viscosity}$$

NOTE : If there is no special space-time point in the fluid (translation invariant) then $\partial_a T^{ab} = 0$

[energy-momentum conservation]
[fluid is a closed system]

* Momentarily Inertial Frame (MIF) : It is defined as such coordinate system for which the elements in the fluids are in rest

$$u^a_{MIF} = (c, \vec{0})$$

* Perfect fluid : It is a fluid for which, in the MIF,

$$T^{ij} \ (i \neq j) = T^{i0} = T^{0j} = 0$$

so that

$$T^{ab}_{MIF} = \begin{bmatrix} \overset{\text{energy density}}{\rho} & \\ & \underset{\text{pressure}}{P}\, \delta^{ij} \end{bmatrix}$$

In an arbitrary Lorentz frame one has

$$T^{ab} = P\,\eta^{ab} + (P+\rho)\,\frac{u^a u^b}{c^2}$$