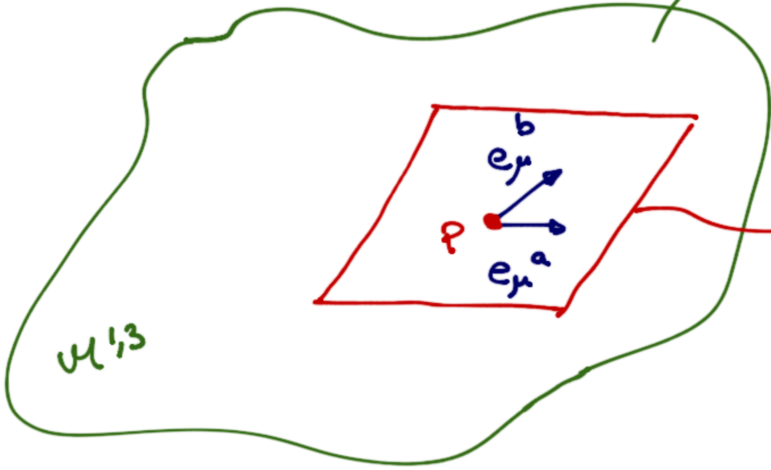
I. Frames and connections

space-time $\mathcal{M}^{1,3}$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$



Tangent space (at point p)

$$ds^2 = \eta_{ab}\, e^a e^b$$

with $e^a = \underbrace{e_\mu{}^a}_{\text{frame field}} dx^\mu$

- Locally one can define a "tetrad" $e_\mu{}^a(x)$ such that

$$g_{\mu\nu}(x) = e_\mu{}^a(x)\; e_\nu{}^b(x)\; \eta_{ab}$$

- GCT vs Local Lorentz transf. $\begin{cases} \text{GCT}: & T'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu}\, T^\nu(x) \\[2ex] \text{LLT}: & T'^a(x) = \Lambda^a{}_b(x)\, T^b(x) \\ [SO(1,3)] & \text{with } \Lambda^{a'}{}_a \Lambda^{b'}{}_b\, \eta_{a'b'} = \eta_{ab} \end{cases}$

Important : Note that frames are not unique (redundancy) as

$$e'_\mu{}^a = \Lambda^a{}_b(x)\; e_\mu{}^b \;\Rightarrow\; g_{\mu\nu}' = e'_\mu{}^a\, e'_\nu{}^b\, \eta_{ab}$$
$$= e_\mu{}^a\, e_\nu{}^b\, \eta_{ab} = g_{\mu\nu}$$

This local (gauge) symmetry will introduce a gauge field for the particles (spinors) transforming under the local Lorentz group

$\Rightarrow$ spin connection $\omega_\mu{}^{ab}$!!

- Connections and covariant derivatives

→ World tensors : $\underbrace{\text{Christoffel}}$: $\Gamma_{\mu\nu}{}^{\rho}(g) = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$

$\Rightarrow \Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$

(no torsion)

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho$$
$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^{\rho} V_\rho$$

→ Tangent space tensors : $\underbrace{\text{Lorentz}}$ SO(1,3) generators $(M_{cd})^a{}_b = 2\delta^a_{[c}\eta_{d]b}$

$$\left.\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^{cd} (M_{cd})^a{}_b V^b = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \\ D_\mu V_a &= \partial_\mu V_a - \omega_\mu{}^{cd} (M_{cd})^b{}_a V_b = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b \end{aligned}\right\} \text{Vector}$$

$$D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \omega_\mu{}^{cd} \underbrace{(\gamma_{cd})_\alpha{}^\beta}_{M_{cd} = \frac{1}{4}\gamma_{cd}} \psi_\beta \quad \} \text{ Spinor}$$

- Vielbein postulate : $\nabla_\mu e_\nu{}^a = 0 \Rightarrow D_\mu V^a = e_\nu{}^a \nabla_\mu V^\nu$

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \qquad (\times e_a{}^\lambda)$$

$$\Rightarrow \quad e_a{}^\lambda\left(\partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_{\nu b}\right) = \Gamma_{\mu\nu}{}^\lambda \quad \Rightarrow$$ One indep connection !!

$\Longrightarrow$ The spin connection is a composite field

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) \qquad \text{[no torsion]}$$

$$T_{\mu\nu}{}^a \equiv D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b \quad \equiv \text{Torsion}$$

Important: $T_{\mu\nu}{}^a \overset{!}{=} 0 \Rightarrow \underbrace{\omega_\mu{}^{ab}(e)}_{} = 2 e^{\nu[a} \partial_{[\mu} e_{\nu]}{}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e_\sigma{}^c$

Levi-Civita connection [torsion free]

II. Actions and symmetries

Symmetries $\begin{cases} \text{space-time symmetries} \\ \text{gauge internal symmetries} \end{cases}$ $\begin{cases} \text{GCT} \longrightarrow \text{Diff}: x'^\mu = x^\mu - \xi^\mu(x) \\ \text{Local Lorentz} \longrightarrow \psi'_\alpha = \psi_\alpha - \lambda^{ab} [\gamma_{ab}]_\alpha{}^\beta \psi_\beta \end{cases}$

$\boxed{s=0}$ Scalar field ϕ [no world or tangent space index]

- G.C.T : $\phi'(x') = \phi(x) \Rightarrow \delta\phi = \mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi$
- gauge : No gauge symmetry (matter field)
- action : $S_\phi = \int d^4x \sqrt{-|g|} \left(-\frac{1}{2} g^{\mu\nu} \underbrace{\partial_\mu \phi}_{\nabla_\mu \phi = \partial_\mu \phi} \partial_\nu \phi \right)$
- EOM : $\Box \phi = 0$

$\boxed{s=1/2}$ Fermion field ψ_α [$\alpha \equiv$ tangent space spinorial index]

- Local Lorentz : $\psi'_\alpha(x) = \Lambda_\alpha{}^\beta \psi_\beta(x) \Rightarrow \delta_\lambda \psi_\alpha = \lambda^{ab} [\gamma_{ab}]_\alpha{}^\beta \psi_\beta$
- gauge : No gauge symmetry (matter field)
- action : $S_\psi = \int d^4x \sqrt{-|g|} \left(+\frac{1}{2} \bar\psi \gamma^\mu \nabla_\mu \psi - \frac{1}{2} \bar\psi \overleftarrow{\nabla}_\mu \gamma^\mu \psi \right)$

where $\nabla_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \underbrace{\omega_\mu{}^{ab}}_{\text{spin connection}} [\gamma_{ab}]_\alpha{}^\beta \psi_\beta \equiv D_\mu \psi_\alpha$

$\bar\psi \equiv \psi^t C \overset{!}{=} i \psi^\dagger \gamma^0$

↳ Majorana condition with $(\gamma^\mu)^t = t_0 t_1 C \gamma^\mu C^{-1}$

NOTE: In the absence of torsion one can integrate by parts S_ψ to write it as

$$S_\psi = \int d^4x \sqrt{-|g|}\, \bar{\Psi} \gamma^\mu D_\mu \Psi \qquad \text{[Dirac action]}$$

- EOM : $\gamma^\mu D_\mu \Psi_\alpha = 0$

$\boxed{s=1}$ Abelian [Maxwell] vector field A_μ [$\mu \equiv$ world index]

- GCT : $A'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \Rightarrow \delta_\xi A_\mu = \mathcal{L}_\xi A_\mu = \xi^\rho \partial_\rho A_\mu + (\partial_\mu \xi^\rho) A_\rho$
- gauge : $A'_\mu(x) = A_\mu(x) + \nabla_\mu \theta(x) \Rightarrow \delta_\theta A_\mu = \nabla_\mu \theta(x) = \partial_\mu \theta(x)$

  └ scalar function $\theta(x)$
- action : $S_A = \int d^4x \sqrt{-|g|} \left(-\frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu}} \right)$

  $\underbrace{F_{\mu\nu}}_{\text{gauge invariant}} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

  └ $\Gamma^\rho_{[\mu\nu]} = 0$ [No torsion]
- EOM : $\nabla_\mu F^{\mu\nu} = 0$

$\boxed{s = 3/2}$ Gravitino (vector-spinor) field $\Psi_{\mu\alpha}$ [$\mu \equiv$ world index, $\alpha \equiv$ tangent space index]

- GCT : $\Psi'_{\mu\alpha}(x') = \dfrac{\partial x^\nu}{\partial x^\mu} \Psi_{\nu\alpha}(x) \Rightarrow \delta_\xi \Psi_{\mu\alpha} = \mathcal{L}_\xi \Psi_{\mu\alpha} = \xi^\rho \partial_\rho \Psi_{\mu\alpha} + (\partial_\mu \xi^\rho) \Psi_{\rho\alpha}$
- Local Lorentz : $\Psi'_{\mu\alpha}(x) = \Lambda_\alpha{}^\beta \Psi_{\mu\beta}(x) \Rightarrow \delta_\Lambda \Psi_{\mu\alpha} = \lambda^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\mu\beta}$
- gauge : $\Psi'_{\mu\alpha}(x) = \Psi_{\mu\alpha}(x) + \nabla_\mu \epsilon_\alpha(x) \Rightarrow \delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$
- action : $S_\Psi = \int d^4x \sqrt{-|g|}\, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \underbrace{\nabla_\nu \Psi_\rho}$ [Rarita-Schwinger]

$$\nabla_\nu \Psi_{\rho\alpha} = \partial_\nu \Psi_{\rho\alpha} + \frac{1}{4} \omega_\nu{}^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\rho\beta} - \Gamma^\lambda_{\nu\rho} \Psi_{\lambda\alpha} = D_\nu \Psi_{\rho\alpha} \underbrace{- \Gamma^\lambda_{\nu\rho} \Psi_{\lambda\alpha}}_{\text{No torsion} \Rightarrow \text{Irrelevant due to } \gamma^{\mu\nu\rho}}$$

Important: $\delta_\epsilon S_{3/2} = \int d^4x \sqrt{-|g|}\left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\delta_\epsilon \Psi_\rho\right) \times 2$ ← variation w.r.t $\bar{\Psi}$

$$= \int d^4x \sqrt{-|g|}\left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\nabla_\rho \epsilon\right) \times 2$$

$$= \int d^4x \sqrt{-|g|}\Big[\underbrace{\nabla_\rho\left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\epsilon\right)}_{\text{boundary term}} + \nabla_\rho \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\epsilon\Big] \times 2$$

(no torsion)

$$= -\int d^4x \sqrt{-|g|}\, \bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_{[\nu}\nabla_{\rho]}\Psi_\mu \times 2$$

$$= -\int d^4x \sqrt{-|g|}\, \bar{\epsilon}\gamma^{\mu\nu\rho}\frac{1}{2}\left(\frac{1}{4}R_{\nu\rho ab}[\gamma^{ab}]\right)\Psi_\mu \times 2$$

$$= -\frac{1}{4}\int d^4x \sqrt{-|g|}\, \bar{\epsilon}\gamma^{\mu\nu\rho}\gamma_{ab}R_{\mu\nu}{}^{ab}\Psi_\rho = (*)$$

NOTE: $\gamma^{\mu\nu\rho}\gamma_{ab} = \gamma^{\mu\nu\rho}{}_{ab} + 6\gamma^{[\mu\nu}{}_b\,\delta^{\rho]}_a + 6\gamma^{[\mu}\delta^{\nu\rho]}_{ba}$

- $\gamma^{\mu\nu\rho}{}_{ab}R_{\mu\nu}{}^{ab} = 0$ [only in D=4]
- $6\gamma^{[\mu\nu}{}_b\,\delta^{\rho]}_a R_{\mu\nu}{}^{ab} = 2\gamma^{\mu\nu}{}_b\,\delta^\rho_a R_{\mu\nu}{}^{ab} + 4\gamma^{\nu\rho}{}_b\,\delta^\mu_a R_{\mu\nu}{}^{ab}$

Torsion-free Bianchi id.

$R_{[\mu\nu\rho]}{}^a = 0$

$$= 2\gamma^{\mu\nu b}R_{\mu\nu}{}^\rho{}_b + 4\gamma^{\nu\rho b}R_{\mu\nu}{}^\mu{}_b$$

$$= 2\gamma^{\mu\nu b}R_{\mu\nu}{}^\rho{}_b + 4\gamma^{\nu\rho b}R_{\nu b} \quad 0 \text{ [sym]}$$

- $6\gamma^{[\mu}\delta^{\nu\rho]}_{ba}R_{\mu\nu}{}^{ab} = 4\gamma^\mu\delta^{\nu\rho}_{ba}R_{\mu\nu}{}^{ab} + 2\gamma^\rho\delta^{\mu\nu}_{ba}R_{\mu\nu}{}^{ab}$

$$= 4\gamma^\mu R_{\mu\nu}{}^{\rho\nu} + 2\gamma^\rho R_{\mu\nu}{}^{\nu\mu}$$

$$= 4\gamma^\mu R_\mu{}^\rho - 2\gamma^\rho R$$

$$= 4\gamma^\mu\left(R_\mu{}^\rho - \frac{1}{2}\delta^\rho_\mu R\right)$$

NOTE: $\bar{\chi}\gamma_{\mu_1\ldots\mu_n}\lambda = t_n\bar{\lambda}\gamma_{\mu_1\ldots\mu_n}\chi$ with $t_0 = t_3 = +1$, $t_1 = t_2 = -1$
(4D)

$$(*) = -\int d^4x \sqrt{-|g|}\, \left(R_\mu{}^\rho - \tfrac{1}{2}\delta_\mu^\rho R \right) \bar{\epsilon}\gamma^\mu \Psi_\rho$$

$$= -\int d^4x \sqrt{-|g|}\, \underbrace{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right)}_{\text{Einstein tensor } G_{\mu\nu}\ !!} \bar{\epsilon}\gamma^\mu \Psi^\nu$$

Therefore :

i) $\delta_\epsilon S_\Psi = 0$ in Minkowski space-time where $\nabla_\mu \Psi_\rho = \partial_\mu \Psi_\rho$ and $\nabla_\mu \epsilon_\alpha = \partial_\mu \epsilon_\alpha$. This is the work by Rarita-Schwinger (free $s=3/2$ field has ϵ_α-gauge invariance)

ii) $\delta_\epsilon S_\Psi \propto \underbrace{G_{\mu\nu}}_{\text{gravity}} \bar{\epsilon}\gamma^\mu \Psi^\nu \neq 0$ $\Rightarrow$ This suggests that gravity has something to say about general ϵ_α-gauge invariance !!

- EOM : $\gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho = 0$

  or $\gamma^{\mu\nu\rho} D_\nu \Psi_\rho = 0$ (no torsion)

$\boxed{s=2}$ Metric field $g_{\mu\nu}$ [$\mu,\nu \equiv$ world indices)

- GCT : $g'_{\mu\nu}(x') = \dfrac{\partial x^\rho}{\partial x'^\mu} \dfrac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \Rightarrow \delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2\,\partial_{(\mu} \xi^\rho g_{\nu)\rho}$
- gauge : No gauge symmetry in GR, but ... ϵ_α-gauge symmetry?
- action : $S_g = \dfrac{1}{2\kappa^2} \int d^4x \sqrt{-|g|}\, g^{\mu\nu} R_{\mu\nu}$ [Einstein - Hilbert] (2nd order formalism)
- EOM : $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$

$\kappa^2 = 8\pi G_N$ = grav coupling cte

Proposal : Can gravity compensate for the lack of ϵ_α-invariance of S'_Ψ ?

(i) The fact that $\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$ appears in $\delta_\epsilon S_\Psi$ is highly remarkable.

(ii) We know from supersymmetry that $s=0$ ($\phi \in \mathbb{C}$) and $s=1/2$ (Weyl ψ) can furnish a susy chiral multiplet transforming as

$$\boxed{\begin{array}{l} \delta_\epsilon \phi \propto \bar{\epsilon}\, \psi \\ \delta_\epsilon \psi \propto \bar{\epsilon}\, \sigma^\mu (\partial_\mu \phi) \end{array}}$$

Free theory or Wess-Zumino model in flat space
- $\delta\mathcal{L}$ = boundary terms
- susy algebra on-shell

(iii) We also know from supersymmetry that $s=1/2$ (Majorana λ_α) and $s=1$ (A_μ) can furnish a susy vector multiplet transforming as

$$\boxed{\begin{array}{l} \delta_\epsilon A_\mu \propto \bar{\epsilon}\, \gamma_\mu \lambda \\ \delta_\epsilon \lambda \propto \bar{\epsilon}\, \gamma_{\mu\nu} F^{\mu\nu} \end{array}}$$

Super Yang-Mills theory in flat space

(iv) Could there be a similar story involving $s=3/2$ (Ψ) and $s=2$ (e) ??

Beautiful idea !!

III. $\mathcal{N}=1$ Supergravity with $\Lambda=0$

We have collected already quite some clues for the local ϵ_α-invariance to actually be local supersymmetry involving not only the $s=3/2$ field but also the $s=2$ metric field. We are going to work this idea out now.

Let's start from:

$$S_g = \frac{1}{2\kappa^2}\int d^4x \sqrt{-|g|}\; g^{\mu\nu} R_{\mu\nu} = \frac{1}{2\kappa^2}\int d^4x \sqrt{-|g|}\; e_a{}^{\mu} e_b{}^{\nu} R_{\mu\nu}{}^{ab}(e)$$

$$S_\Psi = \underbrace{-\frac{1}{2\kappa^2}}_{\text{we have introduced this constant}}\int d^4x \sqrt{-|g|}\; \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \underbrace{\nabla_\nu \Psi_\rho}_{D_\nu[\omega]\Psi_\rho}$$

Important: We are considering a space-time without torsion

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$$

so integration by parts works normally: $\int dx \sqrt{-|g|}\, \nabla_\mu V^\mu = \int dx\, \partial_\mu\left(\sqrt{-|g|}\, V^\mu\right)$

We already know that $\delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$. But the question is: what should be $\delta_\epsilon e_\mu{}^a$?

$$\delta_\epsilon\, e_\mu{}^a = c\, \bar{\epsilon}\gamma^a \Psi_\mu \qquad \text{(what else could it be?)}$$

with c being a constant to be fixed. Note that this combination already appeared in $\delta_\epsilon S_\Psi \Rightarrow$ ✓

Let's compute things explicitely

$$S = S_g(e) + S_\psi(e,\psi)$$

$$\delta_e S = \underbrace{\left(\underbrace{\frac{\delta S_g}{\delta e}}_{A.1} + \underbrace{\frac{\delta S_\psi}{\delta e}}_{A.2} \right)}_{A} \delta_e e + \underbrace{\frac{\delta S_\psi}{\delta \Psi} \delta_e \Psi}_{B \to \text{This we already computed}}$$

<u>B</u>: $\displaystyle \frac{\delta S_\psi}{\delta \Psi_{\mu\alpha}} \delta_e \Psi_{\mu\alpha} = \frac{1}{2K^2} \int d^4x \underbrace{e}_{\sqrt{|-g|}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \bar{\epsilon} \gamma^\mu \Psi^\nu$

<u>A.1</u>: This is, by definition, the computation of the Einstein equations in terms of frames $e_\mu{}^a$:

$$\frac{\delta S_g}{\delta e_\mu{}^a} \delta e_\mu{}^a = \frac{1}{2K^2} \int d^4x \left[\delta(e) R + 2e\, \delta(e_a{}^\mu) e_b{}^\nu R_{\mu\nu}{}^{ab} + e\, e_a{}^\mu e_b{}^\nu \delta(R_{\mu\nu}{}^{ab}) \right] = (*)$$

- $\delta(|M|) = |M| \operatorname{Tr}(M^{-1} \delta M) \Rightarrow \delta(e) = e\, e_a{}^\mu \delta e_\mu{}^a$
- $e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu \Rightarrow \delta e_\mu{}^a e_a{}^\nu + e_\mu{}^a \delta e_a{}^\nu = 0$

$$\Rightarrow \delta e_a{}^\nu e_\mu{}^a = - e_a{}^\nu \delta e_\mu{}^a \qquad (\times e_b{}^\mu)$$

$$\Rightarrow \delta e_b{}^\nu = - e_a{}^\nu e_b{}^\mu \delta e_\mu{}^a$$

- $\delta(R_{\mu\nu}{}^{ab}) = D_\mu \delta\omega_\nu{}^{ab} - D_\nu \delta\omega_\mu{}^{ab} \underset{\hookrightarrow \text{no torsion}}{=} \nabla_\mu \delta\omega_\nu{}^{ab} - \nabla_\nu \omega_\mu{}^{ab} \underset{\hookrightarrow \text{no torsion}}{\Rightarrow}$ Total derivative

$$(*) = \frac{1}{2K^2} \int d^4x\, e \left(e_a{}^\mu R\, \delta e_\mu{}^a - 2\, e_b{}^\nu e_c{}^\mu e_a{}^\rho R_{\mu\nu}{}^{ab} \delta e_\rho{}^c \right)$$

$$= \frac{1}{2K^2} \int d^4x\, e \left(e_a{}^\mu R\, \delta e_\mu{}^a - 2 R_\mu{}^\rho e_c{}^\mu \delta e_\rho{}^c \right) = (\text{relabeling})$$

$$= \frac{1}{2K^2}\int d^4x \; e \left(e_a{}^{\rho} R - 2 R_\mu{}^\rho e_a{}^\mu \right) \delta e_\rho{}^a = (*)$$

Now we plug: $\delta_\epsilon e_\rho{}^a = c \, \bar\epsilon \gamma^a \Psi_\rho$

$$(*) = -\frac{c}{K^2}\int d^4x \; e \left(R_\mu{}^\rho e_a{}^\mu - \frac{1}{2} e_a{}^\rho R \right) \bar\epsilon \gamma^a \Psi_\rho$$

$$= -\frac{c}{K^2}\int d^4x \; e \left(R_\mu{}^\rho - \frac{1}{2}\delta_\mu^\rho R \right) \bar\epsilon \gamma^\mu \Psi_\rho$$

$$= -\frac{c}{K^2}\int d^4x \; e \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \bar\epsilon \gamma^\mu \Psi^\nu$$

lowest order Ψ

A.2: $$\frac{\delta S_\Psi}{\delta e_\rho{}^a}\delta e_\rho{}^a = -\frac{1}{2K^2}\int d^4x \; \delta\left[e \, \bar\Psi_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho \right] = 0$$

NOTE: $\delta e = e \, e_a{}^\mu \delta e_\mu{}^a = c \, e \, e_a{}^\mu \bar\epsilon \gamma^a \Psi_\mu \Rightarrow \epsilon\Psi^3\text{-terms}$

$$\begin{aligned}
\delta(\gamma^{\mu\nu\rho}) &= \delta(e_a{}^\mu e_b{}^\nu e_c{}^\rho)\gamma^{abc} \\
&= \delta e_a{}^\mu e_b{}^\nu e_c{}^\rho \gamma^{abc} + \dots \\
&= - e_d{}^\mu e_a{}^\lambda \delta e_\lambda{}^d \, e_b{}^\nu e_c{}^\rho \gamma^{abc} + \dots \\
&= - e_d{}^\mu \delta e_\lambda{}^d \gamma^{\lambda\nu\rho} + \dots \\
&= - e_d{}^\mu \gamma^{\lambda\nu\rho} (\bar\epsilon \gamma^d \Psi_\lambda) \Rightarrow \epsilon\Psi^3\text{-terms}
\end{aligned}$$

$$\begin{aligned}
\delta(\nabla_\nu \Psi_\rho) = \delta(D_\nu \Psi_\rho) &= \delta\left(\partial_\nu \Psi_\rho + \frac{1}{4}\omega_\nu{}^{ab}[\gamma_{ab}]\psi_\rho\right) \\
&= \frac{1}{4}\delta(\omega_\nu{}^{ab})[\gamma_{ab}]\Psi_\rho \\
&\Rightarrow \epsilon\Psi^3\text{-terms}
\end{aligned}$$

$\downarrow$

schematically: $\omega = e^{-1}\partial e - e^{-1}e^{-1} e \partial e$

Wrapping up B, (A.1) and (A.2) we obtain at lowest order ($\bar{\epsilon}\Psi$-terms) in fermions the following result:

$$\delta_\epsilon S = \delta_\epsilon(S_g + S_\Psi) = \underbrace{\left(\tfrac{1}{2} - c\right)}_{c=\frac{1}{2}\ !!}\frac{1}{\kappa^2}\int d^4x\, e\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right)\bar{\epsilon}\gamma^\mu\Psi^\nu$$

Therefore we have an ϵ_α-invariant action at lowest order in fermionic fields with transformation rules:

$$\text{(i)}\quad \boxed{\begin{aligned}\delta_\epsilon e_\mu{}^a &= \tfrac{1}{2}\,\bar{\epsilon}\gamma^a\Psi_\mu \\ \delta_\epsilon \Psi_\mu &= D_\mu\epsilon\end{aligned}} \Rightarrow \text{SUGRA !!}$$

Remark: It was crucial that $\nabla_\nu\Psi_\rho$ in the action R-S action appears with a $\gamma^{\mu\nu\rho}$. This allowed us to replace $\nabla_\nu\Psi_\rho = D_\nu\Psi_\rho$ fitting well with $D_\mu\epsilon$ and

$$D_{[\mu}D_{\nu]} = R_{\mu\nu} \Rightarrow \text{GR !!}$$

However, there is an easier way to see how GR emerges from the SUSY algebra of supersymmetry:

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]e_\mu{}^a &= \delta_{\epsilon_1}\left(\tfrac{1}{2}\bar{\epsilon}_2\gamma^a\Psi_\mu\right) - (1\leftrightarrow 2) \\
&= \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\nabla_\mu\epsilon_1 - (1\leftrightarrow 2) = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 - (1\leftrightarrow 2) \\
&= \tfrac{1}{2}\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 - \underbrace{\tfrac{1}{2}\bar{\epsilon}_1\gamma^a D_\mu\epsilon_2}_{\equiv\xi^a} \\
&\overset{t_1=-1}{=} \tfrac{1}{2}\left(\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 + D_\mu\bar{\epsilon}_2\gamma^a\epsilon_1\right) = D_\mu\left(\tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1\right) \\
&= D_\mu\xi^a \qquad \text{with } \xi^a = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1
\end{aligned}$$

Therefore we have found that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = D_\mu \xi^a$

Moreover $D_\mu \xi^a = \nabla_\mu \xi^a$ and

$$\delta_\xi e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a = \xi^\rho \partial_\rho e_\mu{}^a + \partial_\mu \xi^\rho e_\rho{}^a = (\text{explicit covariantisation})$$

$$= \xi^\rho \underbrace{\nabla_\rho e_\mu{}^a}_{0} + \cancel{\xi^\rho \Gamma_{\rho\mu}^\lambda e_\lambda{}^a} - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b + \underbrace{\nabla_\mu \xi^\rho e_\rho{}^a}_{\nabla_\mu \xi^a} - \cancel{\Gamma_{\mu\lambda}^\rho \xi^\lambda e_\rho{}^a}$$

$= \text{no torsion}$

$$= \partial_\mu \xi^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b$$

Then : $\nabla_\mu \xi^a = \delta_\xi e_\mu{}^a + \underbrace{\xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b}_{\delta_\Lambda e_\mu{}^a \text{ with } \Lambda^a{}_b = \xi^\rho \omega_\rho{}^a{}_b} = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$

So we have shown that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$

$\Rightarrow$ Local SUSY algebra implies $\left\{\begin{array}{l}\text{GCT} \\ \text{Local Lorentz}\end{array}\right. \Rightarrow$ General Relativity !!

→ Next question : Are the transformations (i) a symmetry of the action to all orders in fermions ?

Unfortunately the answer is no . . . ☹ . . .

It had been surprising otherwise as the Rarita - Schwinger action describes a free $s = 3/2$ field Ψ_ρ whereas the E-H action describing gravity is highly interacting !!

IMPORTANT : The computation we have performed did not rely on the dimension D or any special feature but having a real Ψ. Therefore, it works in the same way in any D. What makes certain dimensions special is that local SUSY can be stated at all orders in fermions . For example : $\mathcal{N}=1$ $D=4$ or $D=11$

To have $\mathcal{N}=1$ and $D=4$ sugra to all orders in fermions one has to introduce Ψ^4-terms both in the R-S action and in the transformation rules :

$$S_\Psi = -\frac{1}{2\kappa^2}\int d^4x\, e\left\{ \bar\Psi_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{16}\Big[(\bar\Psi^\rho \gamma^\mu \Psi^\nu)(\bar\Psi_\rho \gamma_\mu \Psi_\nu + 2\bar\Psi_\rho \gamma_\nu \Psi_\mu) - 4(\bar\Psi_\mu \gamma^\rho \Psi_\rho)(\bar\Psi^\mu \gamma^\rho \Psi_\rho) \Big] \right\}$$

with a torsion-free $D_\mu \Psi_\rho = \partial_\mu \Psi_\rho + \frac{1}{4}\omega_\mu{}^{ab}(e)\,[\gamma_{ab}]\,\Psi_\rho$

The SUSY transformation rules (i) must be also modified with higher-order fermion terms :

$$\delta e_\mu{}^a = \frac{1}{2}\bar\epsilon\gamma^a\Psi_\mu$$

$$\delta\Psi_\mu = D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\left(\omega_\mu{}^{ab}(e) + K_\mu{}^{ab}\right)[\gamma_{ab}]\,\epsilon$$

with $K_{\mu\nu\rho} = -\frac{1}{4}\left(\bar\Psi_\mu\gamma_\rho\Psi_\nu - \bar\Psi_\nu\gamma_\mu\Psi_\rho + \bar\Psi_\rho\gamma_\nu\Psi_\mu\right.$

IMPORTANT : We see that local SUSY at the full fermion label can be very conveniently described using gravitino torsion. This is an example of how torsion appears in Physics. Some 1^{st} and 1.5 order formulation of supergravity make all these structures manifest and render the problem of full local supersymmetry tractable !!

Some involved fermionic manipulations "Fierzing" are also required in the process.

Appendix : Supergravity with extended ($\mathcal{N}>1$) supersymmetry.

- There are theories of supergravity in D=4 with more than one ($\mathcal{N}=1$) gravitino fields : the so called "extended SUGRA's"

- The field content of the SUGRA multiplet includes :

$$\underbrace{e_\mu{}^a}_{\substack{\text{metric}\\(s=2)}} \oplus \underbrace{\begin{matrix}\Psi_{\mu\alpha}{}^1\\ \vdots \\ \Psi_{\mu\alpha}{}^{\mathcal{N}}\end{matrix}}_{\substack{\mathcal{N}\text{ gravitini}\\(s=3/2)}} \oplus \text{EXTRA FIELDS (bosonic and fermionic)}$$

- The most studied cases are $\mathcal{N} = \underbrace{2}_{\substack{\text{Minimally}\\\text{extended}}}\ ,\ \underbrace{4}_{\substack{\text{Half}\\\text{maximal}}}\ ,\ \underbrace{8}_{\substack{\text{Maximally}\\\text{extended}}}$

Number of supercharges in the susy algebra : 8 16 32

* $\mathcal{N}=2$:

$$\underbrace{e_\mu{}^a}_{s=2}\ ,\ \underbrace{A_\mu}_{\substack{s=1\\\text{(graviphoton)}}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,2}}_{s=3/2}$$

* $\mathcal{N}=4$:

$$\underbrace{e_\mu{}^a}_{s=2}\ ,\ \underbrace{A_\mu{}^{1,\dots,6}}_{s=1}\ ,\ \underbrace{\tau}_{s=0}^{\in\mathbb{C}} \in \overbrace{\frac{SL(2)}{SO(2)}}^{\text{"coset space"}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,\dots,4}}_{s=3/2}\ ,\ \underbrace{\psi_\alpha{}^{1,\dots,4}}_{s=1/2}$$

* $\mathcal{N}=8$:

$$\underbrace{e_\mu{}^a}_{s=2}\ ,\ \underbrace{A_\mu{}^{1,\dots,28}}_{s=1}\ ,\ \underbrace{\phi^{1,\dots,70}}_{\substack{s=0\\\in\mathbb{R}}} \in \underbrace{\frac{E_{7(7)}}{SU(8)}}_{\text{"coset space"}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,\dots,8}}_{s=3/2}\ ,\ \underbrace{\psi_\alpha{}^{1,\dots,56}}_{s=1/2}$$

Appendix: Scalar kinetic terms and "coset" spaces

The scalar kinetic terms can be understood geometrically from a "fictitious" (or auxiliary) scalar space perspective where scalar fields $\phi_i \in \mathbb{R}$ $(i=1,\dots,N)$ play the role of coordinates:

$$S_\phi = \int d^4x \sqrt{-|g|} \left[- \underbrace{K_{ij}(\phi)}_{\text{"metric" in field space}} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi) \right]$$

- One canonically normalised scalar:

$$K_{\phi\phi} = \frac{1}{2}$$

- N canonically normalised scalars:

$$K_{ij} = \frac{1}{2} \delta_{ij}$$

The geometrical interpretation becomes obvious when writing the kinetic terms as:

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = - K_{ij}(\phi) \underbrace{\partial_\mu \phi^i}_{d\phi^i} \underbrace{\partial^\mu \phi^j}_{d\phi^j} \quad \Rightarrow \quad \text{"Scalar geometry"}$$

$\Rightarrow$ Line element in field space !!

Important: In supergravity the scalar geometries are of a specific type called coset spaces.

Coset space $\mathcal{M} = \frac{G}{H}$: Coordinates on $\mathcal{M}$ (fields ϕ^i) correspond to an element of G not being an element of $H \subset G$:

- generators of G : $\{ \underbrace{h_1, \ldots, h_{\dim H}}_{\text{generators of } H} ; t_1, \ldots, t_{\dim G - \dim H} \}$
  [in a given representation]

- Coset representative : $\mathcal{V}(\phi) = e^{\sum_{i=1}^{N = \dim G - \dim H} \underbrace{\phi^i}_{} t_i} \in \frac{G}{H}$
  scalars = algebra parameters

- Scalar matrix : $M(\phi) = \mathcal{V}^t \mathcal{V} \in G$

- Scalar kinetic terms :

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = - K_{ij}(\phi)\, \partial_\mu \phi^i \, \partial^\mu \phi^j = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right]$$

Important : $\mathcal{L}_{kin}$ is invariant under the action of $g \in G$

$$\mathcal{V}' = \mathcal{V} g \quad \text{with} \quad g \in G$$

$$\Rightarrow \quad M' = v'^t v' = g^t \underbrace{v^t v}_{M} g = g^t M g$$

$$\Rightarrow \quad \frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}' = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M' \partial^\mu M'^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g^t \partial_\mu M \underbrace{g\, g^{-1}}_{\mathbb{I}} \partial^\mu M^{-1} g^{-t}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g^t \partial_\mu M \partial^\mu M^{-1} g^{-t}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \partial^\mu M^{-1}\right] = \frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}$$

cyclicity

Example : $\mathcal{M} = \frac{SL(2)}{SO(2)} \Rightarrow G = SL(2)$, $H = SO(2) \subset SL(2)$

- Generators of G : $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

[fundamental represent]

$\Rightarrow$ Commutators: $[T, E_\pm] = \pm 2 E_\pm$
$[E_+, E_-] = T$

- Some examples of group elements of $G = SL(2)$

$$g_T = e^{\frac{1}{2}\phi T} = \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix} \quad , \quad g_{E_+} = e^{x E_+} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$g_h = e^{\theta \underbrace{(E_+ - E_-)}_{h \text{ generator}}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2) = H$$

• When constructing $\mathcal{V} \in \frac{SL(2)}{SO(2)}$ one must be careful for not to exponentiate $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One choice is

$$\mathcal{V}(\phi, \chi) = g_T \, g_{E_+} = \begin{bmatrix} e^{\frac{\phi}{2}} & e^{\frac{\phi}{2}} \chi \\ 0 & e^{-\frac{\phi}{2}} \end{bmatrix} \in \frac{SL(2)}{SO(2)}$$

so that

$$M(\phi, \chi) = \mathcal{V}^t \, \mathcal{V} = \begin{bmatrix} e^{\phi} & e^{\phi} \chi \\ e^{\phi} \chi & e^{-\phi} + \chi e^{\phi} \chi \end{bmatrix}$$

and

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right]$$

$$= -\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \chi \, \partial^\mu \chi$$

$$\Rightarrow \quad K_{ij}(\phi, \chi) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\phi} \end{bmatrix}$$

NOTE : Coset spaces of the form $\frac{G}{H}$ with H being the maximal compact subgroup of G (like $\frac{SL(2)}{SO(2)}$) will be important when describing the scalar geometries arising from Kaluza-Klein reductions.

IV. Coupling $\mathcal{N}=1$ Supergravity to SYM and Matter fields

We are now going to couple the $\mathcal{N}=1$ supergravity multiplet to other multiplets of $\mathcal{N}=1$ supersymmetry: $\underbrace{\text{vector mult.}}_{\text{Super Yang-Mills (SYM)}}$ & $\underbrace{\text{chiral mult.}}_{\text{Matter}}$

MULTIPLET		-2	-3/2	-1	1/2	0	1/2	1	3/2	2
Gravity	$g_{\mu\nu}$	1								1
	Ψ_μ		1						1	
Vector	A_μ			1				1		
	λ_α				1		1			
Chiral	ψ_α				1		1			
	$\phi \in \mathbb{C}$					1 1				

Table : $\mathcal{N}=1$ multiplets, fields and helicity states

* Free theory : Local susy transformations [$\epsilon_\alpha(x)$ parameter]

$s = 2, \frac{3}{2}$

$$\boxed{\begin{aligned} \delta_\epsilon e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a \Psi_\mu \\ \delta_\epsilon \Psi_\mu &= D_\mu \epsilon \end{aligned}}$$

SUGRA Multiplet

$s = 1, \frac{1}{2}$

$$\boxed{\begin{aligned} \delta_\epsilon A_\mu &= -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu \lambda \\ \delta_\epsilon \lambda &= \tfrac{1}{4}\bar{\epsilon}\gamma_{\mu\nu} F^{\mu\nu} \end{aligned}}$$

Vector multiplet

$s = \frac{1}{2}, 0$

$$\boxed{\begin{aligned} \delta_\epsilon \phi &= \tfrac{1}{\sqrt{2}}\bar{\epsilon}\psi \\ \delta_\epsilon \psi &= \tfrac{1}{\sqrt{2}}\bar{\epsilon}\sigma^\mu(\partial_\mu \phi) \end{aligned}}$$

Chiral Multiplet

The bosonic Lagrangian

We focus on the bosonic terms in the action $\mathcal{S}$. The fermionic terms then follow from requiring $\mathcal{N}=1$ local supersymmetry.

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, e\Big[R - \frac{1}{4}\overbrace{\mathrm{Re}\, f_{ab}(\phi)}^{\text{“}g^{-2}\delta_{ab}\text{-like”}} F_{\mu\nu}{}^{a}F^{\mu\nu b} - \frac{1}{4}\frac{1}{e}\overbrace{\mathrm{Im}\, f_{ab}(\phi)}^{\text{“}\frac{1}{8\pi^2}\,\theta\,\delta_{ab}\text{-like”}}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^{a}F_{\rho\sigma}{}^{b}$$

$$- \underbrace{K_{i\bar{j}}(\phi,\bar{\phi})}_{\text{“scalar geometry”}}\nabla_\mu\phi^i\nabla^\mu\bar{\phi}^{\bar{j}} - V(\phi,\bar{\phi}) + \text{fermionic terms.}$$

with $\phi^i \in \mathbb{C}$ and $A_\mu^a \in \mathbb{R}$ spanning an internal gauge symmetry G_0

- $i = 1,\dots,n_c$ chiral multiplets
- $a = 1,\dots,n_v$ vector multiplets $\Rightarrow$ $n_v = \dim(G_0)$

We will consider matter charged under G_0 :

$$\nabla_\mu\phi^i = \partial_\mu\phi^i - A_\mu{}^a \underbrace{\delta_a\phi^i}_{\text{linear sym} \Rightarrow \delta_a\phi^i = (t_a)^i{}_j\,\phi^j \equiv \mathcal{K}_a{}^i(\phi)} = \partial_\mu\phi^i - A_\mu{}^a\,\mathcal{K}_a{}^i(\phi)$$

$$\nabla_\mu\bar{\phi}^{\bar{i}} = (\nabla_\mu\phi^i)^*$$

generators of $R[G_0]$: $(t_a)^i{}_j$

NOTE: Demanding gauge invariance of $K_{i\bar{j}}(\phi,\bar{\phi})$ under a a gauge G_0-transformation with parameters θ^a $(a=1,\dots,\dim G_0)$

- $\delta_\theta\phi^i = \theta^a(x)\,(t_a)^i{}_j\,\phi^j = \theta^a\;\mathcal{K}_a{}^i(\phi)$
- $\delta_\theta\bar{\phi}^{\bar{i}} = \theta^a(x)\,(t_a^*)^{\bar{i}}{}_{\bar{j}}\,\bar{\phi}^{\bar{j}} = \theta^a\;\bar{\mathcal{K}}_a{}^{\bar{i}}(\bar{\phi})$ [conjugate]

$\delta_\theta K_{i\bar{j}} = 0 \implies$ $\mathcal{K}_a{}^i(\phi)$ are a subset (labelled by a) of the Killing vectors of the scalar geometry determined by the $K_{i\bar{j}}$ metric !!

↳ isometry of the scalar geometry !!

For cosets $\frac{G}{H} \implies \underbrace{G_0}_{\text{gauge symmetry}} \subset \underbrace{G}_{\text{scalar geometry}}$

* Scalar geometry and $V(\phi, \bar{\phi})$

$\mathcal{N}=1$ supersymmetry requires the scalar geometry to be a complex Kähler manifold. This implies

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} \equiv \text{"Kähler metric"}$$

in terms of a "Kähler potential" $K(\phi, \bar{\phi}) \in \mathbb{R}$. Moreover

$$\mathcal{K}_a{}^i(\phi) = -i\, K^{i\bar{j}}(\phi, \bar{\phi}) \frac{\partial P_a(\phi, \bar{\phi})}{\partial \bar{\phi}^{\bar{j}}}$$

where $P_a(\phi, \bar{\phi})$ are called "moment maps" or "Killing prepotentials" and enter

$$\nabla_\mu \phi^i = \partial_\mu \phi^i - A_\mu{}^a\, \mathcal{K}_a{}^i(\phi)$$

The moment maps P_a are then expressed as :

$$P_a = -\frac{i}{2}\left[\mathcal{K}_a{}^i \partial_i K - \bar{\mathcal{K}}_a{}^{\bar{j}} \partial_{\bar{j}} K\right] - r_a$$

with $r_a = \delta_{a\, U(1)} \underbrace{\xi_{FI}}$ only for Abelian factors in G_0.

Fayet-Iliopoulos ≡ arbitrary parameters

Lastly the scalar potential $V(\phi,\bar\phi)$ is given by

$$V = e^{\kappa^2 K}\Big[\underbrace{\overbrace{D_i W\, K^{i\bar j}\, D_{\bar j}\overline{W}}^{e^{-\frac{1}{2}\kappa^2 K} F^i}}_{F\text{-terms} > 0} - 3\kappa^2 |W|^2\Big] + \underbrace{\overbrace{\tfrac{1}{2} P_a\, \mathrm{Re}(f^{-1})^{ab}\, P_b}^{D^a}}_{D\text{-terms} > 0}$$

in terms of an arbitrary "holomorphic superpotential" $W(\phi)$ where the Kähler derivatives read:

$$D_i W = \partial_i W + \kappa^2 (\partial_i K) W$$

$$D_{\bar i} \overline{W} = \partial_{\bar i} \overline{W} + \kappa^2 (\partial_{\bar i} K) \overline{W}$$

Important : Theory defined by : $\underbrace{G_0}_{\text{gauge group + representation } R[G_0] \text{ for } \phi^i} \oplus f_{ab}(\phi),\ K(\phi,\bar\phi),\ W(\phi) \oplus P_a(\phi,\bar\phi)$

* Local susy transformations & susy breaking.

The local susy transformations take the generic form

$$\delta_\epsilon \text{ Fermion} \sim \bar\epsilon \text{ Boson} \quad , \quad \delta_\epsilon \text{ Boson} \sim \bar\epsilon \text{ Fermion}$$

=> Lorentz invariance at the vacuum requires $\langle \text{Fermion} \rangle = 0$ and consequently $\langle \delta_\epsilon \text{ Boson} \rangle = 0$ always.

=> Lorentz invariance at the vacuum permits $\langle \text{Boson} \rangle \neq 0$ and consequently $\langle \delta_\epsilon \text{ Fermion} \rangle \begin{cases} = 0 \rightarrow \text{susy preserved} \\ \neq 0 \rightarrow \text{susy broken (spontaneously)} \end{cases}$

Let us look at the δ_ϵ Fermions in the interacting theory:

Gravitino : $$\delta_\epsilon \Psi_\mu = D_\mu \epsilon + \frac{1}{2}\kappa^2 e^{\frac{1}{2}\kappa^2 K} W \quad \bar{\epsilon}\gamma_\mu$$

Chiralini : $$\delta_\epsilon \psi^i = \frac{1}{\sqrt{2}}\left[\bar{\epsilon}\sigma^\mu(\partial_\mu \phi^i) - F^i \epsilon\right]$$

Gaugini : $$\delta_\epsilon \lambda^a = \frac{1}{4}\bar{\epsilon}\gamma_{\mu\nu}F^{\mu\nu a} + \frac{1}{2} i \bar{\epsilon} \underbrace{\gamma_*}_{\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3} D^a$$

with

- $F^i = e^{\frac{1}{2}\kappa^2 K} K^{i\bar{j}} D_{\bar{j}} \bar{W}$ => F-term in $V(\phi,\bar{\phi})$
- $D^a = \mathrm{Re}\,(f^{-1})^{ab} P_b$ => D-term in $V(\phi,\bar{\phi})$

As a result one has that

$$\text{SUSY}\begin{cases}\delta_\epsilon \psi^i = 0 \\ \delta_\epsilon \lambda^a = 0\end{cases} \Rightarrow \begin{matrix}\langle F^i\rangle = 0 \\ \langle D^a\rangle = 0\end{matrix} \Rightarrow V = -3 e^{\kappa^2 K}\kappa^2 |W|^2 < 0$$

AdS vacuum !!

NOTE: If SUSY is broken the gravitino $\Psi_{\mu\alpha}$ gets a mass and acquires a longitudinal mode $\partial_\mu \eta_\alpha$ by eating up the goldstino associated to the direction of ~~SUSY~~.

Ex: F-term breaking => $\eta^\alpha F_i \psi^i$ and $m^2_{3/2} = \kappa^4 e^{\kappa^2 K}|W|^2$.

NOTE: The existence of a de Sitter vacuum requires susy to be broken => ϕ^i scalars relevant for cosmology !!

[late-time cosmic acceleration, inflation, ...]

$*$ Global susy: Switching off Gravity $[\kappa^2 \to 0]$

In order to go from local susy (supergravity) to global susy (supersymmetric field theory) one switches off gravity by setting $\kappa^2 \to 0$. As a result:

- $g_{\mu\nu} \to \eta_{\mu\nu}$ (Minkowski space-time)
- $\Psi_{\mu\alpha} \to 0$ (no gauge field for a global symmetry)
- $F_i \to F_i = \partial_i W$
- $D^a \to D^a$ (gauge structure unaffected)

$$\Rightarrow \quad V = K^{i\bar{j}} F_i \bar{F}_{\bar{j}} + \frac{1}{2} \mathrm{Re}\, f_{ab} D^a D^b \geq 0$$

Therefore: $V > 0 \Rightarrow F_i \neq 0$ and/or $D^a \neq 0 \Rightarrow$ Susy breaking !!

NOTE: Susy field theories are very interesting playgrounds where to discover universality classes of phenomena both classically and also at the quantum level.

V. Prelude to superstrings and D=10,11 supergravity

* From strings to $\mathcal{N}=2$, D=10 Supergravity

Particle evolution in D-dimensions

$$\bullet \approx > X^M \underbrace{(\tau)}_{\text{proper time}}$$

String evolution in D-dimensions

$$\approx > X^M(\tau,\sigma)$$

$\ell_s^2 \sim 2\alpha'$

$$+\text{susy} \Rightarrow \left.\begin{matrix}\theta^1(\tau,\sigma)\\ \theta^2(\tau,\sigma)\end{matrix}\right\}\text{ Grassman variables}$$

Set $D=10$ and $\theta^{1,2}$ being $\underbrace{\text{M-W}}_{\text{Majorana - Weyl}}$ fermions

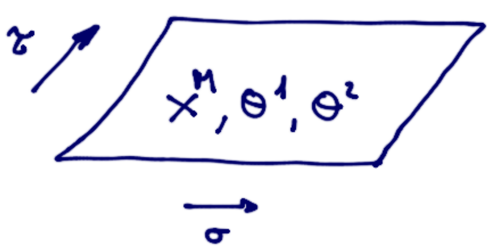
→ 2D conformal field theory : $X^M(\tau,\sigma)$, $\theta^{1,2}(\tau,\sigma)$

$$S_{2D} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^M + \text{fermion terms}$$

with $\sigma^\alpha = (\tau,\sigma)$ $\qquad \eta^{\alpha\beta} = (-1,1)$

↳ gauge fixing : diff + Weyl in 2D

→ Mode expansion and states



$$\Rightarrow \quad X_M = \sum_n \left(a_M^{(n)} e^{-2in(\tau-\sigma)} + \tilde{a}_M^{(n)} e^{-2in(\tau+\sigma)}\right)$$

$$\theta^1 = \sum_n b^{(n)} e^{-2in(\tau-\sigma)} \; ; \; \theta^2 = \sum_n \tilde{b}^{(n)} e^{-2in(\tau+\sigma)}$$

Promote a's, $\tilde{a}$'s, b's, $\tilde{b}$'s to operators with [,] or { , } relations : "dilaton"

$$|state\rangle = a^\dagger_M \, a^\dagger_N \, |0\rangle \Rightarrow \underbrace{G_{MN}}_{\text{metric}} \oplus \underbrace{B_{MN}}_{\text{antisym}} \oplus \underbrace{\Phi}_{\text{scalar trace}}$$

$D=10$: metric antisym scalar trace

→ Mass of a state :

$$M^2 = \frac{1}{l_s^2}\left[N(a,b) + \tilde{N}(\tilde{a},\tilde{b})\right] \Rightarrow \begin{matrix} l_s \to 0 \\ \bigcirc \to \bullet \\ M^2 \to \infty \end{matrix} \Rightarrow$$

ocupation numbers

"low energy": Keep only massless states !!

→ $\mathcal{N}=2$, $D=10$ massles spectrum : $\underbrace{G_{MN}, B_{(2)}, \Phi, C_{(p)}}_{\text{Bosons}}$; $\underbrace{\chi^{1,2}_\alpha, \Psi^{1,2}_{M\alpha}}_{\text{Fermions}}$

$(ch\,\Psi^1 \neq ch\,\Psi^2)$ IIA : $p=1,3 \Rightarrow C_M, C_{MNP}$

$(ch\,\Psi^1 = ch\,\Psi^2)$ IIB : $p=0,2,4 \Rightarrow C_{(0)}, C_{MN}, C_{MNPQ}$

NOTE: A p-form $C_{(p)}$ has p autisymmetric indices $C_{(p)} = C_{[M_1 \cdots M_p]}$

- Lagrangian : a candidate

$$\mathcal{L}_{10D} = \frac{1}{2K_{10}^2}\int d^{10}x \sqrt{G}\left[R - \frac{1}{2}\partial_M \Phi\, \partial^M \Phi - \frac{1}{2\times 3!} e^{-\Phi} \underbrace{H_{MNP}}\, H^{MNP} + \ldots + \text{fermi}\right.$$

with $2K_{10}^2 = \frac{1}{2\pi}(2\pi l_s)^8$

$H_{(3)} \equiv H_{MNP} = \partial_{[M} B_{NP]} = dB_{(2)}$

→ We can also study a probe string propagating in a background $\{ G_{MN}, B_{MN}, \Phi, C_{(p)} \}$ generated by other strings around:

$$S_{\substack{\text{probe}\\ \text{string}}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \Big[(\partial_\alpha X^M)(\partial^\alpha X^N) \underbrace{G_{MN}(X)} + \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \underbrace{B_{MN}(X)} + \dots$$

G_{MN}, $B_{(2)}$, etc can be viewed as couplings in the 2D field theory !!

Conformal invariance $\Rightarrow \underbrace{\beta_G^{MN} = \beta_B^{MN} = \dots = 0}$

At lowest order in $\frac{\sqrt{\alpha'}}{L}$ ~ system scale

$\Rightarrow$ E.O.M of an action !!

→ $\mathcal{N}=2$, $D=10$ Supergravity action:

$$S_{\substack{\text{SUGRA}\\ \mathcal{N}=2,\, D=10}} = \frac{1}{2K_{10}^2} \int d^{10}x \sqrt{G} \Big[R - \frac{1}{2} \partial^M \Phi \, \partial_M \Phi - \frac{1}{2\times 3!} e^{-\Phi} H_{MNP} H^{MNP}$$

$$- \frac{1}{4K_{10}^2} \int d^{10}x \sqrt{G} \begin{cases} \text{IIA}: \frac{1}{2!} e^{3/2\,\Phi} \hat{F}_{MN} \hat{F}^{MN} + \frac{1}{4!} e^{1/2\,\Phi} \hat{F}_{M_1 \dots M_4} \hat{F}^{M_1 \dots M_4} \\ \text{IIB}: e^{2\Phi} \partial_M C_{(0)} \partial^M C_{(0)} + \frac{1}{3!} e^{\Phi} \hat{F}_{MNP} \hat{F}^{MNP} + \frac{1}{5!} \hat{F}_{M_1 \dots M_5} \hat{F}^{M_1 \dots M_5} \end{cases}$$

$$B_{(2)} \wedge F_{(4)} \wedge F_{(4)} \Rightarrow \text{wedge products}$$

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x \begin{cases} \text{IIA: } \overbrace{\epsilon^{M_1 \ldots M_{10}} \, B_{M_1 M_2} \, F_{M_3 \ldots M_6} \, F_{M_7 \ldots M_{10}}} \\ \text{IIB: } \underbrace{\epsilon^{M_1 \ldots M_{10}} \, C_{M_1 \ldots M_4} \, H_{M_5 M_6 M_7} \, F_{M_8 M_9 M_{10}}}_{C_{(4)} \wedge H_{(3)} \wedge F_{(3)}} \end{cases}$$

$$+ \, S_{\text{Fermi}}\left(\chi^{1,2}, \Psi^{1,2}\right)$$

where the gauge invariant field strengths are given by:

IIA: $\hat{F}_{(2)} = F_{(2)} = dC_{(1)}$

$$\hat{F}_{(4)} = \underbrace{F_{(4)}}_{dC_{(3)}} + C_{(1)} \wedge H_{(3)}$$

IIB: $\hat{F}_{(3)} = \underbrace{F_{(3)}}_{dC_{(2)}} - H_{(3)} \wedge C_{(0)}$

$$\hat{F}_{(5)} = \underbrace{F_{(5)}}_{dC_{(4)}} + \frac{1}{2}\left[B_{(2)} \wedge F_{(3)} - C_{(2)} \cdot H_{(3)}\right]$$

$\Rightarrow$ Starting from closed superstrings we have obtained $\mathcal{N}=2$, $D=10$ Supergravities as the low-energy limit !!

$\Rightarrow$ Superstrings live in a ten-dimensional space-time ...

... so what about $10-4=6$ extra dimensions ?

→ The type IIA supergravity can be connected to the one and unique $\mathcal{N}=1$, $D=11$ Supergravity conjectured to be the low-energy limit of a mysterious theory of membranes called "M-theory"

$$\boxed{\begin{aligned} S_{\substack{SUGRA \\ \mathcal{N}=1,\, D=11}} &= \frac{1}{2\kappa_{11}^2}\int d^{11}x\,\sqrt{G}\left[R - \frac{1}{2\times 4!}F_{\hat{M}_1\ldots\hat{M}_4}F^{\hat{M}_1\ldots\hat{M}_4}\right] \\ &- \frac{1}{12\kappa_{11}^2}\int d^{11}x\,\underbrace{\epsilon^{\hat{M}_1\ldots\hat{M}_{11}}A_{\hat{M}_1\hat{M}_2\hat{M}_3}F_{\hat{M}_4\ldots\hat{M}_7}F_{\hat{M}_8\ldots\hat{M}_{11}}}_{A_{(3)}\wedge F_{(4)}\wedge F_{(4)}} \\ &+ S_{Fermi}(\Psi) \end{aligned}}$$

with $2\kappa_{11}^2 = \frac{1}{2\pi}(2\pi\, l_p)^9$

↳ Planck scale

* The field content of the theory is $G_{\hat{M}\hat{N}} \oplus A_{\hat{M}\hat{N}\hat{P}} \oplus \Psi_{\hat{M}\alpha}$

with $F_{(4)} \equiv F_{\hat{M}_1\ldots\hat{M}_4} = \partial_{[\hat{M}_1}A_{\hat{M}_2\hat{M}_3\hat{M}_4]} \equiv dA_{(3)}$. It is invariant under local supersymmetry transformations

$$\delta_\epsilon e_{\hat{M}}{}^{\hat{A}} = \bar{\epsilon}\,\Gamma^{\hat{A}}\,\Psi_{\hat{M}}$$

$$\delta_\epsilon A_{\hat{M}\hat{N}\hat{P}} = -3\,\bar{\epsilon}\,\Gamma_{[\hat{M}\hat{N}}\Psi_{\hat{P}]}$$

$$\delta_\epsilon \Psi_{\hat{M}} = D_{\hat{M}}\epsilon + \frac{1}{12}\left[\Gamma_{\hat{M}}\frac{1}{4!}F_{\hat{Q}\hat{R}\hat{S}\hat{T}}\Gamma^{\hat{Q}\hat{R}\hat{S}\hat{T}} - 3\,\frac{1}{3!}F_{\hat{M}\hat{N}\hat{P}\hat{Q}}\Gamma^{\hat{N}\hat{P}\hat{Q}}\right]\epsilon$$

Important: Note that there is no coupling to be tuned !!

* M-theory $\Rightarrow$ IIA $\begin{cases} G_{\hat{M}\hat{N}} \Rightarrow G_{MN} \oplus G_{M10} \equiv C_M \oplus G_{10\,10} \equiv \Phi \\ A_{\hat{M}\hat{N}\hat{P}} \Rightarrow A_{MNP} \equiv C_{MNP} \oplus A_{MN10} \equiv B_{MN} \end{cases}$

$\hat{M} = (M, 10)$

$\hookrightarrow M = 0, \ldots, 9$

$\hookrightarrow \hat{M} = 0, \ldots, 10$

Then one finds that

$$\underbrace{G_{\hat{M}\hat{N}} \, , \, A_{\hat{M}\hat{N}\hat{P}}}_{\mathcal{N}=1,\ D=11 \text{ SUPERGRAVITY}} \Longrightarrow \underbrace{G_{MN} \, , \, B_{MN} \, , \, \Phi \, , \, C_M \, , \, C_{MNP}}_{\mathcal{N}=2,\ D=10 \text{ Type IIA SUPERGRAVITY}}$$

Important: The 11D action also reduces consistently to the type IIA action (not only the field content)