[School on Holography and Supergravity 2021 : July 2021]

Lectures on Supergravity and Dualities

I. Space-time, frames and connections

II. Actions and symmetries

III.1. $\mathcal{N}=1$ supergravity with $\Lambda = 0$

III.2. $\mathcal{N}=1$ supergravity with $\Lambda \neq 0$

IV. Coupling $\mathcal{N}=1$ supergravity to SYM and matter fields

- Scalar kinetic terms and coset spaces

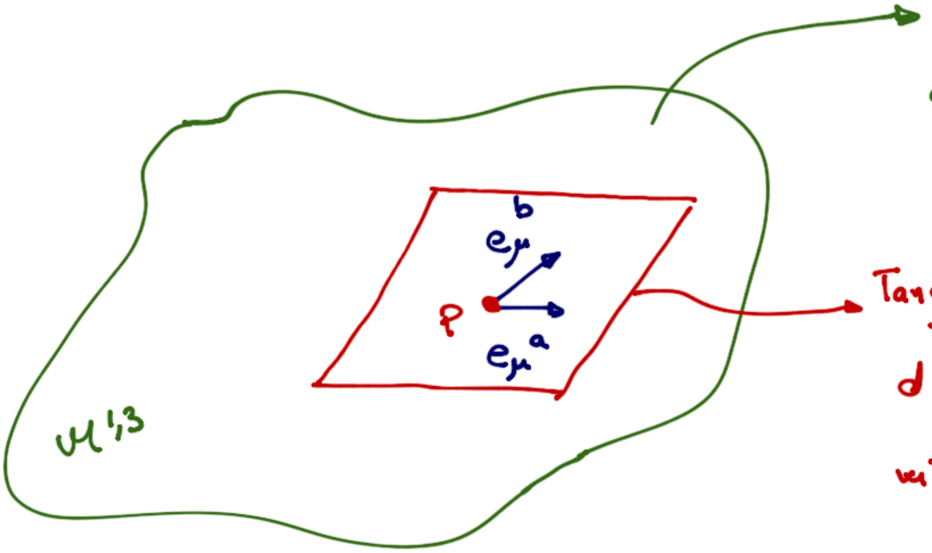
V. Extended $\mathcal{N} \geq 2$ supergravity

VI. Maximal $\mathcal{N}=8$ ungauged supergravity

- Electromagnetic duality

VII. Maximal $\mathcal{N}=8$ gauged supergravity : gaugings and embedding tensor

I. Space-time : frames and connections



space-time $M^{1,3}$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Tangent space (at point p)

$$ds^2 = \eta_{ab}\, e^a e^b$$

with $e^a = \underbrace{e_\mu{}^a}_{\text{frame field}} dx^\mu$

- Locally one can define a "tetrad" $e_\mu{}^a(x)$ such that

$$g_{\mu\nu}(x) = e_\mu{}^a(x)\, e_\nu{}^b(x)\, \eta_{ab}$$

- GCT vs Local Lorentz transf.

$$\begin{cases} \text{GCT}: & T'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu} T^\nu(x) \\ \text{LLT}: & T'^a(x) = \Lambda^a{}_b(x)\, T^b(x) \\ [SO(1,3)] & \text{with } \Lambda^{a'}{}_a \Lambda^{b'}{}_b \eta_{a'b'} = \eta_{ab} \end{cases}$$

Important : Note that frames are not unique (redundancy) as

$$e'_\mu{}^a = \Lambda^a{}_b(x)\, e_\mu{}^b \Rightarrow g_{\mu\nu}' = e'_\mu{}^a e'_\nu{}^b \eta_{ab}$$

$$= e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}$$

This local (gauge) symmetry will introduce a gauge field for the particles (spinors) transforming under the local Lorentz group

=> spin connection $\omega_\mu{}^{ab}$!!

- Connections and covariant derivatives

→ World tensors : Christoffel : $\Gamma_{\mu\nu}{}^{\rho}(g) = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$

$$\Rightarrow \Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$$

(no torsion)

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^{\rho} V_\rho$$

→ Tangent space tensors : Lorentz SO(1,3) generators $(M_{cd})^a{}_b = 2\delta_{[c}^a \eta_{d]b}$

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^{cd} (M_{cd})^a{}_b V^b = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b$$

$$D_\mu V_a = \partial_\mu V_a - \omega_\mu{}^{cd} (M_{cd})^b{}_a V_b = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b$$

} Vector

$$D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \omega_\mu{}^{cd} \underbrace{(\gamma_{cd})_\alpha{}^\beta}_{M_{cd} = \frac{1}{4}\gamma_{cd}} \psi_\alpha$$

} Spinor

- Vielbein postulate : $\nabla_\mu e_\nu{}^a = 0 \Rightarrow D_\mu V^a = e_\nu{}^a \nabla_\mu V^\nu$

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^{\rho} e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \qquad (\times e_a{}^\lambda)$$

$$\Rightarrow e_a{}^\lambda \left(\partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_{\nu b} \right) = \Gamma_{\mu\nu}{}^{\lambda}$$

=> One indep connection !!

⟹ The spin connection is a composite field

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$$

[no torsion]

$$T_{\mu\nu}{}^a \equiv D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b \quad \equiv \text{Torsion}$$

Important: $$T_{\mu\nu}{}^a \overset{!}{=} 0 \Rightarrow \underbrace{\omega_\mu{}^{ab}(e)}_{\text{Levi-Civita connection [torsion free]}} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c$$

II. Actions and symmetries

$$\text{Symmetries}\begin{cases}\text{space-time symmetries}\begin{cases}\text{GCT} \longrightarrow \text{Diff}: x'^\mu = x^\mu - \xi^\mu(x)\\ \text{Local Lorentz} \longrightarrow \psi'_\alpha = \psi_\alpha - \lambda^{ab}[\gamma_{ab}]_\alpha{}^\beta\psi_\beta\end{cases}\\ \text{gauge internal symmetries}\end{cases}$$

$\boxed{s=0}$ Scalar field ϕ [no world or tangent space index]

- G.C.T : $\phi'(x') = \phi(x) \Rightarrow \delta\phi = \mathcal{L}_\xi\phi = \xi^\mu\partial_\mu\phi$
- gauge : No gauge symmetry (matter field)
- action : $$S_\phi = \int d^4x\,\sqrt{-|g|}\,\Big(-\tfrac{1}{2}g^{\mu\nu}\underbrace{\partial_\mu\phi}_{\nabla_\mu\phi = \partial_\mu\phi}\partial_\nu\phi\Big)$$
- EOM : $\Box\phi = 0$

$\boxed{s=1/2}$ Fermion field ψ_α [α ≡ tangent space spinorial index]

- Local Lorentz : $\psi'_\alpha(x) = \Lambda_\alpha{}^\beta\psi_\beta(x) \Rightarrow \delta_\lambda\psi_\alpha = \lambda^{ab}[\gamma_{ab}]_\alpha{}^\beta\psi_\beta$
- gauge : No gauge symmetry (matter field)
- action : $$S_\psi = \int d^4x\,\sqrt{-|g|}\,\Big(+\tfrac{1}{2}\bar{\psi}\gamma^\mu\nabla_\mu\psi - \tfrac{1}{2}\bar{\psi}\overleftarrow{\nabla}_\mu\gamma^\mu\psi\Big)$$

where $$\nabla_\mu\psi_\alpha = \partial_\mu\psi_\alpha + \tfrac{1}{4}\underbrace{\omega_\mu{}^{ab}}_{\text{spin connection}}[\gamma_{ab}]_\alpha{}^\beta\psi_\beta \equiv D_\mu\psi_\alpha$$

$$\bar{\psi} \equiv \psi^t C \overset{!}{=} i\psi^\dagger\gamma^0$$

↳ Majorana condition with $(\gamma^\mu)^t = t_0 t_1 C\gamma^\mu C^{-1}$

NOTE: In the absence of torsion one can integrate by parts S_ψ to write it as

$$S_\psi = \int d^4x \sqrt{-|g|}\; \bar{\Psi}\, \gamma^\mu D_\mu \Psi$$ [Dirac action]

- EOM : $\gamma^\mu D_\mu \Psi_\alpha = 0$

$\boxed{s=1}$ Abelian [Maxwell] vector field A_μ [$\mu \equiv$ world index]

- GCT : $A'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \Rightarrow \delta_\xi A_\mu = \mathcal{L}_\xi A_\mu = \xi^\rho \partial_\rho A_\mu + (\partial_\mu \xi^\rho) A_\rho$
- gauge : $A'_\mu(x) = A_\mu(x) + \nabla_\mu \theta(x) \Rightarrow \delta_\theta A_\mu = \nabla_\mu \theta(x) = \partial_\mu \theta(x)$

  ↳ scalar function $\theta(x)$
- action : $S_A = \int d^4x \sqrt{-|g|} \left(-\frac{1}{4} \underbrace{F_{\mu\nu}} F^{\mu\nu} \right)$

  $\underbrace{F_{\mu\nu}}_{\text{gauge invariant}} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

  ↳ $\Gamma^\rho_{[\mu\nu]} = 0$ [No torsion]
- EOM : $\nabla_\mu F^{\mu\nu} = 0$

$\boxed{s=3/2}$ Gravitino (vector-spinor) field $\Psi_{\mu\alpha}$ [$\mu \equiv$ world index, $\alpha \equiv$ tangent space index]

- GCT : $\Psi'_{\mu\alpha}(x') = \dfrac{\partial x^\nu}{\partial x^\mu} \Psi_{\nu\alpha}(x) \Rightarrow \delta_\xi \Psi_{\mu\alpha} = \mathcal{L}_\xi \Psi_{\mu\alpha} = \xi^\rho \partial_\rho \Psi_{\mu\alpha} + (\partial_\mu \xi^\rho) \Psi_{\rho\alpha}$
- Local Lorentz : $\Psi'_{\mu\alpha}(x) = \Lambda_\alpha{}^\beta \Psi_{\mu\beta}(x) \Rightarrow \delta_\Lambda \Psi_{\mu\alpha} = \lambda^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\mu\beta}$
- gauge : $\Psi'_{\mu\alpha}(x) = \Psi_{\mu\alpha}(x) + \nabla_\mu \epsilon_\alpha(x) \Rightarrow \delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$
- action : $S_\Psi = \int d^4x \sqrt{-|g|}\; \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \underbrace{\nabla_\nu \Psi_\rho}$ [Rarita-Schwinger]

$$\nabla_\nu \Psi_{\rho\alpha} = \partial_\nu \Psi_{\rho\alpha} + \frac{1}{4} \omega_\nu{}^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\rho\beta} - \Gamma^\lambda_{\nu\rho} \Psi_{\lambda\alpha} = D_\nu \Psi_{\rho\alpha} \underbrace{- \Gamma^\lambda_{\nu\rho} \Psi_{\lambda\alpha}}$$

No torsion ⇒ Irrelevant due to $\gamma^{\mu\nu\rho}$

Important : $\delta_{\epsilon} S^{r}_{\Psi} = \int d^4x \sqrt{-|g|} \left(- \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \delta_\epsilon \Psi_\rho \right) \times 2$ ↱ variation w.r.t $\bar{\Psi}$

$$= \int d^4x \sqrt{-|g|} \left(- \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\rho \epsilon \right) \times 2$$

$$= \int d^4x \sqrt{-|g|} \left[\nabla_\rho \underbrace{\left(- \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \epsilon \right)}_{\text{boundary term}} + \nabla_\rho \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \epsilon \right] \times 2$$

(no torsion)

$$= - \int d^4x \sqrt{-|g|}\ \bar{\epsilon} \gamma^{\mu\nu\rho} \nabla_{[\nu} \nabla_{\rho]} \Psi_\mu \times 2$$

$$= - \int d^4x \sqrt{-|g|}\ \bar{\epsilon} \gamma^{\mu\nu\rho} \frac{1}{2} \left(\frac{1}{4} R_{\nu\rho ab} [\gamma^{ab}] \right) \Psi_\mu \times 2$$

$$= - \frac{1}{4} \int d^4x \sqrt{-|g|}\ \bar{\epsilon} \gamma^{\mu\nu\rho} \gamma_{ab} R_{\mu\nu}{}^{ab} \Psi_\rho = (*)$$

NOTE: $\gamma^{\mu\nu\rho} \gamma_{ab} = \gamma^{\mu\nu\rho}{}_{ab} + 6 \gamma^{[\mu\nu}{}_b \delta^{\rho]}_a + 6 \gamma^{[\mu} \delta^{\nu\rho]}_{ba}$

- $\gamma^{\mu\nu\rho}{}_{ab} R_{\mu\nu}{}^{ab} = 0$ [only in D=4]
- $6 \gamma^{[\mu\nu}{}_b \delta^{\rho]}_a R_{\mu\nu}{}^{ab} = 2 \gamma^{\mu\nu}{}_b \delta^\rho_a R_{\mu\nu}{}^{ab} + 4 \gamma^{\nu\rho}{}_b \delta^\mu_a R_{\mu\nu}{}^{ab}$

Torsion-free Bianchi id.

$$= 2 \gamma^{\mu\nu b} R_{\mu\nu}{}^{\rho}{}_b + 4 \gamma^{\nu\rho b} R_{\mu\nu}{}^{\mu}{}_b$$

$R_{[\mu\nu\rho]}{}^a = 0$

$$= \boxed{2 \gamma^{\mu\nu b} R_{\mu\nu}{}^{\rho}{}_b} + \boxed{4 \gamma^{\nu\rho b} R_{\nu b}} \rightarrow 0$$

[sym]

- $6 \gamma^{[\mu} \delta^{\nu\rho]}_{ba} R_{\mu\nu}{}^{ab} = 4 \gamma^{\mu} \delta^{\nu\rho}_{ba} R_{\mu\nu}{}^{ab} + 2 \gamma^{\rho} \delta^{\mu\nu}_{ba} R_{\mu\nu}{}^{ab}$

$$= 4 \gamma^\mu R_{\mu\nu}{}^{\rho\nu} + 2 \gamma^\rho R_{\mu\nu}{}^{\nu\mu}$$

$$= 4 \gamma^\mu R_\mu{}^\rho - 2 \gamma^\rho R$$

$$= 4 \gamma^\mu \left(R_\mu{}^\rho - \frac{1}{2} \delta^\rho_\mu R \right)$$

NOTE: $\bar{\chi} \gamma_{\mu_1 \dots \mu_n} \lambda = t_n \bar{\lambda} \gamma_{\mu_1 \dots \mu_n} \chi$ with $t_0 = t_3 = +1$, $t_1 = t_2 = -1$
(4D)

$$(*) = -\int d^4x \sqrt{-|g|}\,\left(R_\mu{}^\rho - \tfrac{1}{2}\,\delta_\mu^\rho R \right) \bar{\epsilon}\,\gamma^\mu \Psi_\rho$$

$$= -\int d^4x \sqrt{-|g|}\,\underbrace{\left(R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R \right)}_{\text{Einstein tensor } G_{\mu\nu}\ !!} \bar{\epsilon}\,\gamma^\mu \Psi^\nu$$

Therefore:

i) $\delta_\epsilon S_\Psi = 0$ in Minkowski space-time where $\nabla_\mu \Psi_\rho = \partial_\mu \Psi_\rho$ and $\nabla_\mu \epsilon_\alpha = \partial_\mu \epsilon_\alpha$.
This is the work by Rarita-Schwinger
(free $s=3/2$ field has ϵ_α-gauge invariance)

ii) $\delta_\epsilon S_\Psi \propto \underbrace{G_{\mu\nu}}_{\text{gravity}} \bar{\epsilon}\,\gamma^\mu \Psi^\nu \neq 0$ $\Rightarrow$ This suggests that gravity has something to say about general ϵ_α-gauge invariance !!

- EOM : $\gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho = 0$

  or $\gamma^{\mu\nu\rho} D_\nu \Psi_\rho = 0$ (no torsion)

$\boxed{s=2}$ Metric field $g_{\mu\nu}$ [$\mu,\nu \equiv$ world indices)

- GCT : $g'_{\mu\nu}(x') = \dfrac{\partial x^\rho}{\partial x'^\mu}\dfrac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x) \Rightarrow \delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2\,\partial_{(\mu}\xi^\rho\, g_{\nu)\rho}$
- gauge : No gauge symmetry in GR, but ... ϵ_α-gauge symmetry?
- action : $S_g = \dfrac{1}{2\kappa^2}\displaystyle\int d^4x \sqrt{-|g|}\, g^{\mu\nu} R_{\mu\nu}$ [Einstein - Hilbert]
(2nd order formalism)
- EOM : $G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R = 0$

$\kappa^2 = 8\pi G_N$ = grav coupling cte

Proposal : Can gravity compensate for the lack of ϵ_α-invariance of S'_Ψ ?

(i) The fact that $\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$ appears in $\delta_\epsilon S_\Psi$ is highly remarkable.

(ii) We know from supersymmetry that $\overset{(\phi \in \mathbb{C})}{s=0}$ and $\overset{(\text{Weyl } \psi)}{s=1/2}$ can furnish a susy chiral multiplet transforming as

$$\boxed{\begin{aligned} \delta_\epsilon \phi &\propto \bar{\epsilon}\, \psi \\ \delta_\epsilon \psi &\propto \bar{\epsilon}\, \sigma^\mu (\partial_\mu \phi) \end{aligned}}$$

} Free theory or Wess-Zumino model in flat space {
- $\delta \mathcal{L}$ = boundary terms
- susy algebra on-shell

(iii) We also know from supersymmetry that $\overset{(\text{Majorana } \lambda_\alpha)}{s=1/2}$ and $\overset{(A_\mu)}{s=1}$ can furnish a susy vector multiplet transforming as

$$\boxed{\begin{aligned} \delta_\epsilon A_\mu &\propto \bar{\epsilon}\, \gamma_\mu \lambda \\ \delta_\epsilon \lambda &\propto \bar{\epsilon}\, \gamma_{\mu\nu} F^{\mu\nu} \end{aligned}}$$

} Super Yang-Mills theory in flat space

(iv) Could there be a similar story involving $\overset{(\Psi)}{s=3/2}$ and $\overset{(e)}{s=2}$??

Beautiful idea !!

III.1 $\mathcal{N}=1$ Supergravity with $\Lambda=0$

We have collected already quite some clues for the local ϵ_α-invariance to actually be local supersymmetry involving not only the $s=3/2$ field but also the $s=2$ metric field. We are going to work this idea out now.

Let's start from:

$$S_g = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\,g^{\mu\nu}R_{\mu\nu} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\,e_a{}^{\mu}\,e_b{}^{\nu}\,R_{\mu\nu}{}^{ab}(e)$$

$$S_\Psi = -\underbrace{\frac{1}{2\kappa^2}}_{\text{we have introduced this constant}}\int d^4x\,\sqrt{-|g|}\,\bar{\Psi}_\mu\,\gamma^{\mu\nu\rho}\,\underbrace{\nabla_\nu\Psi_\rho}_{D_\nu[\omega]\Psi_\rho}$$

Important: We are considering a space-time without torsion

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$$

so integration by parts works normally: $\int dx\,\sqrt{-|g|}\,\nabla_\mu V^\mu = \int dx\,\partial_\mu\left(\sqrt{-|g|}\,V^\mu\right)$

We already know that $\delta_\epsilon\Psi_{\mu\alpha} = \nabla_\mu\epsilon_\alpha = D_\mu\epsilon_\alpha$. But the question is: what should be $\delta_\epsilon e_\mu{}^a$?

$$\delta_\epsilon\, e_\mu{}^a = c\,\bar{\epsilon}\,\gamma^a\,\Psi_\mu$$ (what else could it be?)

with c being a constant to be fixed. Note that this combination already appeared in $\delta_\epsilon S_\Psi \Rightarrow$ ✓

Let's compute things explicitely

$$S = S_g(e) + S_\psi(e,\psi)$$

$$\delta_e S = \underbrace{\left(\underbrace{\frac{\delta S_g}{\delta e}}_{A.1} + \underbrace{\frac{\delta S_\psi}{\delta e}}_{A.2}\right)}_{A}\delta_e e + \underbrace{\frac{\delta S_\psi}{\delta \Psi}\delta_e \Psi}_{B \to \text{This we already computed}}$$

B :
$$\frac{\delta S_\psi}{\delta \Psi_{\mu\alpha}}\delta_e \Psi_{\mu\alpha} = \frac{1}{2K^2}\int d^4x \underbrace{e}_{\sqrt{|-g|}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\bar{\epsilon}\gamma^\mu \Psi^\nu$$

A.1 : This is, by definition, the computation of the Einstein equations in terms of frames $e_\mu{}^a$:

$$\frac{\delta S_g}{\delta e_\mu{}^a}\delta e_\mu{}^a = \frac{1}{2K^2}\int d^4x \left[\delta(e) R + 2e\, \delta(e_a{}^\mu) e_b{}^\nu R_{\mu\nu}{}^{ab} + e\, e_a{}^\mu e_b{}^\nu \delta(R_{\mu\nu}{}^{ab})\right] = (*)$$

- $\delta(|M|) = |M| \operatorname{Tr}(M^{-1}\delta M) \Rightarrow \delta(e) = e\, e_a{}^\mu \delta e_\mu{}^a$
- $e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu \Rightarrow \delta e_\mu{}^a e_a{}^\nu + e_\mu{}^a \delta e_a{}^\nu = 0$

$$\Rightarrow \delta e_a{}^\nu e_\mu{}^a = - e_a{}^\nu \delta e_\mu{}^a \qquad (\times e_b{}^\mu)$$

$$\Rightarrow \delta e_b{}^\nu = - e_a{}^\nu e_b{}^\mu \delta e_\mu{}^a$$

- $\delta(R_{\mu\nu}{}^{ab}) = D_\mu \delta\omega_\nu{}^{ab} - D_\nu \delta\omega_\mu{}^{ab} \underset{\hookrightarrow \text{no torsion}}{=} \nabla_\mu \delta\omega_\nu{}^{ab} - \nabla_\nu \omega_\mu{}^{ab} \underset{\hookrightarrow \text{no torsion}}{\Rightarrow}$ Total derivative

$$(*) = \frac{1}{2K^2}\int d^4x\, e \left(e_a{}^\mu R\, \delta e_\mu{}^a - 2\, e_b{}^\nu e_c{}^\mu e_a{}^\rho R_{\mu\nu}{}^{ab} \delta e_\rho{}^c\right)$$

$$= \frac{1}{2K^2}\int d^4x\, e \left(e_a{}^\mu R\, \delta e_\mu{}^a - 2 R_\mu{}^\rho e_c{}^\mu \delta e_\rho{}^c\right) = (\text{relabeling})$$

$$= \frac{1}{2K^2}\int d^4x\, e\left(e_a{}^\rho R - 2\, R_\mu{}^\rho\, e_a{}^\mu\right)\delta e_\rho{}^a = (*)$$

Now we plug: $\delta_\epsilon e_\rho{}^a = c\,\bar\epsilon\gamma^a\Psi_\rho$

$$(*) = -\frac{c}{K^2}\int d^4x\, e\left(R_\mu{}^\rho e_a{}^\mu - \frac{1}{2}e_a{}^\rho R\right)\bar\epsilon\gamma^a\Psi_\rho$$

$$= -\frac{c}{K^2}\int d^4x\, e\left(R_\mu{}^\rho - \frac{1}{2}\delta_\mu^\rho R\right)\bar\epsilon\gamma^\mu\Psi_\rho$$

$$= -\frac{c}{K^2}\int d^4x\, e\left(R_{\mu\sigma} - \frac{1}{2}g_{\mu\sigma} R\right)\bar\epsilon\gamma^\mu\Psi^\sigma$$

<u>A.2:</u> $$\frac{\delta S_\Psi}{\delta e_\rho{}^a}\,\delta e_\rho{}^a = -\frac{1}{2K^2}\int d^4x\,\delta\left[e\,\bar\Psi_\mu\gamma^{\mu\sigma\rho}\nabla_\sigma\Psi_\rho\right] \overset{\substack{\text{lowest order }\Psi\\ \downarrow}}{=} 0$$

NOTE: $\delta e = e\, e_a{}^\mu\,\delta e_\mu{}^a = c\, e\, e_a{}^\mu\,\bar\epsilon\gamma^a\Psi_\mu \Rightarrow \epsilon\Psi^3\text{-terms}$

$$\delta(\gamma^{\mu\sigma\rho}) = \delta(e_a{}^\mu e_b{}^\sigma e_c{}^\rho)\gamma^{abc}$$
$$= \delta e_a{}^\mu\, e_b{}^\sigma e_c{}^\rho\gamma^{abc} + \ldots$$
$$= -e_d{}^\mu e_a{}^\lambda\,\delta e_\lambda{}^d\; e_b{}^\sigma e_c{}^\rho\gamma^{abc} + \ldots$$
$$= -e_d{}^\mu\,\delta e_\lambda{}^d\,\gamma^{\lambda\sigma\rho} + \ldots$$
$$= -e_d{}^\mu\,\gamma^{\lambda\sigma\rho}(\bar\epsilon\gamma^d\Psi_\lambda) \Rightarrow \epsilon\Psi^3\text{-terms}$$

$$\delta(\nabla_\sigma\Psi_\rho) = \delta(D_\sigma\Psi_\rho) = \delta\left(\partial_\sigma\Psi_\rho + \frac{1}{4}\omega_\sigma{}^{ab}[\gamma_{ab}]\psi_\rho\right)$$
$$= \frac{1}{4}\,\delta(\omega_\sigma{}^{ab})\,[\gamma_{ab}]\Psi_\rho$$
$$\Rightarrow \epsilon\Psi^3\text{-terms}$$

$\downarrow$

schematically: $\omega = e^{-1}\partial e - e^{-1}e^{-1}e\,\partial e$

Wrapping up B, (A.1) and (A.2) we obtain at lowest order ($\bar\epsilon\Psi$-terms) in fermions the following result:

$$\delta_\epsilon S = \delta_\epsilon(S_g + S_\Psi) = \underbrace{\left(\tfrac{1}{2} - c\right)}_{c = \frac{1}{2}\ !!}\frac{1}{\kappa^2}\int d^4x\, e\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right)\bar\epsilon\gamma^\mu\Psi^\nu$$

Therefore we have an ϵ_α-invariant action at lowest order in fermionic fields with transformation rules:

$$(i)\quad \boxed{\begin{aligned}\delta_\epsilon e_\mu{}^a &= \tfrac{1}{2}\,\bar\epsilon\gamma^a\Psi_\mu\\ \delta_\epsilon\Psi_\mu &= D_\mu\epsilon\end{aligned}} \quad \Rightarrow \quad \text{SUGRA}\ !!$$

Remark: It was crucial that $\nabla_\nu\Psi_\rho$ in the action R-S action appears with a $\gamma^{\mu\nu\rho}$. This allowed us to replace $\nabla_\nu\Psi_\rho = D_\nu\Psi_\rho$ fitting well with $D_\mu\epsilon$ and

$$D_{[\mu}D_{\nu]} = R_{\mu\nu} \quad \Rightarrow \quad GR\ !!$$

However, there is an easier way to see how GR emerges from the SUSY algebra of supersymmetry:

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]e_\mu{}^a &= \delta_{\epsilon_1}\left(\tfrac{1}{2}\bar\epsilon_2\gamma^a\Psi_\mu\right) - (1\leftrightarrow 2)\\
&= \tfrac{1}{2}\bar\epsilon_2\gamma^a\nabla_\mu\epsilon_1 - (1\leftrightarrow 2) = \tfrac{1}{2}\bar\epsilon_2\gamma^a D_\mu\epsilon_1 - (1\leftrightarrow 2)\\
&= \tfrac{1}{2}\bar\epsilon_2\gamma^a D_\mu\epsilon_1 - \tfrac{1}{2}\bar\epsilon_1\gamma^a D_\mu\epsilon_2\\
t_1 = -1 \quad &= \tfrac{1}{2}\left(\bar\epsilon_2\gamma^a D_\mu\epsilon_1 + D_\mu\bar\epsilon_2\gamma^a\epsilon_1\right) = D_\mu\overbrace{\left(\tfrac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1\right)}^{\equiv\,\xi^a}\\
&= D_\mu\xi^a \quad \text{with } \xi^a = \tfrac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1\end{aligned}$$

Therefore we have found that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = D_\mu \xi^a$

Moreover $D_\mu \xi^a = \nabla_\mu \xi^a$ and

$$\delta_\xi e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a = \xi^\rho \partial_\rho e_\mu{}^a + \partial_\mu \xi^\rho e_\rho{}^a = \text{(explicit covariantisation)}$$

$$= \xi^\rho \underbrace{\nabla_\rho e_\mu{}^a}_{0} + \cancel{\xi^\rho \Gamma_{\rho\mu}{}^\lambda} e_\lambda{}^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b + \underbrace{\nabla_\mu \xi^\rho e_\rho{}^a}_{\nabla_\mu \xi^a} - \cancel{\Gamma_{\mu\lambda}{}^\rho \xi^\lambda} e_\rho{}^a$$

$\equiv$ no torsion

$$= \nabla_\mu \xi^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b$$

Then : $\nabla_\mu \xi^a = \delta_\xi e_\mu{}^a + \underbrace{\xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b}_{\delta_\Lambda e_\mu{}^a \text{ with } \Lambda^a{}_b = \xi^\rho \omega_\rho{}^a{}_b} = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$

So we have shown that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$

$\Rightarrow$ Local SUSY algebra implies $\begin{cases} \text{GCT} \\ \text{Local Lorentz} \end{cases} \Rightarrow$ General Relativity !!

→ Next question : Are the transformations (i) a symmetry of the action to all orders in fermions ?

Unfortunately the answer is no . . . ☹ . . .

It had been surprising otherwise as the Rarita - Schwinger action describes a free $s = 3/2$ field Ψ_ρ whereas the E-H action describing gravity is highly interacting !!

IMPORTANT : The computation we have performed did not rely on the dimension D or any special feature but having a real Ψ. Therefore, it works in the same way in any D. What makes certain dimensions special is that local SUSY can be stated at all orders in fermions . For example : $\mathcal{N}=1$ D=4 or D=11

To have $\mathcal{N}=1$ and $D=4$ sugra to all orders in fermions one has to introduce Ψ^4-terms both in the R-S action and in the transformation rules :

$$S_{\Psi} = -\frac{1}{2\kappa^2}\int d^4x\, e\left\{ \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{16}\left[(\bar{\Psi}^\rho \gamma^\mu \Psi^\nu)(\bar{\Psi}_\rho \gamma_\mu \Psi_\nu + 2\bar{\Psi}_\rho \gamma_\nu \Psi_\mu) - 4(\bar{\Psi}_\mu \gamma^\rho \Psi_\rho)(\bar{\Psi}^\mu \gamma^\rho \Psi_\rho) \right] \right\}$$

with a torsion-free $D_\mu \Psi_\rho = \partial_\mu \Psi_\rho + \frac{1}{4}\omega_\mu{}^{ab}(e)\,[\gamma_{ab}]\,\Psi_\rho$

The SUSY transformation rules (i) must be also modified with higher-order fermion terms :

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\epsilon}\gamma^a \Psi_\mu$$
$$\delta \Psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\left(\omega_\mu{}^{ab}(e) + K_\mu{}^{ab}\right)[\gamma_{ab}]\,\epsilon$$

with $K_{\mu\nu\rho} = -\frac{1}{4}\left(\bar{\Psi}_\mu \gamma_\rho \Psi_\nu - \bar{\Psi}_\nu \gamma_\mu \Psi_\rho + \bar{\Psi}_\rho \gamma_\nu \Psi_\mu \right)$

IMPORTANT : We see that local SUSY at the full fermion level can be very conveniently described using gravitino torsion. This is an example of how torsion appears in Physics. Some 1st and 1.5 order formulations of supergravity make all these structures manifest and render the problem of full local supersymmetry tractable !!

Some involved fermionic manipulations "Fierzing" are also required in the process.

III.2 $\mathcal{N}=1$ Supergravity with $\Lambda \neq 0$

Question : Is local supersymmetry compatible with a cosmological constant Λ ? What modifications are required ?

$$S_g = S_{E-H} + S_\Lambda = \frac{1}{2\kappa^2}\int d^4x \; e \left(e_a{}^\mu e_a{}^\nu R_{\mu\nu}{}^{ab} - \Lambda \right)$$

$$S_\Psi = S_{R-S} + S_{mass} = -\frac{1}{2\kappa^2}\int d^4x \; e \left(\bar\Psi_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho + m\, \bar\Psi_\mu \gamma^{\mu\nu} \Psi_\nu \right)$$

NOTE : A mass term of the form $g^{\mu\nu}\bar\Psi_\mu \Psi_\nu$ does not describe the right number of d.o.f.

SUSY transformation rules :

(ii)
$$\boxed{\begin{aligned} \delta e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon \gamma^a \Psi_\mu \\ \delta \Psi_\mu &= D_\mu \epsilon - g\, \gamma_\mu \epsilon \end{aligned}}$$
=> SUGRA + Λ !!

Let's repeat the computation we performed in the case $\Lambda = m = g = 0$

The new pieces to compute are :

- $\dfrac{\delta S_{EH}}{\delta e_\mu{}^a}$ = same computation $= -\dfrac{1}{2\kappa^2}\displaystyle\int d^4x \; e \; G_{\mu\nu}\, \bar\epsilon \gamma^\mu \Psi_\nu$ ▲

- $\dfrac{\delta S_\Lambda}{\partial e_\mu{}^a} = \dfrac{-1}{2\kappa^2}\displaystyle\int d^4x \; \Lambda \; \delta(e) = \dfrac{-1}{2\kappa^2}\int d^4x \; e \Lambda e_a{}^\mu \delta e_\mu{}^a =$

  $= \dfrac{-\Lambda}{4\kappa^2}\displaystyle\int d^4x \; e \; \bar\epsilon \gamma^\mu \Psi_\mu$ ■

- $\dfrac{\delta S_{R-S}}{\delta e_\mu{}^a} = 0$ (same computation at lowest order in fermions)

- $$\frac{\delta S_{R-S}}{\delta \Psi_\mu} = -\frac{1}{2\kappa^2}\int d^4x\, e \left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \overbrace{\delta_\epsilon \Psi_\rho}^{\text{new transf (ii)}}\right)\times 2 = \text{same computation}$$

$$= \underbrace{\frac{1}{2\kappa^2}\int d^4x\, e\, G_{\mu\nu}\, \bar{\epsilon}\gamma^\mu \Psi^\nu}_{\text{Piece from } D_\rho \epsilon} - \underbrace{\frac{g}{\kappa^2}\int d^4x\, e\, \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\gamma_\rho \epsilon}_{A.3} = (a)$$

A.3 : $$-\frac{g}{\kappa^2}\int d^4x\, e\, \nabla_\nu \bar{\Psi}_\mu \underbrace{\gamma^{\mu\nu\rho}\gamma_\rho}_{2\gamma^{\mu\nu}} \epsilon = +\frac{2g}{\kappa^2}\int d^4x\, e\, \nabla_\mu \bar{\Psi}_\nu \gamma^{\mu\nu}\epsilon$$

$t_2 = -1$
$$= -\frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu \Psi_\nu = -\frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu} D_\mu \Psi_\nu$$

$$(a) = \frac{1}{2\kappa^2}\int d^4x\, e\, \underset{\blacktriangle}{G_{\mu\nu}}\, \bar{\epsilon}\gamma^\mu \Psi^\nu - \frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\underset{\bullet}{\gamma^{\mu\nu}} D_\mu \Psi_\nu$$

- $$\frac{\delta S'_{mass}}{\delta e_a{}^\mu} = -\frac{m}{2\kappa^2}\int d^4x\, \delta\left[e\, e_a{}^\mu e_b{}^\nu \bar{\Psi}_\mu \gamma^{ab}\Psi_\nu\right] \Rightarrow \epsilon\Psi^3\text{-terms}$$

- $$\frac{\delta S_{mass}}{\delta \Psi_\nu} = -\frac{m}{2\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \gamma^{\mu\nu}\delta\Psi_\nu \times 2 =$$

$$= -\frac{m}{\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \gamma^{\mu\nu} D_\nu \epsilon + \frac{mg}{\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \overbrace{\gamma^{\mu\nu}\gamma_\nu}^{3\gamma^\mu}\epsilon$$

$t_1 = t_2 = -1$
$$= +\frac{m}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\underset{\bullet}{\gamma^{\mu\nu}} D_\mu \Psi_\nu - \frac{3mg}{\kappa^2}\int d^4x\, e\, \underset{\blacksquare}{\bar{\epsilon}}\gamma^\mu \Psi_\mu$$

For the various terms to cancel one needs:

$m = 2g$, $\frac{\Lambda}{4} = -3mg \Rightarrow \Lambda = -12mg = -24g^2 < 0$

IMPORTANT: Invariance to all order in fermions is achieved by adding the torsion terms to $\delta\Psi_\mu$ as in the $\Lambda = 0$ case.

IV. Coupling $\mathcal{N}=1$ Supergravity to SYM and Matter fields

We are now going to couple the $\mathcal{N}=1$ supergravity multiplet to other multiplets of $\mathcal{N}=1$ supersymmetry: $\underbrace{\text{vector mult.}}_{\text{Super Yang-Mills (SYM)}}$ & $\underbrace{\text{chiral mult.}}_{\text{Matter}}$

MULTIPLET		-2	-3/2	-1	1/2	0	1/2	1	3/2	2
Gravity	$g_{\mu\nu}$	1								1
	Ψ_μ		1						1	
Vector	A_μ			1				1		
	λ_α				1		1			
Chiral	ψ_α				1		1			
	$\phi \in \mathbb{C}$					11				

Table : $\mathcal{N}=1$ multiplets, fields and helicity states

* Free theory : Local susy transformations [$\epsilon_\alpha(x)$ parameter]

$s = 2, \frac{3}{2}$

$$\boxed{\begin{aligned} \delta_\epsilon e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_\mu \\ \delta_\epsilon \Psi_\mu &= D_\mu \epsilon \end{aligned}}$$

SUGRA Multiplet

$s = 1, \frac{1}{2}$

$$\boxed{\begin{aligned} \delta_\epsilon A_\mu &= -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda \\ \delta_\epsilon \lambda &= \frac{1}{4}\bar{\epsilon}\gamma_{\mu\nu}F^{\mu\nu} \end{aligned}}$$

Vector multiplet

$s = \frac{1}{2}, 0$

$$\boxed{\begin{aligned} \delta_\epsilon \phi &= \frac{1}{\sqrt{2}}\bar{\epsilon}\psi \\ \delta_\epsilon \psi &= \frac{1}{\sqrt{2}}\bar{\epsilon}\sigma^\mu(\partial_\mu\phi) \end{aligned}}$$

Chiral Multiplet

The bosonic Lagrangian

We focus on the bosonic terms in the action $\mathcal{S}$. The fermionic terms then follow from requiring $\mathcal{N}=1$ local supersymmetry.

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, e\Big[R - \frac{1}{4}\overbrace{\mathrm{Im}\,\mathcal{N}_{ab}(\phi)}^{\text{“}g^{-2}\delta_{ab}\text{-like”}} F_{\mu\nu}{}^{a} F^{\mu\nu b} - \frac{1}{4}\frac{1}{e}\overbrace{\mathrm{Re}\,\mathcal{N}_{ab}(\phi)}^{\text{“}\frac{1}{8\pi^2}\Theta\,\delta_{ab}\text{-like”}} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{a} F_{\rho\sigma}{}^{b}$$

$$- \underbrace{K_{i\bar{j}}(\phi,\bar{\phi})}_{\text{“scalar geometry”}} \nabla_\mu \phi^i \nabla^\mu \bar{\phi}^{\bar{j}} - V(\phi,\bar{\phi}) + \text{fermionic terms.}$$

with $\phi^i \in \mathbb{C}$ and $A_\mu^a \in \mathbb{R}$ spanning an internal gauge symmetry G_0

- $i = 1,\ldots,n_c$ chiral multiplets
- $a = 1,\ldots,n_v$ vector multiplets $\Rightarrow$ $n_v = \dim(G_0)$

We will consider matter charged under G_0 :

$$\nabla_\mu \phi^i = \partial_\mu \phi^i - A_\mu{}^a \underbrace{\delta_a \phi^i}_{\text{linear sym} \Rightarrow \delta_a\phi^i = \overbrace{(t_a)}^{\text{generators of } R[G_0]}{}^{i}{}_{j}\,\phi^j \equiv k_a{}^i(\phi)} = \partial_\mu \phi^i - A_\mu{}^a k_a{}^i(\phi)$$

$$\nabla_\mu \bar{\phi}^{\bar{i}} = (\nabla_\mu \phi^i)^*$$

* Scalar geometry and $V(\phi,\bar{\phi})$

$\mathcal{N}=1$ supersymmetry requires the scalar geometry to be a complex Kähler manifold. This implies

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial\phi^i\,\partial\bar{\phi}^{\bar{j}}} \equiv \text{“Kähler metric”}$$

in terms of a “Kähler potential” $K(\phi,\bar{\phi}) \in \mathbb{R}$.

NOTE: Demanding gauge invariance of $K_{i\bar{j}}(\phi,\bar{\phi})$ under a a gauge G_0 transformation with parameters θ^a $(a=1,\dots,\dim G_0)$

- $\delta_\theta \phi^i = \theta^a(x)\,(t_a)^i{}_j\,\phi^j = \theta^a\,\mathcal{K}_a{}^i(\phi)$
- $\delta_\theta \bar{\phi}^{\bar{i}} = \theta^a(x)\,(t_a^*)^{\bar{i}}{}_{\bar{j}}\,\bar{\phi}^{\bar{j}} = \theta^a\,\bar{\mathcal{K}}_a{}^{\bar{i}}(\bar{\phi})$ [conjugate]

$\delta_\theta K_{i\bar{j}} = 0$ ⇒ $\mathcal{K}_a{}^i(\phi)$ are a subset (labelled by a) of the Killing vectors of the scalar geometry determined by the $K_{i\bar{j}}$ metric !!

↳ isometry of the scalar geometry !!

For cosets $\frac{G}{H}$ ⇒ $G_0 \subset G$

gauge symmetry ↲ ↳ scalar geometry

Moreover

$$\mathcal{K}_a{}^i(\phi) = -i\,K^{i\bar{j}}(\phi,\bar{\phi})\,\frac{\partial P_a(\phi,\bar{\phi})}{\partial\bar{\phi}^{\bar{j}}}$$

where $P_a(\phi,\bar{\phi})$ are called “moment maps” or “Killing prepotentials” and enter

$$\mathcal{D}_\mu\phi^i = \partial_\mu\phi^i - A_\mu{}^a\,\mathcal{K}_a{}^i(\phi)$$

The moment maps P_a are then expressed as:

$$P_a = -\frac{i}{2}\left[\mathcal{K}_a{}^i\,\partial_i K - \bar{\mathcal{K}}_a{}^{\bar{j}}\,\partial_{\bar{j}} K\right] - r_a$$

with $r_a = \delta_{a\,U(1)}\,\underbrace{\xi_{FI}}_{\text{Fayet-Iliopoulos} \equiv \text{arbitrary parameters}}$ only for Abelian factors in G_0.

Lastly the scalar potential $V(\phi,\bar{\phi})$ is given by

$\kappa^2 = 8\pi G_N$

$$V = e^{\kappa^2 K}\Big[\underbrace{\overbrace{D_i W}^{e^{-\frac{1}{2}\kappa^2 K} F^i} K^{i\bar{j}} D_{\bar{j}} \overline{W}}_{\text{F-terms} > 0} - 3\kappa^2 |W|^2\Big] + \underbrace{\frac{1}{2} \overbrace{P_a \, \mathrm{Im}(N^{-1})^{ab} P_b}^{D^a}}_{\text{D-terms} > 0}$$

in terms of an arbitrary "holomorphic superpotential" $W(\phi)$ where the Kähler derivatives read:

$$D_i W = \partial_i W + \kappa^2 (\partial_i K) W$$
$$D_{\bar{i}} \overline{W} = \partial_{\bar{i}} \overline{W} + \kappa^2 (\partial_{\bar{i}} K) \overline{W}$$

Important : Theory defined by : $\underbrace{G_0}_{\text{gauge group + representation } R[G_0] \text{ for } \phi^i} \oplus N_{ab}(\phi), K(\phi,\bar{\phi}), W(\phi) \oplus P_a(\phi,\bar{\phi})$

* Local susy transformations & susy breaking.

The local susy transformations take the generic form

$$\delta_\epsilon \text{ Fermion} \sim \bar{\epsilon} \text{ Boson} \quad , \quad \delta_\epsilon \text{ Boson} \sim \bar{\epsilon} \text{ Fermion}$$

=> Lorentz invariance at the vacuum requires $\langle \text{Fermion} \rangle = 0$ and consequently $\langle \delta_\epsilon \text{ Boson} \rangle = 0$ always.

=> Lorentz invariance at the vacuum permits $\langle \text{Boson} \rangle \neq 0$ and consequently $\langle \delta_\epsilon \text{ Fermion} \rangle \begin{array}{l} = 0 \rightarrow \text{susy preserved} \\ \neq 0 \rightarrow \text{susy broken (spontaneously)} \end{array}$

Let us look at the δ_ϵ Fermions in the interacting theory:

Gravitino : $$\delta_\epsilon \Psi_\mu = D_\mu \epsilon + \frac{1}{2}\kappa^2 e^{\frac{1}{2}\kappa^2 K} W \bar{\epsilon}\gamma_\mu$$

Chiralini : $$\delta_\epsilon \psi^i = \frac{1}{\sqrt{2}}\left[\bar{\epsilon}\sigma^\mu(\partial_\mu \phi^i) - F^i \epsilon\right]$$

Gaugini : $$\delta_\epsilon \lambda^a = \frac{1}{4}\bar{\epsilon}\gamma_{\mu\nu}F^{\mu\nu a} + \frac{1}{2} i \bar{\epsilon}\underbrace{\gamma_*}_{\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3} D^a$$

with

- $F^i = e^{\frac{1}{2}\kappa^2 K} K^{i\bar{j}} D_{\bar{j}}\bar{W}$ => F-term in $V(\phi,\bar{\phi})$
- $D^a = \mathrm{Im}(N^{-1})^{ab} P_b$ => D-term in $V(\phi,\bar{\phi})$

As a result one has that

$$\text{SUSY}\begin{cases}\delta_\epsilon \psi^i = 0 \\ \delta_\epsilon \lambda^a = 0\end{cases} \Rightarrow \begin{matrix}\langle F^i\rangle = 0 \\ \langle D^a\rangle = 0\end{matrix} \Rightarrow V = -3e^{\kappa^2 K}\kappa^2|W|^2 < 0$$

AdS vacuum !!

NOTE : If SUSY is broken the gravitino $\Psi_{\mu\alpha}$ gets a mass and acquires a longitudinal mode $\partial_\mu \eta_\alpha$ by eating up the goldstino associated to the direction of ~~SUSY~~.

Ex: F-term breaking => $\eta \propto F_i \psi^i$ and $m_{3/2}^2 = \kappa^4 e^{\kappa^2 K}|W|^2$.

NOTE: The existence of a de Sitter vacuum requires susy to be broken => ϕ^i scalars relevant for cosmology !!

[late-time cosmic acceleration, inflation, ...]

* Global susy : Switching off Gravity [$\kappa^2 \to 0$]

In order to go from local susy (supergravity) to global susy (supersymmetric field theory) one switches off gravity by setting $\kappa^2 \to 0$. As a result :

- $g_{\mu\nu} \to \eta_{\mu\nu}$ (Minkowski space-time)
- $\Psi_{\mu\alpha} \to 0$ (no gauge field for a global symmetry)
- $F_i \to F_i = \partial_i W$
- $D^a \to D^a$ (gauge structure unaffected)

$$\Rightarrow \quad V = K^{i\bar{j}} F_i \bar{F}_{\bar{j}} + \frac{1}{2} \operatorname{Im} N_{ab} D^a D^b \geq 0$$

Therefore : $V > 0 \Rightarrow F_i \neq 0$ and/or $D^a \neq 0 \Rightarrow$ Susy breaking !!

NOTE: Susy field theories are very interesting playgrounds where to discover universality classes of phenomena both classically and also at the quantum level.

* Scalar kinetic terms and "coset" spaces

The scalar kinetic terms can be understood geometrically from a "fictitious" (or auxiliary) scalar space perspective where scalar fields $\phi_i \in \mathbb{R}$ $(i=1,\ldots,N)$ play the role of coordinates :

$$S_\phi = \int d^4x \sqrt{-|g|} \left[- \underbrace{K_{ij}(\phi)}_{\text{"metric" in fied space}} \partial_\mu \phi^i \partial^\mu \phi^j \right]$$

- One canonically normalised scalar :

$$K_{\phi\phi} = \frac{1}{2}$$

- N canonically normalised scalars :

$$K_{ij} = \frac{1}{2} \delta_{ij}$$

The geometrical interpretation becomes obvious when writing the kinetic terms as :

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = - K_{ij}(\phi) \underbrace{\partial_\mu \phi^i}_{d\phi^i} \underbrace{\partial^\mu \phi^j}_{d\phi^j} \quad \Rightarrow \quad [\sigma\text{-model}]$$

$$\Rightarrow \quad ds^2_\phi = K_{ij}(\phi)\, d\phi^i\, d\phi^j \quad \Rightarrow \quad \text{"Scalar geometry"}$$

If $\mathcal{L}_{\text{kin}}$ (or the theory) has some Lie group symmetry G then :

- Linear action on ϕ^i : Then $K_{ij} = \text{cte}$ and also invariant under the linear action of G

- Non-linear action on ϕ^i : Then the scalar geometry is described by a coset G/H and K_{ij} is the corresponding G-invariant metric

Important : In supergravity the scalar geometries are often coset spaces $\frac{G}{H}$ as global symmetries G are non-linearly realised on the scalar sector.

Important: Every two points connected by $g \in G \Rightarrow$ "Homogeous space"

* Coset space $\mathcal{M} = \frac{G}{H}$: Coordinates on $\mathcal{M}$ (fields ϕ^i) correspond to an element of G not being an element of its maximal compact subgroup $H \subset G$ (Isotropy group) :

- generators of G : $\left\{ \underbrace{h_1, \ldots, h_{\dim H}}_{H \subset G} \,;\, \underbrace{t_1, \ldots, t_{\dim G - \dim H}}_{M = A \oplus N \to \text{Nilpotent}} \right\}$

[in a given representation]

max. compact

max abelian (Cartan)

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{“Iwasawa decomposition”}}$$

NOTE: Split real form of a complex Lie algebra is a maximally non-compact version whose Cartan abelian subalgebra A can be chosen along non-compact directions.

- G algebra structure: Let us denote $h \in H$ and $m \in M$ so that

$$\underbrace{[h,h] = h}_{H \subset G \text{ (subgroup)}} \quad , \quad [h,m] = \underbrace{\cancel{h}}_{\text{Reductive}} \oplus m \quad , \quad [m,m] = h \oplus \underbrace{\cancel{m}}_{\text{Symmetric}}$$

↱ suitable for higher-dim origin

- Coset representative (Borel or triangular gauge)

$$V(\phi) = \underbrace{e^{\sum_{i=1}^{\dim N} x_i \overset{\text{Positive roots}}{\overset{\uparrow}{E_i}}}}_{g_N} \cdot \underbrace{e^{\sum_{i=1}^{\dim A} \phi_i \overset{\text{Cartan}}{\overset{\uparrow}{T_i}}}}_{g_A}$$

- Maurer-Cartan one-form :

$$J_\mu = \mathcal{V}^{-1}\partial_\mu \mathcal{V} = \underbrace{Q_\mu}_{\in H} + \underbrace{P_\mu}_{\in M} \in \mathrm{Lie}(G)$$

NOTE: Q_μ plays the role of a composite connection for the local H symmetry. It enters the covariant derivatives of the fermions transforming linearly under H :

$$D_\mu \Psi_\circ \equiv \partial_\mu \Psi_\circ + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Psi_\circ - \overbrace{Q_\mu}^{Q_\mu(\phi)}\Psi_\circ$$

Moreover : $D_\mu \mathcal{V} \equiv \partial_\mu \mathcal{V} - \overset{h \in H}{\mathcal{V}} Q_\mu \Rightarrow \mathcal{V}^{-1} D_\mu \mathcal{V} = \underbrace{\mathcal{V}^{-1}\partial_\mu \mathcal{V}}_{Q_\mu + P_\mu} - Q_\mu$

$= P_\mu$

- Scalar matrix : $M(\phi) = \mathcal{V}\underbrace{\Delta}_{\text{H-invariant positive def. matrix}}\mathcal{V}^t$

- Scalar kinetic terms :

$$\frac{1}{\sqrt{-|g|}}\mathcal{L}_{kin} = -\mathrm{Tr}\left[D_\mu \mathcal{V}\, D^\mu \mathcal{V}^{-1}\right] = -\mathrm{Tr}\left[P_\mu P^\mu\right] = \frac{1}{4}\mathrm{Tr}\left[\partial_\mu M\, \partial^\mu M^{-1}\right]$$

$$= -K_{ij}(\phi)\,\partial_\mu \phi^i \partial^\mu \phi^j$$

Important : Coset representatives transform as

$$\mathcal{V}' = g\,\mathcal{V}\,h(x) \qquad \text{with} \quad \underset{\hookrightarrow \text{global}}{g \in G}\,, \quad \underset{\hookrightarrow \text{local}}{h(x) \in H}$$

$$\Rightarrow \quad M' = v' \Delta v'^{t} = g\, v\, \overbrace{h \Delta h^{t}}^{\Delta}\, v^{t} g^{t} = g\, \overbrace{v \Delta v^{t}}^{M} g^{t} = g M g^{t}$$

As a result, $\mathcal{L}_{kin}$ is invariant under the action of $g \in G$

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}' = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M' \partial^\mu M'^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g\, \partial_\mu M\, \underbrace{g^{t} g^{-t}}_{\mathbb{I}}\, \partial^\mu M^{-1} g^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[g\, \partial_\mu M\, \partial^\mu M^{-1} g^{-1}\right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M\, \partial^\mu M^{-1}\right] = \frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin}$$

cyclicity ↲

Example : $\mathcal{M} = \frac{SL(2)}{so(2)} \Rightarrow G = SL(2)\,,\; H = \overbrace{SO(2)}^{\Delta = \mathbb{I}} \subset SL(2)$

- Generators of G : $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

  [fundamental represent]

  $\Rightarrow$ Commutators: $[T, E_\pm] = \pm 2 E_\pm$
  $[E_+, E_-] = T$

- Some examples of group elements of $G = SL(2)$

$$g_T = e^{-\frac{1}{2}\phi T} = \begin{pmatrix} e^{-\frac{\phi}{2}} & 0 \\ 0 & e^{\frac{\phi}{2}} \end{pmatrix} \;, \quad g_{E_+} = e^{x E_+} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$g_h = e^{\underbrace{\theta (E_+ - E_-)}_{h \text{ generator}}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2) = H$$

- When constructing $\mathcal{V} \in \frac{SL(2)}{SO(2)}$ one must be careful for not to exponentiate $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using Borel gauge

$$\mathcal{V}(\phi, \chi) = g_{E_+} \, g_T = \begin{bmatrix} e^{-\frac{\phi}{2}} & e^{\frac{\phi}{2}} \chi \\ 0 & e^{\frac{\phi}{2}} \end{bmatrix} \in \frac{SL(2)}{SO(2)}$$

so that

$$M(\phi, \chi) = \mathcal{V} \, \mathbb{I} \, \mathcal{V}^t = \begin{bmatrix} e^{-\phi} + \chi e^{\phi} \chi & e^{\phi} \chi \\ e^{\phi} \chi & e^{\phi} \end{bmatrix}$$

and

$$\frac{1}{\sqrt{-|g|}} \mathcal{L}_{kin} = \frac{1}{4} \mathrm{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right]$$

$$= -\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \chi \, \partial^\mu \chi$$

$$\Rightarrow \quad K_{ij}(\phi, \chi) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\phi} \end{bmatrix}$$

NOTE: Coset spaces of the form $\frac{G}{H}$ with H being the maximal compact subgroup of G (like $\frac{SL(2)}{SO(2)}$) are important when describing the scalar geometries arising from Kaluza-Klein reductions.

V Extended $\mathcal{N} \geq 2$ supergravity

- There are theories of supergravity in D=4 with more than one ($\mathcal{N}=1$) gravitino fields : the so called "extended SUGRA's"

- The field content of the SUGRA multiplet includes :

$$\underbrace{e_\mu{}^a}_{\substack{\text{metric}\\ (s=2)}} \;\oplus\; \underbrace{\begin{matrix}\Psi_{\mu\alpha}{}^1\\ \vdots\\ \Psi_{\mu\alpha}{}^{\mathcal{N}}\end{matrix}}_{\substack{\mathcal{N}\text{ gravitini}\\ (s=3/2)}} \;\oplus\; \text{EXTRA FIELDS (bosonic and fermionic)}$$

	Minimally extended	Half maximal	Maximally extended
The most studied cases are $\mathcal{N}=$	2 ,	4 ,	8
Number of supercharges in the susy algebra :	8	16	32

$*$ $\mathcal{N}=2$:

$$\underbrace{e_\mu{}^a}_{s=2},\ \underbrace{A_\mu}_{\substack{s=1\\ \text{(graviphoton)}}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,2}}_{s=3/2}$$

$*$ $\mathcal{N}=4$:

$$\underbrace{e_\mu{}^a}_{s=2},\ \underbrace{A_\mu{}^{1,\dots,6}}_{s=1},\ \underbrace{\overset{\in\mathbb{C}}{\tau}}_{s=0} \in \overset{\text{↱ ``coset space''}}{\frac{SL(2)}{SO(2)}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,\dots,4}}_{s=3/2},\ \underbrace{\psi_\alpha{}^{1,\dots,4}}_{s=1/2}$$

$*$ $\mathcal{N}=8$:

$$\underbrace{e_\mu{}^a}_{s=2},\ \underbrace{A_\mu{}^{1,\dots,28}}_{s=1},\ \underbrace{\phi^{1,\dots,70}}_{\substack{s=0\\ \in\mathbb{R}}} \in \underset{\substack{\downarrow\\ \text{``coset''}\\ \text{space}}}{\frac{E_{7(7)}}{SU(8)}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,\dots,8}}_{s=3/2},\ \underbrace{\psi_\alpha{}^{1,\dots,56}}_{s=1/2}$$

- Depending on the number $\mathcal{N}$ of supersymmetries, the sugra multiplet can be coupled to various types of matter multiplets

Majorana

* $\mathcal{N} = 2$: Vector multiplet : $(A_\mu,\ z \in \mathbb{C},\ \lambda_{1,2})$

Hypermultiplet : $(q_{1,\dots,4} \in \mathbb{R},\ \zeta^{1,2})$

Majorana

Majorana

* $\mathcal{N} = 4$: Vector multiplet : $(A_\mu,\ \phi_{1,\dots,6} \in \mathbb{R},\ \lambda_{1,\dots,4})$

* $\mathcal{N} = 8$: No matter multiplets

- The various scalar fields parameterise different classes of scalar manifolds

[Sp($2n_v+2$) vector bundle]

* $\mathcal{N} = 2$: Vector multiplets $z^{i=1,\dots,n_v}$ span $\overbrace{\mathcal{M}_{SK}}^{\text{Special Kähler}}$

$\mathcal{M} = \mathcal{M}_{SK} \times \mathcal{M}_{QK}$ Hypermultiplets $q_{u=1,\dots,4n_h}$ span $\underbrace{\mathcal{M}_{QK}}_{\text{Quaternionic Kähler}}$

n = # vector multiplets

* $\mathcal{N} = 4$: $$\mathcal{M} = \frac{SL(2)}{SO(2)} \times \frac{SO(6,n)}{SO(6) \times SO(n)}$$

[Holonomy : SU(2) × Sp($2n_h$)]

* $\mathcal{N} = 8$: $$\mathcal{M} = \frac{E_{7(7)}}{SU(8)}$$

VI. Maximal $\mathcal{N}=8$ ungauged supergravity

* Action : ungauged theory $\Rightarrow$ $D_\mu = \partial_\mu$

$$F_{\mu\nu}{}^{\Lambda} = \partial_\mu A_\nu{}^{\Lambda} - \partial_\nu A_\mu{}^{\Lambda}$$

$[\Lambda = 1, \dots, 28]$

$$\begin{aligned} S^{\text{ungauged}}_{\mathcal{N}=8} = \int d^4x \sqrt{-|g|} \Big\{ & \frac{R}{2} \\ & + \frac{1}{96} \operatorname{Tr}\left[\partial_\mu M \, \partial^\mu M^{-1} \right] \\ & + \frac{1}{4} \overbrace{\mathcal{I}_{\Lambda\Sigma}(\phi)}^{\text{“}g^{-2}\delta_{\Lambda\Sigma}\text{ like”}} F_{\mu\nu}{}^{\Lambda} F^{\mu\nu\,\Sigma} \\ & + \frac{1}{4} \frac{1}{2\sqrt{-|g|}} \overbrace{\mathcal{R}_{\Lambda\Sigma}(\phi)}^{\text{“}\frac{1}{8\pi^2}\theta\,\delta_{\Lambda\Sigma}\text{ like”}} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma} \\ & + \text{fermi-terms} \Big\} = \int d^4x \sqrt{-|g|}\, \mathcal{L} \end{aligned}$$

$M_{MN} \in \dfrac{E_{7(7)}}{SU(8)} = \dfrac{G}{H}$

↳ $M = 1, \dots, 56$

[fund. rep. $E_{7(7)}$]

↙ Lagrangian

NOTE: KK reduction of 11D/Type II SUGRA on T^7/T^6 yields ungauged $\mathcal{N}=8$ (maximal) supergravity in 4D

Symmetries :

* Global $G = E_{7(7)}$ of the scalar sector $M \in \dfrac{G}{H}$

* Local $H = SU(8)$ R-symmetry linearly acting on fermions

* $U(1)^{28}$ gauge theory with $\underbrace{\text{uncharged matter}}_{D_\mu M_{MN} = \partial_\mu M_{MN}}$

Electric-magnetic Sp(56) duality

As in classical electromagnetism we can associate with the electric $F_{\mu\nu}{}^{\Lambda}$ their magnetic duals $G_{\mu\nu\Lambda}$

$$G_{\mu\nu\Lambda} \equiv -\varepsilon_{\mu\nu\rho\sigma}\frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}{}^{\Lambda}} = R_{\Lambda\Sigma}(\phi)\, F_{\mu\nu}{}^{\Sigma} - I_{\Lambda\Sigma}(\phi)\, \underbrace{* F_{\mu\nu}{}^{\Sigma}}_{\text{4D Hodge dual}}$$

with

$$* F_{\mu\nu}{}^{\Sigma} \equiv \frac{\sqrt{-|g|}}{2!}\,\varepsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma\,\Sigma}$$

↳ $** = -1$

NOTE: In ordinary Maxwell theory without scalars ($\phi^{i} = 0$) one has $I_{\Lambda\Sigma} = -\delta_{\Lambda\Sigma}$, $R_{\Lambda\Sigma} = 0$.

In terms of $(F_{\mu\nu}{}^{\Lambda},\, G_{\mu\nu\Lambda})$ the $\underbrace{\text{vacuum}}_{\text{no charged matter}}$ Maxwell equations are

$$\nabla^{\mu}\left(* F_{\mu\nu}{}^{\Lambda} \right) = 0 \quad , \quad \nabla^{\mu}\left(* G_{\mu\nu\Lambda} \right) = 0$$

which can be expressed as

$$d\,\mathcal{G}_{\mu\nu}{}^{M} = 0 \quad \text{with} \quad \mathcal{G}_{\mu\nu}{}^{\overbrace{M}^{M=1,\ldots,56}} = \begin{pmatrix} F_{\mu\nu}{}^{\Lambda} \\ G_{\mu\nu\Lambda} \end{pmatrix}$$

Using $G_{\mu\nu}{}^{M}$ the vector sector of the Lagrangian can be expressed as

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}\sqrt{-|g|}\; M_{MN}(\phi)\; G_{\mu\nu}{}^{M}\, G^{\mu\nu\, N}$$

with

$$\underbrace{M_{MN}}_{\text{symmetric}} = \begin{bmatrix} M_{\Lambda\Sigma} & M_{\Lambda}{}^{\Sigma} \\ M^{\Lambda}{}_{\Sigma} & M_{\Lambda\Sigma} \end{bmatrix} = \begin{bmatrix} -(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma} \\ (\mathcal{I}^{-1}\mathcal{R})^{\Lambda}{}_{\Sigma} & -(\mathcal{I}^{-1})^{\Lambda\Sigma} \end{bmatrix}$$

Importantly the electric $F_{\mu\nu}{}^{\Lambda}$ and magnetic $G_{\mu\nu\,\Lambda}$ field strengths do NOT carry independent dynamics as they obey (by construction) twisted self-duality conditions

$$* G^{M} = -\overbrace{\Omega^{MN}}^{\text{Sp(56)-inv matrix } \Omega_{MN} = \begin{bmatrix} 0 & \mathbb{I}_{28} \\ -\mathbb{I}_{28} & 0 \end{bmatrix}} M_{NP}(\phi)\; G^{P}$$

NOTE: The scalar matrix satisfies $M(\phi)\,\Omega\, M(\phi) = \Omega$

Important: The reformulation of the vector sector in terms of $G_{\mu\nu}{}^{M}$ allows to elevate the $G = E_{7(7)}$ global symmetry of the scalar sector to global symmetries of field equations and Bianchi identities. [on-shell]

More concretely

$$g \in G = E_{7(7)} \begin{cases} \phi \to \overset{\text{action on scalars}}{g \circ \phi} & \text{(non-linear action)} \\ \mathcal{G}^{\mathbb{M}} \to [R(g)]^{\mathbb{M}}{}_{\mathbb{N}}\, \mathcal{G}^{\mathbb{N}} & \text{(linear action)} \end{cases}$$

and invariance of $d\mathcal{G} = 0$ and $*\mathcal{G} = -\Omega M \mathcal{G}$

imposes (sufficient conditions)

i) $R(g) \in Sp(56) \Rightarrow R(g)^t\, \Omega\, R(g) = \Omega$

ii) $$M(\underset{\substack{\downarrow \\ \text{non-linear action}}}{g \circ \phi}) = R(g)^{-t} \underset{\substack{\downarrow \\ \text{linear action}}}{M(\phi)} R(g)^{-1}$$

These two conditions are verified by virtue of supersymmetry.

Important: At the level of the action [off-shell] the symmetry is reduced to $G_{elec} \subset G$

$$g \in G_{elec} : \quad R(g) = \begin{bmatrix} A & 0 \\ C & D = A^{-t} \end{bmatrix} \qquad \text{Boundary term}$$

$$\widehat{|||}$$

$$\Rightarrow \delta\mathcal{L}_{bos} = \frac{1}{8} (CA)_{\Lambda\Sigma}\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma}$$

Symplectic frame : It is a choice of embedding of $R(g) \subset Sp(56)$

$\Rightarrow$ NOT UNIQUE $\Rightarrow$ Important consequences when having a "gauging"

VII. Maximal $\mathcal{N}=8$ gauged supergravity

The ungauged maximal supergravity can be deformed by means of the so-called gauging procedure.

Gauging : Promote a subgroup $G_0 \subset G = E_{7(7)}$ from global to local (gauge)

$\alpha = 1, \dots, 133$ [Adj rep $E_{7(7)}$]

$$\nabla_\mu = \partial_\mu - \overset{\text{gauge coupling}}{g} \; \underbrace{A_\mu^{\mathbb{P}}}_{\substack{\text{gauge} \\ \text{fields}}} \; \underbrace{\Theta_{\mathbb{P}}{}^{\alpha}}_{\substack{\text{embedding} \\ \text{tensor}}} \; \underbrace{t_\alpha}_{E_{7(7)} \text{ generators } t_{\alpha=1,\dots,133}}$$

Both electric/magnetic [dyonic gaugings]

$$\Downarrow$$

"Selector"

Ex : $$\nabla_\mu M_{\mathbb{M}\mathbb{N}} = \partial_\mu M_{\mathbb{M}\mathbb{N}} - g \, A^{\mathbb{P}} \underbrace{\Theta_{\mathbb{P}}{}^{\alpha} [t_\alpha]_{(\mathbb{M}}{}^{\mathbb{Q}}}_{X_{\mathbb{P}\mathbb{M}}{}^{\mathbb{Q}} \;\Leftrightarrow\; \text{“Charges”}} M_{\mathbb{N})\mathbb{Q}}$$

Important : The constant embedding tensor charges $X_{\mathbb{M}\mathbb{N}}{}^{\mathbb{P}}$ encodes all the information about the 4D theory !!

* Action : gauged theory $\approx>$ $\partial_\mu \longrightarrow \nabla_\mu$

$F_{\mu\nu}{}^{\Lambda} \longrightarrow H_{\mu\nu}{}^{\Lambda}$ (non-Abelian)

$[\Lambda = 1, \dots, 28]$

$$\begin{aligned}
S^{\text{gauged}}_{\mathcal{N}=8} = \int d^4x \sqrt{-|g|} \Big\{ & \frac{R}{2} \\
& + \frac{1}{96} \mathrm{Tr}\Big[\overset{\text{charged fields}}{\nabla_\mu} M \; \nabla^\mu M^{-1} \Big] - \overbrace{V(\phi, \underbrace{X}_{\text{couplings}})}^{\text{Scalar potential}} \\
& + \frac{1}{4} \mathcal{I}_{\Lambda\Sigma}(\phi) \; \underbrace{H_{\mu\nu}{}^{\Lambda} \; H^{\mu\nu\,\Sigma}}_{\text{non-abelian vectors}} \\
& + \frac{1}{4} \frac{1}{2\sqrt{-|g|}} \; R_{\Lambda\Sigma}(\phi) \; \varepsilon^{\mu\nu\rho\sigma} H_{\mu\nu}{}^{\Lambda} \; H_{\rho\sigma}{}^{\Sigma} \\
& + \underbrace{\mathcal{L}_{top}}_{\text{topological terms}} + \text{fermi-terms} + \underbrace{\text{fermi masses}}_{\text{to restore susy}} \Big\}
\end{aligned}$$

NOTE: Fermi masses are required to restore susy as when we saw in the context of $\mathcal{N}=1$ SUGRA in the presence of a C.C $\Lambda \neq 0$.

* Consistency conditions on $X_{MN}{}^{P}$

i) Linear constraint ≡ Representation constraint

$\mathcal{N}=8$ susy
[tensor hierarchy] $\Rightarrow$ $X_{MN}{}^{P} \in \underbrace{912}$ of $E_{7(7)}$

$X_{M[NP]} = 0 \; , \; X_{(MNP)} = 0$

$X_{PN}{}^{P} = X_{MP}{}^{P} = 0$

ii) Quadratic constraints

Closure of the gauge group G_0 $\Rightarrow$ $\Omega^{MN} X_{MP}{}^{Q} X_{NR}{}^{S} = 0$

Then i) and ii) imply

$$[X_M , X_N] = - X_{MN}{}^{P} X_P$$

↳ close gauge algebra in the 4D theory

* Vector-tensor sector : Vectors fields $A_\mu{}^M$ span now a non-abelian gauge group $G_0 \subset G = E_{7(7)}$

$$\mathcal{G}_{\mu\nu}{}^{M} \longrightarrow H_{\mu\nu}{}^{M} = 2\,\partial_{[\mu} A_{\nu]}{}^{M} + g\, X_{[PQ]}{}^{M} A_\mu{}^{P} A_\nu{}^{Q}$$
$$+ g \underbrace{\tfrac{1}{2}\, \Omega^{MN}\, \Theta_N{}^{\alpha}}_{Z^{M\alpha}} B_{\mu\nu\alpha}$$

Required for $H_{\mu\nu}{}^{M}$ to be gauge covariant !!

↳ with tensor gauge param. $\Xi_{\mu\alpha}$

NOTE: Jacobi identity for $X_{MN}{}^{P}$ does not hold $\Rightarrow$ $\delta A_\mu^M = D_\mu \Lambda^M - g\, Z^{M\alpha}\, \Xi_{\mu\alpha}$

To gauge away vectors in the sector where Jacobi fails ↲

$\underbrace{\qquad}$

Auxiliary two-forms $B_{\mu\nu\alpha}$ dual to scalars

$\Rightarrow$ Non-dynamical

[They also enter $\mathcal{L}_{top}$]

relevant when magnetic charges are present ↲

NOTE: Duality relation:

$$H_{(3)\alpha} = \frac{1}{12} (t_\alpha)_M{}^P \underbrace{M_{NP} * \nabla M^{MN}}_{\text{scalar current (adjoint)}}$$

$*$ Scalar potential: This is probably the most distinctive feature of a gauged supergravity. It takes the form:

$$V(M,X) = \frac{g^2}{672} \Big[X_{MN}{}^{R} X_{PQ}{}^{S} M^{MP} M^{NQ} M_{RS} + 7\, X_{MN}{}^{Q} X_{PQ}{}^{N} M^{MP} \Big]$$

NOTE: The scalar potential makes the gauged theory more interesting for Phenomenology: dS vacua, ...

Important: Maximal gauged supergravities in D=4 appear when compactifying 11D / Type II supergravity on tori and spheres possibly with fluxes, ...

$$\text{Geometry} \oplus \text{fluxes} \oplus \ldots = X_{MN}{}^{P}$$

VIII. Kaluza-Klein reduction on S^1

In this section we are working out the dimensional reduction of gravity in D+1 dimension down to D dimensions. As we will see, this provides a unification of the form:

D+1 Gravity $\Longrightarrow$ Gravity + Maxwell + scalar in D

We will describe gravity in D+1 dimensions:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \sqrt{-|\hat{g}|} \; \hat{R}$$

with $\hat{g}_{MN}$ and $\hat{R}_{MN}$ being the metric and Ricci scalar in a (D+1) dimensional space-time $\underbrace{M}_{x^M} = \underbrace{0, 1, \dots, D-1}_{x^\mu}, \underbrace{z}_{z}$.

Let's take the z-coordinate to be S^1 $\Longrightarrow$ Fourier expansion

$$\hat{g}_{MN}(x,z) = \sum_{\substack{n=0 \\ \Uparrow \\ \text{Fourier} \\ \text{mode}}}^{\infty} \hat{g}^{(n)}_{MN}(x) \, e^{i \frac{n}{L} z}$$

$(z \to z + 2\pi L)$

$\Longrightarrow$ The zero-mode $(n=0)$ is a massless mode whereas $n \neq 0$ correspondes to a tower of massive modes (KK tower).

Example: Scalar field $\hat{\phi}$ in D+1 dimensional flat space-time

$$S_{\phi} = \int d^{D+1}x \, \partial_M \hat{\phi} \, \partial^M \hat{\phi} \quad \underset{\text{E.O.M}}{\Longrightarrow} \quad \hat{\Box} \hat{\phi} = \partial_M \partial^M \hat{\phi} = 0$$

Fourier expansion along S^1 : $\hat{\phi}(x,z) = \sum_{n=0}^{\infty} \phi^{(n)}(x)\, e^{i\frac{n}{L}z}$

so that

$$\hat{\Box}\hat{\phi} = (\underbrace{\partial_\mu \partial^\mu}_{\Box} + \partial_z^2)\hat{\phi} = \sum_{n=0}^{\infty} \underbrace{\left[\Box\phi^{(n)} - \frac{n^2}{L^2}\phi^{(n)}\right]} e^{i\frac{n}{L}z} = 0$$

Each mode must satisfy

$$\Box\phi^{(n)} - \underbrace{\frac{n^2}{L^2}}\phi^{(n)} = 0$$

$m^2 \equiv \frac{n^2}{L^2}$ => Massive modes !!

$$m = \frac{|n|}{L}$$

Important : The KK phylosophy is to assome a very small L (we don't observe S^1) so that all the modes with $n \neq 0$ are very massive $m = \frac{|n|}{L}$ and we have to reach very high energies in order to produce such KK modes :

$$L \sim m_p \sim 10^{-33}\,\text{cm} \Rightarrow m \sim 10^{-5}\,\text{gr}$$

(Higgs mass $\sim m_{top} \sim 10^{-22}$ gr)

Important : KK reduction = Truncation to $n=0$ massless modes

=> z - independence !!

$$\boxed{\hat{g}_{MN}(x) = \hat{g}^{(0)}_{MN}(x)}$$

The first step to perform a KK reduction is to split the (D+1)-metric in D-dimensional blocks:

$$\hat{g}_{MN}(x) = \begin{bmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{z\nu} & g_{zz} \end{bmatrix} \equiv \begin{bmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix} \Rightarrow \text{Too naive (big mess)}$$

$$\equiv \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix}$$

$\Downarrow$

Much more convenient !!

(see discussion on symmetries)

Therefore we parameterise the (D+1) metric $\hat{g}_{MN}$ as

$\phi \equiv$ "Dilaton"

$$\boxed{d\hat{s}^2_{D+1} = e^{2\alpha\phi} ds^2_D + e^{2\beta\phi} (dz + A_\mu dx^\mu)^2}$$

with α and β being constants to be chosen later on. The associated (D+1) frame then reads:

$$\hat{e}_M{}^A = \begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix}$$

$$\boxed{\begin{matrix} M = \mu, z \\ A = a, \underline{z} \end{matrix}}$$

Equivalently: $\hat{e}^a = e^{\alpha\phi} \underbrace{e^a}_{e_\mu{}^a dx^\mu}$ and $\hat{e}^{\underline{z}} = e^{\beta\phi}(dz + A)$ with $A \equiv A_\mu dx^\mu$

Ex: Check that $\hat{e}_M{}^A \, \hat{e}_N{}^B \, \hat{\eta}_{AB} = \hat{g}_{MN}$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times4} & e^{\beta\phi}\end{bmatrix} \underbrace{\begin{bmatrix} \eta_{ab} & 0_{4\times1} \\ 0_{1\times4} & +1\end{bmatrix}\begin{bmatrix} e^{\alpha\phi} e_\nu{}^b & 0_{4\times1} \\ e^{\beta\phi}A_\nu & e^{\beta\phi}\end{bmatrix}}_{\begin{bmatrix} e^{\alpha\phi} e_{\nu a} & 0_{4\times1}\\ e^{\beta\phi}A_\nu & e^{\beta\phi}\end{bmatrix}}$$

$$= \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi}A_\mu A_\nu & e^{2\beta\phi}A_\mu \\ e^{2\beta\phi}A_\nu & e^{2\beta\phi}\end{bmatrix} = \hat{g}_{MN}(x)$$

In the following our goal will be to compute S_{D+1} using the (D+1)-dimensional frame $\hat{e}_M{}^A$ given above:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}}\int d^{D+1}x \; \hat{e} \; \hat{e}_A{}^M \, \hat{e}_B{}^N \, \hat{R}_{MN}{}^{AB}(\hat{e})$$

⊙ $\hat{e} = e^{(\alpha D+\beta)\phi} \, e$

$A_a = e_a{}^\nu A_\nu$

⊙ We need the inverse (D+1)-dim frame $\hat{e}_A{}^N$

$$\hat{e}_M{}^A \cdot \hat{e}_A{}^N = \delta_M{}^N \Rightarrow \hat{e}_A{}^N = \begin{bmatrix} e^{-\alpha\phi}e_a{}^\nu & -e^{-\alpha\phi}\boxed{A_a} \\ 0_{1\times4} & e^{-\beta\phi}\end{bmatrix}$$

Ex: check that $\hat{e}_M{}^A \, \hat{e}_A{}^N = \delta_M{}^N$

$$\begin{bmatrix} e^{\alpha\phi}e_\mu{}^a & e^{\beta\phi}A_\mu\\ 0_{1\times4} & e^{\beta\phi}\end{bmatrix}\begin{bmatrix} e^{-\alpha\phi}e_a{}^\nu & -e^{-\alpha\phi}A_a\\ 0_{1\times4} & e^{-\beta\phi}\end{bmatrix} = \begin{bmatrix}\delta_\mu{}^\nu & 0_{4\times1}\\ 0_{1\times4} & 1\end{bmatrix}$$

⊙ Now we perform the computation of the Ricci scalar $\hat{R}$.

▲ First we compute the aholonomy coefficients $\hat{\Omega}$:

$$\hat{\Omega}_{[MN]P} = \left(\partial_M \hat{e}_N{}^A - \partial_N \hat{e}_M{}^A\right) \hat{e}_{PA}$$

- $\hat{\Omega}_{[\mu\nu]\rho} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\,\hat{e}_{\rho A}$

$$= (\partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a)\,\hat{e}_{\rho a} + (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\,\hat{e}_{\rho\underline{z}}$$

$$= \left[\partial_\mu (e^{\alpha\phi} e_\nu{}^a) - \partial_\nu (e^{\alpha\phi} e_\mu{}^a)\right] (e^{\alpha\phi} e_{\rho a})$$

$$+ \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right] (e^{\beta\phi} A_\rho)$$

$$= e^{2\alpha\phi}\left[(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a)\, e_{\rho a} + \alpha\, (\partial_\mu \phi\, e_\nu{}^a - \partial_\nu \phi\, e_\mu{}^a)\, e_{\rho a} \right]$$

$$+ e^{2\beta\phi}\left[F_{\mu\nu} A_\rho + \beta\, (\partial_\mu \phi\, A_\nu - \partial_\nu \phi\, A_\mu)\, A_\rho \right]$$

$$= e^{2\alpha\phi}\left[\Omega_{[\mu\nu]\rho} + 2\alpha\, \partial_{[\mu} \phi\, e_{\nu]}{}^a e_{\rho a} \right]$$

$$+ e^{2\beta\phi}\left[F_{\mu\nu} A_\rho + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} A_\rho \right]$$

- $\hat{\Omega}_{[\mu\nu]z} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A)\,\hat{e}_{zA} = (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}})\,\hat{e}_{z\underline{z}}$

$$= \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right] e^{\beta\phi}$$

$$= e^{2\beta\phi}\left[F_{\mu\nu} + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} \right]$$

- $\hat{\Omega}_{[\mu z]\rho} = \partial_\mu \hat{e}_z{}^A\, \hat{e}_{\rho A} = \partial_\mu \hat{e}_z{}^{\underline{z}}\, \hat{e}_{\rho\underline{z}} = \partial_\mu (e^{\beta\phi})\, (e^{\beta\phi} A_\rho)$

$$= e^{2\beta\phi}\, \beta\, \partial_\mu \phi\, A_\rho$$

- $$\hat{\Omega}_{[\mu z]z} = \partial_\mu \hat{e}_z{}^A \, \hat{e}_{zA} = \partial_\mu \hat{e}_z{}^{\underline{z}} \, \hat{e}_{z\underline{z}} = \partial_\mu \left(e^{\beta\phi}\right) e^{\beta\phi}$$
$$= e^{2\beta\phi} \beta \, \partial_\mu \phi$$

- $$\hat{\Omega}_{[z\nu]\rho} = - \partial_\nu \hat{e}_z{}^A \, \hat{e}_{\rho A} = - \partial_\nu \hat{e}_z{}^{\underline{z}} \, \hat{e}_{\rho \underline{z}} = -\partial_\nu \left(e^{\beta\phi}\right)\left(e^{\beta\phi} A_\rho\right)$$
$$= - e^{2\beta\phi} \beta \, \partial_\nu \phi \, A_\rho$$

- $$\hat{\Omega}_{[z\nu]z} = - \partial_\nu \hat{e}_z{}^A \, \hat{e}_{zA} = - \partial_\nu \hat{e}_z{}^{\underline{z}} \, \hat{e}_{z\underline{z}} = - \partial_\nu \left(e^{\beta\phi}\right)\left(e^{\beta\phi}\right)$$
$$= - e^{2\beta\phi} \beta \, \partial_\nu \phi$$

- $$\hat{\Omega}_{[zz]\rho} = \hat{\Omega}_{[zz]z} = 0$$

- Using $\hat{\Omega}$ we compute the spin connection with all indices curved

$$\hat{\omega}_{M[NP]}(\hat{e}) = \frac{1}{2}\left(\hat{\Omega}_{[MN]P} - \hat{\Omega}_{[NP]M} + \hat{\Omega}_{[PM]N}\right)$$
$$= \hat{\omega}_M{}^{BC}(\hat{e}) \, \hat{e}_{NB} \, \hat{e}_{PC}$$

- $$\hat{\omega}_{\mu[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]\rho} - \hat{\Omega}_{[\nu\rho]\mu} + \hat{\Omega}_{[\rho\mu]\nu}\right)$$

$$= \frac{1}{2}\Big[e^{2\alpha\phi}\Big(2\,\omega_{\mu[\nu\rho]} + 2\alpha\left(\partial_{[\mu}\phi \, e_{\nu]}{}^a e_{\rho a} - \partial_{[\nu}\phi \, e_{\rho]}{}^a e_{\mu a} + \partial_{[\rho}\phi \, e_{\mu]}{}^a e_{\nu a}\right)\Big)$$
$$+ e^{2\beta\phi}\Big(F_{\mu\nu} A_\rho - F_{\nu\rho} A_\mu + F_{\rho\mu} A_\nu + 2\beta\big(\partial_{[\mu}\phi \, A_{\nu]} A_\rho - \partial_{[\nu}\phi \, A_{\rho]} A_\mu$$
$$+ \partial_{[\rho}\phi \, A_{\mu]} A_\nu\big)\Big)\Big]$$

- $\hat{\omega}_{z[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]\rho} - \hat{\Omega}_{[\nu\rho]z} + \hat{\Omega}_{[\rho z]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\partial_\nu\phi A_\rho - (F_{\nu\rho} + 2\beta\partial_{[\nu}\phi A_{\rho]}) + \beta\partial_\rho\phi A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(-F_{\nu\rho} - 4\beta\partial_{[\nu}\phi A_{\rho]}\right)$$

- $\hat{\omega}_{\mu[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]z} - \hat{\Omega}_{[\nu z]\mu} + \hat{\Omega}_{[z\mu]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left((F_{\mu\nu} + 2\beta\partial_{[\mu}\phi A_{\nu]}) - \beta\partial_\nu\phi A_\mu - \beta\partial_\mu\phi A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\partial_\nu\phi A_\mu\right)$$

- $\hat{\omega}_{z[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]z} - \hat{\Omega}_{[\nu z]z} + \overbrace{\hat{\Omega}_{[zz]\nu}}^{0}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\partial_\nu\phi - \beta\partial_\nu\phi\right)\right] = -e^{2\beta\phi}\beta\partial_\nu\phi$$

- $\hat{\omega}_{\mu[z\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu z]\rho} - \hat{\Omega}_{[z\rho]\mu} + \hat{\Omega}_{[\rho\mu]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(\beta\partial_\mu\phi A_\rho + \beta\partial_\rho\phi A_\mu + (F_{\rho\mu} + 2\beta\partial_{[\rho}\phi A_{\mu]})\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\rho\mu} + 2\beta\partial_\rho\phi A_\mu\right)$$

- $\hat{\omega}_{z[z\rho]} = \frac{1}{2}\left(\underbrace{\hat{\Omega}_{[zz]\rho}}_{0} - \hat{\Omega}_{[z\rho]z} + \hat{\Omega}_{[\rho z]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(+\beta\partial_\rho\phi + \beta\partial_\rho\phi\right)\right] = e^{2\beta\phi}\beta\partial_\rho\phi$$

- $\hat{\omega}_{\mu[zz]} = \hat{\omega}_{z[zz]} = 0$

▲ Using the above results we compute the standard spin connection

$$\hat{\omega}_M{}^{BC} = \hat{\omega}_{M[NP]}\,\hat{e}^{BN}\,\hat{e}^{CP}$$

- $\hat{\omega}_{\mu}{}^{bc} = \hat{\omega}_{\mu[NP]}\,\hat{e}^{bN}\,\hat{e}^{cP}$

$$= \hat{\omega}_{\mu[\nu\rho]}\hat{e}^{b\nu}\hat{e}^{c\rho} + \hat{\omega}_{\mu[\nu z]}\hat{e}^{b\nu}\hat{e}^{cz} + \hat{\omega}_{\mu[z\rho]}\hat{e}^{bz}\hat{e}^{c\rho} + \overbrace{\hat{\omega}_{\mu[zz]}}^{0}\hat{e}^{bz}\hat{e}^{cz}$$

$$= \hat{\omega}_{\mu[\nu\rho]}\,e^{-2\alpha\phi}e^{b\nu}e^{c\rho} - \hat{\omega}_{\mu[\nu z]}e^{-2\alpha\phi}e^{b\nu}A^{c} - \hat{\omega}_{\mu[z\rho]}e^{-2\alpha\phi}A^{b}e^{c\rho}$$

$$= \frac{1}{2}\Big[2\,\omega_{\mu}{}^{[bc]} + 2\alpha\Big(\partial_{[\mu}\phi\, e_{\nu]}{}^{a}e_{\rho a}e^{b\nu}e^{c\rho} - \partial_{[\nu}\phi\, e_{\rho]}{}^{a}e_{\mu a}e^{b\nu}e^{c\rho}$$

$$+ \partial_{[\rho}\phi\, e_{\mu]}{}^{a}e_{\nu a}e^{b\nu}e^{c\rho}\Big) + e^{2(\beta-\alpha)\phi}\Big(F_{\mu\nu}A_{\rho}e^{b\nu}e^{c\rho} -$$

$$- F_{\nu\rho}A_{\mu}e^{b\nu}e^{c\rho} + F_{\rho\mu}A_{\nu}e^{b\nu}e^{c\rho}\Big) +$$

$$+ 2\beta e^{2(\beta-\alpha)\phi}\Big(\partial_{[\mu}\phi A_{\nu]}A_{\rho}e^{b\nu}e^{c\rho} - \partial_{[\nu}\phi A_{\rho]}A_{\mu}e^{b\nu}e^{c\rho}$$

$$+ \partial_{[\rho}\phi A_{\mu]}A_{\nu}e^{b\nu}e^{c\rho}\Big)\Big]$$

$$- \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\mu\nu} - 2\beta\partial_{\nu}\phi A_{\mu})e^{b\nu}A^{c}\Big] - \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\rho\mu} + 2\beta\,\partial_{\rho}\phi A_{\mu})A^{b}e^{c\rho}\Big]$$

$$= (*)$$

NOTE 1: $2\alpha\,\frac{1}{2}\Big(\partial_{\mu}\phi\, e_{\nu}{}^{a}e_{\rho a}e^{b\nu}e^{c\rho} - \partial_{\nu}\phi\, e_{\mu}{}^{a}e_{\rho a}e^{b\nu}e^{c\rho}$

$$-\partial_{\nu}\phi\, e_{\rho}{}^{a}e_{\mu a}e^{b\nu}e^{c\rho} + \partial_{\rho}\phi\, e_{\nu}{}^{a}e_{\mu a}e^{b\nu}e^{c\rho}$$

$$+\partial_{\rho}\phi\, e_{\mu}{}^{a}e_{\nu a}e^{b\nu}e^{c\rho} - \partial_{\mu}\phi\, e_{\rho}{}^{a}e_{\nu a}e^{b\nu}e^{c\rho}\Big)$$

$$= \alpha\Big(\underline{\partial_{\mu}\phi\,\eta^{bc}} - \partial_{\nu}\phi\, e^{b\nu}e_{\mu}{}^{c} - \partial_{\nu}\phi\, e_{\mu}{}^{c}e^{b\nu} + \partial_{\rho}\phi\, e_{\mu}{}^{b}e^{c\rho}$$

$$+ \partial_{\rho}\phi\, e_{\mu}{}^{b}e^{c\rho} - \underline{\partial_{\mu}\phi\,\eta^{bc}}\Big)$$

$$= \alpha\left(2\,\partial_{\rho}\phi\, e_{\mu}{}^{b}e^{c\rho} - 2\,\partial_{\nu}\phi\, e_{\mu}{}^{c}e^{b\nu}\right) = \left[\partial^{a} \equiv e^{a\rho}\partial_{\rho}\right]$$

$$= 2\alpha\left(e_{\mu}{}^{b}\partial^{c}\phi - e_{\mu}{}^{c}\partial^{b}\phi\right) = 4\alpha\, e_{\mu}{}^{[b}\partial^{c]}\phi$$

$$= 4\alpha\,\partial^{[c}\phi\, e_{\mu}{}^{b]}$$

NOTE 2:
$$e^{2(\beta-\alpha)\phi}\left(F_{\mu\nu}A_\rho e^{b\nu}e^{c\rho}-F_{\nu\rho}A_\mu e^{b\nu}e^{c\rho}+F_{\rho\mu}A_\nu e^{b\nu}e^{c\rho}\right)$$
$$=e^{2(\beta-\alpha)\phi}\left(F_\mu{}^bA^c-F^{bc}A_\mu+F^c{}_\mu A^b\right)$$
$$=e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]}-F^{bc}A_\mu\right)$$

NOTE 3:
$$2\beta e^{2(\beta-\alpha)\phi}\frac{1}{2}\Big(\partial_\mu\phi A_\nu A_\rho e^{b\nu}e^{c\rho}-\partial_\nu\phi A_\mu A_\rho e^{b\nu}e^{c\rho}$$
$$-\partial_\nu\phi A_\rho A_\mu e^{b\nu}e^{c\rho}+\partial_\rho\phi A_\nu A_\mu e^{b\nu}e^{c\rho}$$
$$+\partial_\rho\phi A_\mu A_\nu e^{b\nu}e^{c\rho}-\partial_\mu\phi A_\rho A_\nu e^{b\nu}e^{c\rho}\Big)$$
$$=\beta e^{2(\beta-\alpha)\phi}\Big(\partial_\mu\phi A^bA^c-\partial^b\phi A_\mu A^c-\partial^b\phi A^cA_\mu+\partial^c\phi A^bA_\mu$$
$$+\partial^c\phi A_\mu A^b-\partial_\mu\phi A^cA^b\Big)$$
$$=\beta e^{2(\beta-\alpha)\phi}\left(-2A_\mu\partial^b\phi A^c+2A_\mu\partial^c\phi A^b\right)$$
$$=-4\beta e^{2(\beta-\alpha)\phi}A_\mu\partial^{[b}\phi A^{c]}$$

$$(*)=\frac{1}{2}\Big[2\omega_\mu{}^{[bc]}-4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi+e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]}-F^{bc}A_\mu\right)$$
$$-4\beta e^{2(\beta-\alpha)\phi}A_\mu\partial^{[b}\phi A^{c]}\Big]$$
$$-\frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[\underbrace{F_\mu{}^bA^c-F_\mu{}^cA^b}_{2F_\mu{}^{[b}A^{c]}}-2\beta\underbrace{\left(\partial^b\phi A^cA_\mu-\partial^c\phi A^b\right)}_{2\partial^{[b}\phi A^{c]}}A_\mu\Big]$$
$$=\frac{1}{2}\Big[2\omega_\mu{}^{[bc]}-4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi-e^{2(\beta-\alpha)\phi}F^{bc}A_\mu$$
$$+e^{2(\beta-\alpha)\phi}\left(\underline{2F_\mu{}^{[b}A^{c]}}\underline{-4\beta A_\mu\partial^{[b}\phi A^{c]}}\underline{-2F_\mu{}^{[b}A^{c]}}\underline{+4\beta\partial^{[b}\phi A^{c]}}\right)\Big]$$
$$=\frac{1}{2}\Big[2\omega_\mu{}^{[bc]}+4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]}-e^{2(\beta-\alpha)\phi}F^{bc}A_\mu\Big]=$$

$$= \omega_\mu{}^{[bc]} + \alpha\left(\partial^c\phi\, e_\mu{}^b - \partial^b\phi\, e_\mu{}^c\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} A_\mu .$$

- $\hat{\omega}_z{}^{bc} = \hat{\omega}_{z[NP]}\, \hat{e}^{bN}\, \hat{e}^{cP}$

$$= \hat{\omega}_{z[\nu\rho]}\hat{e}^{b\nu}\hat{e}^{c\rho} + \hat{\omega}_{z[\nu z]}\hat{e}^{b\nu}\hat{e}^{cz} + \hat{\omega}_{z[z\rho]}\hat{e}^{bz}\hat{e}^{c\rho} + \overbrace{\hat{\omega}_{z[zz]}}^{0}\hat{e}^{bz}\hat{e}^{cz}$$

$$= \hat{\omega}_{z[\nu\rho]} e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat{\omega}_{z[\nu z]} e^{-2\alpha\phi} e^{b\nu} A^c - \hat{\omega}_{z[z\rho]} e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2} e^{2(\beta-\alpha)\phi}\left[-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right] e^{b\nu} e^{c\rho} + e^{2(\beta-\alpha)\phi}\beta\,\partial_\nu\phi\, e^{b\nu} A^c$$

$$- e^{2(\beta-\alpha)\phi}\beta\,\partial_\rho\phi\, A^b e^{c\rho} =$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} + e^{2(\beta-\alpha)\phi}\beta\left[\underline{-\partial^b\phi A^c} + \uwave{\partial^c\phi A^b} + \underline{\partial^b\phi A^c} - \uwave{\partial^c\phi A^b}\right]$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc}.$$

Therefore, using compact notation, we find that

$$\boxed{\hat{\omega}^{bc} = \omega^{[bc]} + \alpha e^{-\alpha\phi}\left(\partial^c\phi\,\hat{e}^b - \partial^b\phi\,\hat{e}^c\right) - \frac{1}{2} F^{bc} e^{(\beta-2\alpha)\phi}\hat{e}^{\underline{z}}}$$

- $\hat{\omega}_\mu{}^{b\underline{z}} = \hat{\omega}_{\mu[NP]}\,\hat{e}^{bN}\,\hat{e}^{\underline{z}P}$

$$= \hat{\omega}_{\mu[\nu\rho]}\hat{e}^{b\nu}\underbrace{\hat{e}^{\underline{z}\rho}}_{0} + \hat{\omega}_{\mu[\nu z]}\hat{e}^{b\nu}\hat{e}^{\underline{z}z} + \hat{\omega}_{\mu[z\rho]}\hat{e}^{bz}\overbrace{\hat{e}^{\underline{z}\rho}}^{0} + \overbrace{\hat{\omega}_{\mu[zz]}}^{0}\hat{e}^{bz}\hat{e}^{\underline{z}z}$$

$$= \hat{\omega}_{\mu[\nu z]} e^{-(\alpha+\beta)\phi} e^{b\nu} = \frac{1}{2} e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi A_\mu\right) e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= \frac{1}{2} e^{(\beta-\alpha)\phi}\left(F_\mu{}^b - 2\beta\,\partial^b\phi A_\mu\right) \quad \longrightarrow F^b{}_c\, e_\mu{}^c$$

$$= -\beta e^{(\beta-\alpha)\phi}\partial^b\phi A_\mu - \frac{1}{2} e^{(\beta-\alpha)\phi} F^b{}_\mu$$

$$= -e^{(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi A_\mu + \frac{1}{2} F^b{}_\mu\right] = -\hat{\omega}_\mu{}^{\underline{z}b}$$

- $\hat{\omega}_z{}^{b\underline{z}} = \hat{\omega}_{z[NP]}\hat{e}^{bN}\hat{e}^{\underline{z}P}$

$$= \hat{\omega}_{z[\nu\rho]}\hat{e}^{b\nu}\underbrace{\hat{e}^{\underline{z}\rho}}_{0} + \hat{\omega}_{z[\nu\underline{z}]}\hat{e}^{b\nu}\hat{e}^{\underline{z}\underline{z}} + \hat{\omega}_{z[\underline{z}\rho]}\hat{e}^{b\underline{z}}\overbrace{\hat{e}^{\underline{z}\rho}}^{0} + \overbrace{\hat{\omega}_{z[\underline{z}\underline{z}]}}^{0}\hat{e}^{b\underline{z}}\hat{e}^{\underline{z}\underline{z}}$$

$$= \hat{\omega}_{z[\nu\underline{z}]}\, e^{-(\alpha+\beta)\phi} e^{b\nu} = -e^{2\beta\phi}\beta\partial_\nu\phi\, e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= -\beta e^{(\beta-\alpha)\phi}\partial^b\phi = -\hat{\omega}_z{}^{\underline{z}b}$$

Using again compact notation we find:

$$\boxed{\hat{\omega}^{b\underline{z}} = -\omega^{\underline{z}b} = -\beta e^{-\alpha\phi}\partial^b\phi\,\hat{e}^z - \frac{1}{2}{}^{(\beta-2\alpha)\phi}F^b{}_c\,\hat{e}^c}$$

- $\hat{\omega}_\mu{}^{\underline{z}\underline{z}} = \hat{\omega}_z{}^{\underline{z}\underline{z}} = 0$

▲ Once we have the spin connection we compute the Riemann tensor

$$\hat{R}_{MN}{}^{BC} = \partial_M\hat{\omega}_N{}^{BC} - \partial_N\hat{\omega}_M{}^{DC} + \hat{\omega}_M{}^B{}_D\,\hat{\omega}_N{}^{DC} - \hat{\omega}_N{}^B{}_D\,\hat{\omega}_M{}^{DC}$$

- $\hat{R}_{\mu\nu}{}^{bc} = \partial_\mu\hat{\omega}_\nu{}^{bc} + \hat{\omega}_\mu{}^b{}_D\,\hat{\omega}_\nu{}^{Dc} - \partial_\nu\hat{\omega}_\mu{}^{bc} - \hat{\omega}_\nu{}^b{}_D\,\hat{\omega}_\mu{}^{Dc}$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}\partial_\mu\left(e^{2(\beta-\alpha)\phi}F^{bc}A_\nu\right)$$
$$+ \hat{\omega}_\mu{}^b{}_d\,\hat{\omega}_\nu{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\,\hat{\omega}_\nu{}^{\underline{z}c} - (\mu\leftrightarrow\nu)$$
$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[2(\beta-\alpha)\partial_\mu\phi F^{bc}A_\nu$$
$$+ \partial_\mu F^{bc}A_\nu + F^{bc}\partial_\mu A_\nu\Big]$$
$$+ \left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d A_\mu\right]$$
$$\left[\omega_\nu{}^{dc} + \alpha(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\nu\right]$$

$$- e^{2(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi\, A_\mu + \frac{1}{2} F^b{}_\mu\right]\left[\beta\,\partial^c\phi\, A_\nu + \frac{1}{2} F^c{}_\nu\right] - (\mu \leftrightarrow \nu)$$

$$= \overbrace{\partial_\mu \omega_\nu{}^{bc} + \omega_\mu{}^b{}_d\, \omega_\nu{}^{dc}}^{R_{\mu\nu}{}^{bc}} + \alpha\, \partial_\mu \left(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c\right)$$

$$- e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} A_\nu + \frac{1}{2}\partial_\mu F^{bc} A_\nu + \frac{1}{2} F^{bc}\partial_\mu A_\nu$$

$$+ \beta^2\, \underline{\partial^b\phi\,\partial^c\phi\, A_\mu A_\nu} + \frac{1}{2}\beta\,\partial^b\phi\, A_\mu F^c{}_\nu + \frac{1}{2}\beta\,\partial^c\phi\, A_\nu F^b{}_\mu + \frac{1}{4} F^b{}_\mu F^c{}_\nu\Big]$$

$$+ \alpha\,\omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi}\,\omega_\mu{}^b{}_d\, F^{dc} A_\nu$$

$$+ \alpha\,\omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right) - \frac{1}{2}\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right) F^{dc} A_\nu$$

$$+ \alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - \underline{\partial^b\phi\,\partial^c\phi\, g_{\mu\nu}} + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c\right)$$

$$- \frac{1}{2} e^{2(\beta-\alpha)\phi}\,\omega_\nu{}^{dc} F^b{}_d A_\mu + \underline{\frac{1}{4} e^{4(\beta-\alpha)\phi} F^b{}_d F^{dc} A_\mu A_\nu}$$

$$- \frac{1}{2}\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) F^b{}_d A_\mu - \underline{(\mu \leftrightarrow \nu)}$$

NOTE: Underlined terms vanish because they are $\mu \leftrightarrow \nu$ symmetric

$$= \frac{1}{2} R_{\mu\nu}{}^{bc} + \alpha\left(\partial_\mu\partial^c\phi\, e_\nu{}^b + \partial^c\phi\,\partial_\mu e_\nu{}^b - \partial_\mu\partial^b\phi\, e_\nu{}^c - \partial^b\phi\,\partial_\mu e_\nu{}^c\right)$$

$$- e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} A_\nu + \frac{1}{2}\beta\,\partial^b\phi\, F^c{}_\nu A_\mu + \frac{1}{2}\beta\,\partial^c\phi\, F^b{}_\mu A_\nu$$

$$+ \frac{1}{2}\alpha\,\partial_d\phi\, F^{dc} A_\nu e_\mu{}^b - \frac{1}{2}\alpha\,\partial^b\phi\, F_\mu{}^c A_\nu$$

$$+ \frac{1}{2}\alpha\,\partial^c\phi\, F^b{}_\nu A_\mu - \frac{1}{2}\alpha\,\partial^d\phi\, F^b{}_d A_\mu e_\nu{}^c$$

$$+ \frac{1}{2}\partial_\mu F^{bc} A_\nu + \frac{1}{2} F^{bc}\partial_\mu A_\nu + \frac{1}{4} F^b{}_\mu F^c{}_\nu$$

$$+ \frac{1}{2}\omega_\mu{}^b{}_d F^{dc} A_\nu + \frac{1}{2}\omega_\nu{}^{dc} F^b{}_d A_\mu\Big]$$

$$+ \alpha\,\omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) + \alpha\,\omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right)$$

$$+ \alpha^2 \big(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - \underline{(\mu \leftrightarrow \nu)}$$

- $\hat{R}_{\mu\nu}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{D}\, \hat{\omega}_\nu{}^{D\underline{z}} - (\mu \leftrightarrow \nu)$

$$= \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{c}\, \hat{\omega}_\nu{}^{c\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{\underline{z}}\, \overbrace{\hat{\omega}_\nu{}^{\underline{z}\underline{z}}}^{0} - (\mu \leftrightarrow \nu)$$

$$= -\beta e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \partial_\mu\partial^b\phi\, A_\nu + \partial^b\phi\, \partial_\mu A_\nu\right]$$

$$-\frac{1}{2} e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \partial_\mu F^b{}_\nu\right]$$

$$-\left[\omega_\mu{}^b{}_c + \alpha\,(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\right]$$

$$e^{(\beta-\alpha)\phi}\left[\beta\,\partial^c\phi\, A_\nu + \frac{1}{2} F^c{}_\nu\right] - (\mu \leftrightarrow \nu)$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi\, A_\nu + \beta\,\partial_\mu\partial^b\phi\, A_\nu + \beta\,\partial^b\phi\,\partial_\mu A_\nu$$

$$+\frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \frac{1}{2}\partial_\mu F^b{}_\nu + \beta\,\omega_\mu{}^b{}_c\,\partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu$$

$$+\alpha\beta\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\,\partial_c\phi\, e_\mu{}^b F^c{}_\nu$$

$$-\alpha\beta\,\partial_\mu\phi\,\partial^b\phi\, A_\nu - \frac{1}{2}\alpha\,\partial^b\phi\, F_{\mu\nu}\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\left[\underline{\frac{1}{2}\beta F^b{}_c\,\partial^c\phi\, A_\mu A_\nu} + \frac{1}{4} F^b{}_c F^c{}_\nu A_\mu\right] - \underline{(\mu \leftrightarrow \nu)}$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta^2\partial_\mu\phi\,\partial^b\phi\, A_\nu - 2\alpha\beta\,\partial_\mu\phi\,\partial^b\phi\, A_\nu + \beta\,\partial_\mu\partial^b\phi\, A_\nu$$

$$+\alpha\beta\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\,\partial_c\phi\, F^c{}_\nu\, e_\mu{}^b + \frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu$$

$$+\beta\,\partial^b\phi\,\partial_\mu A_\nu - \frac{1}{2}\alpha\,\partial^b\phi\, F_{\mu\nu} + \frac{1}{2}\partial_\mu F^b{}_\nu$$

$$+\beta\,\omega_\mu{}^b{}_c\,\partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\,\frac{1}{4} F^b{}_c F^c{}_\nu A_\mu - (\mu \leftrightarrow \nu) = -\hat{R}_{\mu\nu}{}^{\underline{z}b}$$

- $\hat{R}_{\mu z}{}^{bc} = \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_D \hat{\omega}_z{}^{Dc} - (\mu \leftrightarrow z)$

$$= \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_d \, \hat{\omega}_z{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}} \, \hat{\omega}_z{}^{\underline{z}c}$$

$$- \underbrace{\partial_z \hat{\omega}_\mu{}^{bc}}_{0} - \hat{\omega}_z{}^b{}_d \, \hat{\omega}_\mu{}^{dc} - \hat{\omega}_z{}^b{}_{\underline{z}} \, \hat{\omega}_\mu{}^{\underline{z}c}$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} \left[2(\beta-\alpha)\partial_\mu\phi \, F^{bc} + \partial_\mu F^{bc} \right]$$

$$- \left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi \, e_\mu{}^b - \partial^b\phi \, e_{\mu d}) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_d A_\mu \right] \frac{1}{2} e^{2(\beta-\alpha)\phi} F^{dc}$$

$$- e^{(\beta-\alpha)\phi} \left[\beta \partial^b\phi \, A_\mu + \frac{1}{2} F^b{}_\mu \right] \beta e^{(\beta-\alpha)\phi} \partial^c\phi$$

$$+ \frac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_d \left[\omega_\mu{}^{dc} + \alpha(\partial^c\phi \, e_\mu{}^d - \partial^d\phi \, e_\mu{}^c) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^{dc} A_\mu \right]$$

$$+ \beta e^{(\beta-\alpha)\phi} \partial^b\phi \; e^{(\beta-\alpha)\phi} \left[\beta \partial^c\phi \, A_\mu + \frac{1}{2} F^c{}_\mu \right]$$

$$= -e^{2(\beta-\alpha)\phi} \Big[(\beta-\alpha)\partial_\mu\phi \, F^{bc} + \frac{1}{2}\partial_\mu F^{bc} + \frac{1}{2}\omega_\mu{}^b{}_d F^{dc}$$

$$+ \frac{1}{2}\alpha(\partial_d\phi \, e_\mu{}^b - \partial^b\phi \, e_{\mu d}) F^{dc} \underline{+ \beta^2 \partial^b\phi \, \partial^c\phi \, A_\mu} + \frac{1}{2}\beta F^b{}_\mu \partial^c\phi$$

$$- \frac{1}{2}\omega_\mu{}^{dc} F^b{}_d - \frac{1}{2}\alpha(\partial^c\phi \, e_\mu{}^d - \partial^d\phi \, e_\mu{}^c) F^b{}_d \underline{- \beta^2 \partial^b\phi \, \partial^c\phi \, A_\mu}$$

$$- \frac{1}{2}\beta \partial^b\phi \, F^c{}_\mu \Big]$$

$$+ e^{4(\beta-\alpha)\phi} \left[\underline{\frac{1}{4} F^b{}_d F^{dc} A_\mu} - \underline{\frac{1}{4} F^b{}_d F^{dc} A_\mu} \right]$$

$$= -e^{2(\beta-\alpha)\phi} \Big[(\beta-\alpha)\partial_\mu\phi \, F^{bc} - \frac{1}{2}\alpha \partial^d\phi \, F^c{}_d e_\mu{}^b + \frac{1}{2}\alpha\partial^b\phi \, F^c{}_\mu$$

$$- \frac{1}{2}\alpha\partial^c\phi \, F^b{}_\mu + \frac{1}{2}\alpha\partial^d\phi \, F^b{}_d e_\mu{}^c + \frac{1}{2}\beta\partial^c\phi \, F^b{}_\mu - \frac{1}{2}\beta\partial^b\phi \, F^c{}_\mu$$

$$- \frac{1}{2}\omega_\mu{}^b{}_d F^{cd} + \frac{1}{2}\omega_\mu{}^c{}_d F^{bd} + \frac{1}{2}\partial_\mu F^{bc} \Big]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi F^{bc} + \alpha\,\partial^{[b}\phi\, F^{c]}{}_\mu + \alpha\,\partial^d\phi\, F^{[b}{}_d\, e_\mu{}^{c]}$$

$$+ \beta F^{[b}{}_\mu\,\partial^{c]}\phi - \omega_\mu{}^{[b}{}_d F^{c]d} + \frac{1}{2}\partial_\mu F^{bc}\Big]$$

$$= -\hat{R}_{z\mu}{}^{bc}$$

- $$\hat{R}_{\mu z}{}^{b\underline{z}} = \partial_\mu\hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_D\,\hat{\omega}_z{}^{D\underline{z}} \quad - \quad (\mu \leftrightarrow z)$$

$$= \partial_\mu\hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\,\hat{\omega}_z{}^{c\underline{z}} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\,\overbrace{\hat{\omega}_z{}^{\underline{z}\underline{z}}}^{0}$$

$$- \underbrace{\partial_z\hat{\omega}_\mu{}^{b\underline{z}}}_{0} - \hat{\omega}_z{}^b{}_c\,\hat{\omega}_\mu{}^{c\underline{z}} - \omega_z{}^b{}_{\underline{z}}\,\underbrace{\hat{\omega}_\mu{}^{\underline{z}\underline{z}}}_{0}$$

$$= \partial_\mu\hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\,\hat{\omega}_z{}^{c\underline{z}} - \hat{\omega}_z{}^b{}_c\,\hat{\omega}_\mu{}^{c\underline{z}}$$

$$= -\beta e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi\right]$$

$$-\left[\omega_\mu{}^b{}_c + \alpha(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_c A_\mu\right]\beta e^{(\beta-\alpha)\phi}\partial^c\phi$$

$$- \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_c \cdot e^{(\beta-\alpha)\phi}\left[\beta\partial^c\phi A_\mu + \frac{1}{2}F^c{}_\mu\right]$$

$$= -\beta e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \omega_\mu{}^b{}_c\,\partial^c\phi$$

$$+ \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b - \alpha\,\partial^b\phi\,\partial_\mu\phi\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\Big[\underline{\frac{1}{2}\beta F^b{}_c\,\partial^c\phi A_\mu} - \underline{\frac{1}{2}\beta F^b{}_c\,\partial^c\phi A_\mu}$$

$$- \frac{1}{4}F^b{}_c F^c{}_\mu\Big]$$

$$= -\beta e^{(\beta-\alpha)\phi}\left[(\beta-2\alpha)\partial_\mu\phi\,\partial^b\phi + \partial_\mu\partial^b\phi + \alpha\,\partial_c\phi\,\partial^c\phi\, e_\mu{}^b + \omega_\mu{}^b{}_c\,\partial^c\phi\right]$$

$$- \frac{1}{4}e^{3(\beta-\alpha)\phi}F^b{}_c F^c{}_\mu = -\hat{R}_{\mu z}{}^{\underline{z}b} = -\hat{R}_{z\mu}{}^{b\underline{z}} = \hat{R}_{z\mu}{}^{\underline{z}b}$$

▲ With the Riemann tensor we compute now the curved / flat Ricci tensor

$$\hat{R}_{NC} = \hat{R}_{MN}{}^{B}{}_{C}\, \hat{e}_{B}{}^{M}$$

- $$\begin{aligned}
\hat{R}_{\nu c} &= \hat{R}_{M\nu}{}^{B}{}_{c}\, \hat{e}_{B}{}^{M} \\
&= \hat{R}_{\mu\nu}{}^{b}{}_{c}\, \hat{e}_{b}{}^{\mu} + \hat{R}_{\mu\nu}{}^{\underline{z}}{}_{c}\, \overbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0} + \hat{R}_{z\nu}{}^{b}{}_{c}\, e_{b}{}^{z} + \hat{R}_{z\nu}{}^{\underline{z}}{}_{c}\, \hat{e}_{\underline{z}}{}^{z} \\
&= e^{-\alpha\phi} e_{b}{}^{\mu}\, \hat{R}_{\mu\nu}{}^{b}{}_{c} - e^{-\alpha\phi} A_{b}\, \hat{R}_{z\nu}{}^{b}{}_{c} + e^{-\beta\phi} R_{z\nu}{}^{\underline{z}}{}_{c}
\end{aligned}$$

$$\begin{aligned}
= e^{-\alpha\phi} R_{\nu c} + e^{-\alpha\phi} \alpha \Big(& \partial_b \partial_c \phi\, e_\nu{}^{b} - \partial_\nu \partial_c \phi\, D + \partial_c \phi\, \partial_b e_\nu{}^{b} - \partial_c \phi\, \partial_\nu e_\mu{}^{b} e_b{}^{\mu} \\
& - \partial^2 \phi\, e_{\nu c} + \partial_\nu \partial^b \phi\, \eta_{bc} - \partial^b \phi\, \partial_b e_{\nu c} \\
& + \partial^b \phi\, \partial_\nu e_{\mu c}\, e_b{}^{\mu} \Big) \\
- e^{(2\beta - 3\alpha)\phi} \Big[& (\beta-\alpha) \left(\underline{\partial_b \phi\, F^{b}{}_{c} A_\nu} - \partial_\nu \phi\, F^{b}{}_{c} A_b \right) \\
& + \tfrac{1}{2} \beta \left(\underline{\partial^b \phi\, F_{c\nu} A_b} + \underline{\partial_b \phi\, F^{b}{}_{c} A_\nu} \right) \\
& + \tfrac{1}{2} \beta \left(\cancelto{0}{\partial_c \phi\, F^{b}{}_{b} A_\nu} - \partial_c \phi\, F^{b}{}_{\nu} A_b \right) \\
& + \tfrac{1}{2} \alpha\, \underline{\partial_b \phi\, F^{b}{}_{c}} \big(A_\nu \underbrace{\delta_d{}^{d}}_{D} - \underbrace{A_d\, e_\nu{}^{d}}_{A_\nu} \big) \\
& - \tfrac{1}{2} \alpha\, \partial_b \phi \left(\underline{F^{b}{}_{c} A_\nu} - \underline{F_{\nu c} A^{b}} \right) \\
& + \tfrac{1}{2} \alpha\, \partial_c \phi \left(F^{b}{}_{\nu} A_b - \cancelto{0}{F^{b}{}_{b} A_\nu} \right) \\
& + \tfrac{1}{2} \alpha\, \partial_b \phi\, F^{bd} \left(A_d\, e_{\nu c} - \underline{A_\nu\, \eta_{dc}} \right) \\
& + \tfrac{1}{2} \left(\partial_b F^{b}{}_{c} A_\nu - \partial_\nu F^{b}{}_{c} A_b \right) + \tfrac{1}{2} F^{b}{}_{c} F_{b\nu} \\
& + \tfrac{1}{4} \left(\cancelto{0}{F^{b}{}_{b} F_{c\nu}} - F^{b}{}_{\nu} F_{cb} \right) \\
& + \tfrac{1}{2} F^{d}{}_{c} \left(\omega_b{}^{b}{}_{d} A_\nu - \omega_\nu{}^{b}{}_{d} A_b \right) \\
& + \tfrac{1}{2} F^{b}{}_{d} \left(\omega_\nu{}^{d}{}_{c} A_b - \omega_b{}^{d}{}_{c} A_\nu \right) \Big]
\end{aligned}$$

SO(1,D-1) generators
0 are antisymmetric

$$+ e^{-\alpha\phi}\Big[\alpha\,\omega_b{}^b{}_d(\partial_c\phi\, e_\nu{}^d - \partial^d\phi\, e_{\nu c}) - \alpha\,\omega_\nu{}^b{}_d(\cancel{\partial_c\phi}\,\delta_b^d - \partial^d\phi\,\eta_{bc})$$

$$+\alpha\,\omega_\nu{}^d{}_c(\partial_d\phi\,\overbrace{\delta_\nu^b}^{D} - \partial^b\phi\,\eta_{bd}) - \alpha\,\omega_b{}^d{}_c(\partial_d\phi\, e_\nu{}^b - \partial^b\phi\, e_{\nu d})$$

$$+\alpha^2\big(\underline{\partial_\nu\phi\,\partial_c\phi}\,\overbrace{\delta_b^b}^{D} - \underline{\partial_b\phi\,\partial_c\phi\, e_\nu{}^b} + \underline{\partial^b\phi\,\partial_b\phi\, e_{\nu c}} - \underline{\partial^b\phi\,\partial_\nu\phi\,\eta_{bc}}$$

$$-\underline{\partial_d\phi\,\partial^d\phi}\,\underbrace{\delta_b^b}_{D} e_{\nu c} + \underline{\partial_d\phi\,\partial^d\phi\, e_\nu{}^b\eta_{bc}}\big)\Big]$$

$$- A_b\, e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\partial_\nu\phi\, F^b{}_c - \tfrac{1}{2}\alpha\,\partial^d\phi\, F_{cd}\, e_\nu{}^b + \underline{\tfrac{1}{2}\alpha\,\partial^b\phi\, F_{c\nu}}$$

$$-\underline{\tfrac{1}{2}\alpha\,\partial_c\phi\, F^b{}_\nu} + \tfrac{1}{2}\alpha\,\partial^d\phi\, F^b{}_d\, e_{\nu c} + \underline{\tfrac{1}{2}\beta\,\partial_c\phi\, F^b{}_\nu}$$

$$-\underline{\tfrac{1}{2}\beta\,\partial^b\phi\, F_{c\nu}} - \tfrac{1}{2}\omega_\nu{}^b{}_d\, F_c{}^d + \tfrac{1}{2}\omega_{\nu cd}\, F^{bd}$$

$$+\tfrac{1}{2}\partial_\nu F^b{}_c\Big]$$

$$-\beta\, e^{-\alpha\phi}\Big[(\beta-2\alpha)\partial_\nu\phi\,\partial_c\phi + \partial_\nu\partial_c\phi + \alpha\,\partial_d\phi\,\partial^d\phi\, e_{\nu c} + \omega_{\nu cd}\,\partial^d\phi\Big]$$

$$-\tfrac{1}{4}e^{(2\beta-3\alpha)\phi}\, F_{cd}F^d{}_\nu$$

$$= e^{-\alpha\phi}\Big[R_{\nu c} + \alpha\big(\partial_b\partial_c\phi\, e_\nu{}^b - \partial^2\phi\, e_{\nu c} - (D-1)\partial_\nu\partial_c\phi$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b - \partial^b\phi\,\partial_b e_{\nu c} - \partial_c\phi\,\partial_\nu e_\mu{}^b e_\nu{}^\mu + \partial^b\phi\,\partial_\nu e_{\mu c}\, e_b{}^\mu\big)$$

$$+\omega_b{}^b{}_\nu\,\partial_c\phi - \omega_b{}^{bd}\partial_d\phi\, e_{\nu c} + (3-D)\,\omega_{\nu c}{}^d\partial_d\phi$$

$$+\omega^d{}_{\nu c}\,\partial_d\phi\big) + \alpha^2(D-2)\big(\partial_\nu\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\, e_{\nu c}\big)$$

$$-\beta^2\partial_\nu\phi\,\partial_c\phi + \alpha\beta\big(2\,\partial_\nu\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\, e_{\nu c}\big)$$

$$-\beta\big(\partial_\nu\partial_c\phi + \omega_{\nu c}{}^d\,\partial_d\phi\big)\Big]$$

$$-e^{(2\beta-3\alpha)\phi}\Big[\alpha\frac{D-4}{2}\partial_b\phi\, F^b{}_c A_\nu + \beta\frac{3}{2}\partial_b\phi\, F^b{}_c A_\nu + \tfrac{1}{2}\partial_b F^b{}_c A_\nu + \tfrac{1}{2}F_{b\nu}F^b{}_c$$

$$+\tfrac{1}{2}\omega_b{}^b{}_d\, F^d{}_c A_\nu - \tfrac{1}{2}\omega_b{}^d{}_c\, F^b{}_d A_\nu\Big]$$

- $\hat{R}_{zc} = \hat{R}_{Mz}{}^{B}{}_{c}\, \hat{e}_{B}{}^{M}$

$$= \hat{R}_{\mu z}{}^{b}{}_{c}\, \hat{e}_{b}{}^{\mu} + \hat{R}_{\mu z}{}^{\underline{z}}{}_{c}\, \overbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0} + \overbrace{\hat{R}_{zz}{}^{b}{}_{c}}^{0}\, \hat{e}_{b}{}^{z} + \overbrace{\hat{R}_{zz}{}^{\underline{z}}{}_{c}}^{0}\, \hat{e}_{\underline{z}}{}^{z}$$

$$= e^{-\alpha\phi} e_{b}{}^{\mu}\, \hat{R}_{\mu z}{}^{b}{}_{c}$$

$$= -e^{(2\beta-3\alpha)\phi}\Big[\alpha\Big(-\partial_b\phi F^{b}{}_{c} + \frac{D}{2}\partial^{d}\phi F_{dc} - \partial^{b}\phi F_{bc}\Big) + \frac{1}{2}\partial_b F^{b}{}_{c}$$

$$+ \frac{3}{2}\partial_b\phi\, F^{b}{}_{c} + \frac{1}{2}\omega_{b}{}^{b}{}_{d} F^{d}{}_{c} - \frac{1}{2}\omega_{bcd} F^{db}\Big]$$

- $\hat{R}_{v\underline{z}} = \hat{R}_{Mv}{}^{B}{}_{\underline{z}}\, \hat{e}_{B}{}^{M}$

$$= \hat{R}_{\mu v}{}^{b}{}_{\underline{z}}\, \hat{e}_{b}{}^{\mu} + \hat{R}_{\mu v}{}^{\underline{z}}{}_{\underline{z}}\, \overbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0} + \hat{R}_{zv}{}^{b}{}_{\underline{z}}\, \hat{e}_{b}{}^{z} + \overbrace{\hat{R}_{zv}{}^{\underline{z}}{}_{\underline{z}}}^{0}\, \hat{e}_{\underline{z}}{}^{z}$$

$$= e^{-\alpha\phi} e_{b}{}^{\mu}\, \hat{R}_{\mu v}{}^{b}{}_{\underline{z}} - e^{-\alpha\phi} A_b\, \hat{R}_{zv}{}^{b}{}_{\underline{z}}$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^{2}\big(\partial_b\phi\,\partial^{b}\phi A_v - \underline{\partial_v\phi\,\partial^{b}\phi A_b}\big)$$

$$+ \alpha\beta\big(\underset{\sim\sim\sim\sim\sim}{-2\partial_b\phi\,\partial^{b}\phi A_v} + \underset{\cdots\cdots\cdots}{2\partial_v\phi\,\partial^{b}\phi A_b} + (D-1)\partial_c\phi\,\partial^{c}\phi A_v\big)$$

$$+ \alpha\Big(\frac{1}{2}\partial_c\phi\,(D-1)F^{c}{}_{v} - \frac{1}{2}\partial_b\phi F^{b}{}_{v} - \partial^{b}\phi F_{bv}\Big)$$

$$+ \beta\Big(\partial^{2}\phi A_v - \underset{\circ\circ\circ\circ\circ\circ}{\partial_v\partial^{b}\phi A_b} + \frac{1}{2}\partial_b\phi F^{b}{}_{v} + \partial^{b}\phi F_{bv}$$

$$+ \omega_{b}{}^{b}{}_{c}\,\partial^{c}\phi A_v - \underset{\triangle\triangle\triangle\triangle\triangle\triangle}{\omega_{v}{}^{b}{}_{c}\,\partial^{c}\phi A_b}\Big) + \frac{1}{2}\partial_b F^{b}{}_{v}$$

$$-\frac{1}{2}\partial_v F^{b}{}_{\mu} e_{b}{}^{\mu} + \frac{1}{2}\omega_{b}{}^{b}{}_{c} F^{c}{}_{v} - \frac{1}{2}\omega_{v}{}^{b}{}_{c} F^{c}{}_{b}\Big]$$

$$+ e^{(3\beta-4\alpha)\phi}\Big[\underline{\frac{1}{4}F^{b}{}_{c}F^{c}{}_{v}A_b} - \frac{1}{4}F^{b}{}_{c}F^{c}{}_{b}A_v\Big]$$

$$-\beta e^{(\beta-2\alpha)\phi} A_b\Big[\alpha\big(\underset{\cdots\cdots\cdots}{-2\partial_v\phi\,\partial^{b}\phi} + \underset{\sim\sim\sim\sim\sim}{\partial_c\phi\,\partial^{c}\phi\, e_{v}{}^{b}}\big) + \underline{\beta\,\partial_v\phi\,\partial^{b}\phi}$$

$$+ \underset{\circ\circ\circ\circ\circ}{\partial_v\partial^{b}\phi} + \underset{\triangle\triangle\triangle\triangle\triangle\triangle}{\omega_{v}{}^{b}{}_{c}\,\partial^{c}\phi}\Big] - \underline{\frac{1}{4}e^{(3\beta-4\alpha)\phi} A_b F^{b}{}_{c} F^{c}{}_{v}}$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\partial_b\phi\,\partial^b\phi\,A_\nu + \alpha\beta(D-2)\partial_b\phi\,\partial^b\phi\,A_\nu$$

$$+\alpha\frac{D-4}{2}\partial_b\phi\,F^b{}_\nu + \beta\Big(\partial^2\phi\,A_\nu + \frac{3}{2}\partial_b\phi\,F^b{}_\nu + \omega_b{}^b{}_c\,\partial^c\phi\,A_\nu\Big)$$

$$+\frac{1}{2}\partial_b F^b{}_\nu - \frac{1}{2}\partial_\nu F^b{}_\mu\,e_b{}^\mu + \frac{1}{2}\omega_b{}^b{}_c F^c{}_\nu - \frac{1}{2}\omega_\nu{}^b{}_c F^c{}_b\Big]$$

$$-\frac{1}{4}e^{(3\beta-4\alpha)\phi}F^b{}_c F^c{}_b A_\nu$$

- $\hat R_{z\underline{z}} = \hat R_{Mz}{}^B{}_{\underline z}\,\hat e_B{}^M$

$$= \hat R_{\mu z}{}^b{}_{\underline z}\,\hat e_b{}^\mu + \hat R_{\mu z}{}^{\underline z}{}_{\underline z}\,\overbrace{\hat e_{\underline z}{}^\mu}^{0} + \overbrace{\hat R_{zz}{}^b{}_{\underline z}}^{0}\,\hat e_b{}^z + \overbrace{\hat R_{zz}{}^{\underline z}{}_{\underline z}}^{0}\,e_{\underline z}{}^z$$

$$= e^{-\alpha\phi}e_b{}^\mu\,\hat R_{\mu z}{}^b{}_{\underline z}$$

$$= -\beta e^{(\beta-2\alpha)\phi}\Big[(\beta-2\alpha)\partial_b\phi\,\partial^b\phi + \partial^2\phi + \alpha D\,\partial_c\phi\,\partial^c\phi + \omega_b{}^b{}_c\,\partial^c\phi\Big]$$

$$-\frac{1}{4}e^{(3\beta-4\alpha)\phi}F^b{}_c F^c{}_b$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\partial_b\phi\,\partial^b\phi + \alpha\beta(D-2)\partial_b\phi\,\partial^b\phi + \beta(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi)\Big]$$

$$-\frac{1}{4}e^{(3\beta-4\alpha)\phi}F^b{}_c F^c{}_b$$

▲ Now we compute the vielbein components of the Ricci tensor

$$\hat R_{AC} = \hat e_A{}^N\,\hat R_{NC}$$

- $\hat R_{ac} = \hat e_a{}^N\,\hat R_{Nc} = \hat e_a{}^\nu\,\hat R_{\nu c} + \hat e_a{}^z\,\hat R_{zc}$

$$= e^{-\alpha\phi}e_a{}^\nu\,\hat R_{\nu c} - e^{-\alpha\phi}A_a\,\hat R_{zc}$$

$$
\begin{aligned}
= e^{-2\alpha\phi} \Big[& R_{ac} + \alpha\Big(\partial_a \partial_c \phi - \partial^2 \phi\, \eta_{ac} - (D-1)\, \partial_a \partial_c \phi \\
& + \partial_c \phi\, \partial_b e_{\nu}{}^{b} e_a{}^{\nu} - \partial^b \phi\, \partial_b e_{\nu c} e_a{}^{\nu} - \partial_c \phi\, \partial_a e_{\mu}{}^{b} e_b{}^{\mu} + \partial^b \phi\, \partial_a e_{\mu c}\, e_b{}^{\mu} \\
& + \omega_b{}^{b}{}_a\, \partial_c \phi - \omega_b{}^{bd}\, \partial_d \phi\, \eta_{ca} + (3-D)\, \omega_{ac}{}^{d}\, \partial_d \phi + \omega^{d}{}_{ac}\, \partial_d \phi \Big) \\
& + \alpha^2 (D-2) \Big(\partial_a \phi\, \partial_c \phi - \partial^d \phi\, \partial_d \phi\, \eta_{ac} \Big) - \beta^2 \partial_a \phi\, \partial_c \phi \\
& + \alpha\beta \left(2\, \partial_a \phi\, \partial_c \phi - \partial_d \phi\, \partial^d \phi\, \eta_{ac} \right) - \beta \left(\partial_a \partial_c \phi + \omega_{ac}{}^{d}\, \partial_d \phi \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
- e^{(2\beta - 4\alpha)\phi} \Big[& \alpha\, \frac{D-4}{2}\, \partial_b \phi\, F^{b}{}_{c} A_a + \beta\, \frac{3}{2}\, \partial_b \phi\, F^{b}{}_{c} A_a + \frac{1}{2} \partial_b F^{b}{}_{c} A_a \\
& + \frac{1}{2} F_{ba} F^{b}{}_{c} + \frac{1}{2} \omega_b{}^{b}{}_{d} F^{d}{}_{c} A_a - \frac{1}{2} \omega_b{}^{d}{}_{c} F^{b}{}_{d} A_a \Big]
\end{aligned}
$$

$$
\begin{aligned}
+ e^{(2\beta - 4\alpha)\phi} \Big[& \alpha \left(- \partial_b \phi\, F^{b}{}_{c} A_a + \frac{D}{2} \partial^d \phi\, F_{dc} A_a - \partial^b \phi\, F_{bc} A_a \right) \\
& + \frac{1}{2} \partial_b F^{b}{}_{c} A_a + \beta\, \frac{3}{2}\, \partial_b \phi\, F^{b}{}_{c} A_a \\
& + \frac{1}{2} \omega_b{}^{b}{}_{d} F^{d}{}_{c} A_a - \frac{1}{2} \omega_{bcd} F^{db} A_a \Big]
\end{aligned}
$$

$$
\begin{aligned}
= e^{-2\alpha\phi} \Big[& R_{ac} + \alpha\Big(- (D-2)\, \partial_a \partial_c \phi - \partial^2 \phi\, \eta_{ac} \\
& + \partial_c \phi\, \partial_b e_{\nu}{}^{b} e_a{}^{\nu} - \partial^b \phi\, \partial_b e_{\nu c} e_a{}^{\nu} - \partial_c \phi\, \partial_a e_{\mu}{}^{b} e_b{}^{\mu} + \partial^b \phi\, \partial_a e_{\mu c}\, e_b{}^{\mu} \\
& + \omega_b{}^{b}{}_a\, \partial_c \phi - \omega_b{}^{bd}\, \partial_d \phi\, \eta_{ca} + (3-D)\, \omega_{ac}{}^{d}\, \partial_d \phi + \omega^{d}{}_{ac}\, \partial_d \phi \Big) \\
& + \alpha^2 (D-2) \Big(\partial_a \phi\, \partial_c \phi - \partial^d \phi\, \partial_d \phi\, \eta_{ac} \Big) - \beta^2 \partial_a \phi\, \partial_c \phi \\
& + \alpha\beta \left(2\, \partial_a \phi\, \partial_c \phi - \partial_d \phi\, \partial^d \phi\, \eta_{ac} \right) - \beta \left(\partial_a \partial_c \phi + \omega_{ac}{}^{d}\, \partial_d \phi \right) \Big] \\
- e^{(2\beta - 4\alpha)\phi} \Big[& \partial_b \phi\, F^{b}{}_{c} A_a\, \frac{\overbrace{(D-4)\alpha} + \overbrace{3\beta}}{2} + \underline{\frac{1}{2} \partial_b F^{b}{}_{c} A_a} + \frac{1}{2} F_{ba} F^{b}{}_{c} \\
& + \underline{\frac{1}{2} \omega_b{}^{b}{}_{d} F^{d}{}_{c} A_a} - \underline{\frac{1}{2} \omega_b{}^{d}{}_{c} F^{b}{}_{d} A_a} \\
& - \underline{\alpha\, \frac{D-4}{2}\, \partial_b \phi\, F^{b}{}_{c} A_a} - \underline{\frac{1}{2} \partial_b F^{b}{}_{c} A_a}
\end{aligned}
$$

$$\underline{-\beta\,\tfrac{3}{2}\,\partial_b\phi\, F^b{}_c\, A_a}$$

$$\underline{-\tfrac{1}{2}\,\omega_b{}^b{}_d\, F^d{}_c\, A_a} + \underline{\tfrac{1}{2}\,\omega_b{}^d{}_c\, F^b{}_d\, A_a}\,\Big]$$

$$= e^{-2\alpha\phi}\Big[R_{ac} + \alpha\Big(\underline{-(D-2)\,\partial_a\partial_c\phi} \;\; \underline{-\partial^2\phi\,\eta_{ac}} \xrightarrow{} \Box\phi$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial^b\phi\,\partial_b e_{\nu c} e_a{}^\nu - \partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu + \partial^b\phi\,\partial_a e_{\mu c}\, e_b{}^\mu$$

$$+\omega_b{}^b{}_a\,\partial_c\phi - \underline{\omega_b{}^{bd}\,\partial_d\phi\,\eta_{ac}} \; \underbrace{-(D-3)\,\omega_{ac}{}^d\,\partial_d\phi}_{-(D-2)+1} + \omega^d{}_{ac}\,\partial_d\phi\Big)$$

$$+\alpha^2(D-2)\Big(\partial_a\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\,\eta_{ac}\Big) - \beta^2\,\partial_a\phi\,\partial_c\phi$$

$$+\alpha\beta\,\Big(2\,\partial_a\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\,\eta_{ac}\Big) - \beta\Big(\underbrace{\partial_a\partial_c\phi + \omega_{ac}{}^d\,\partial_d\phi}_{\nabla_a\nabla_c\phi}\Big)\Big]$$

$$-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb} = (*)$$

NOTE 4: We will see later that one must set $\beta = -(D-2)\alpha$

$$(*) = -\tfrac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb} + e^{-2\alpha\phi}\Big[R_{ac} + \underline{\overbrace{(D-2)\,\alpha}^{-\beta}\,\nabla_a\nabla_c\phi}$$

$$+\partial_a\phi\,\partial_c\phi\,\Big(\underbrace{\alpha^2(D-2) - \beta^2 + 2\alpha\beta}_{\alpha\beta-\beta^2 = -(D-2)(D-1)\alpha^2}\Big) - \partial^b\phi\,\partial_b\phi\,\eta_{ac}\Big(\underbrace{\alpha^2(D-2)+\alpha\beta}_{0}\Big)$$

$$+\alpha\,\Big(-\Box\phi\,\eta_{ac} \underline{-(D-2)\,\nabla_a\nabla_c\phi} + \omega_{ac}{}^d\,\partial_d\phi + \omega^d{}_{ac}\,\partial_d\phi$$

$$+\omega_b{}^b{}_a\,\partial_c\phi + \partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu - \partial_d\phi\,\partial^d e_{\nu c}\, e_a{}^\nu$$

$$-\partial_c\phi\,\partial_a e_\nu{}^b e_b{}^\nu + \partial_d\phi\,\partial_a e_{\nu c}\, e^{d\nu}\Big)\Big]$$

NOTE 5: $\alpha^2 = \dfrac{1}{2(D-2)(D-1)}$ [We will see later]

$$= -\tfrac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb} + e^{-2\alpha\phi}\Big[R_{ac} - \tfrac{1}{2}\,\partial_a\phi\,\partial_c\phi - \alpha\,\Box\phi\,\eta_{ac}$$

$$+\partial_d\phi\,\Big(\underbrace{\omega_{ac}{}^d + \omega^d{}_{ac} - \partial^d e_{\nu c} e_a{}^\nu + \partial_a e_{\nu c}\, e^{d\nu}}_{0}\Big) + \partial_c\phi\Big(\underbrace{\omega_b{}^b{}_a + \partial_b e_\nu{}^b e_a{}^\nu - \partial_a e_\nu{}^b e_b{}^\nu}_{0}\Big)$$

$$= (*)$$

Remark 1

$$\omega_b{}^{ba} = e_b{}^\mu \omega_\mu{}^{ba} = -e_b{}^\mu \omega_\mu{}^{ab}(e)$$

$$= -e_b{}^\mu \frac{1}{2}\Big[e^{\nu a}\partial_\mu e_\nu{}^b - e^{\nu b}\partial_\mu e_\nu{}^a - e^{\nu a}\partial_\nu e_\mu{}^b + e^{\nu b}\partial_\nu e_\mu{}^a$$
$$- e^{\nu a} e^{b\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c + e^{\nu b} e^{a\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c\Big]$$

$$= -\frac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial_b e_\nu{}^a e^{\nu b} - \partial^a e_\mu{}^b e_b{}^\mu + \partial^b e_\mu{}^a e_b{}^\mu$$
$$- e^{\nu a} e^{b\sigma}\partial_\nu e_{\sigma b} + e^{\nu b} e^{a\sigma}\partial_\nu e_{\sigma b}\Big]$$

$$= -\frac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial^\nu e_\nu{}^a - \partial^a e_\mu{}^b e_b{}^\mu + \partial^\mu e_\mu{}^a$$
$$- \partial^a e_{\nu b} e^{b\nu} + \partial^b e_{\nu b} e^{a\nu}\Big]$$

$$= -\frac{1}{2}\Big[2\partial_b e_\nu{}^b e^{\nu a} - \partial^a e_{\nu b} e^{b\nu}\Big] = \partial^a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e^{\nu a}$$

$$\Rightarrow \omega_b{}^b{}_a = \partial_a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e_a{}^\nu$$

Remark 2

[We can compute faster by using anholonomy coefficients]

$$\omega_{acd} + \omega_{dac} = \frac{1}{2}\Big[\underline{\Omega_{[ac]d}} - \underline{\Omega_{[cd]a}} + \Omega_{[da]c}$$
$$+ \Omega_{[da]c} - \underline{\Omega_{[ac]d}} + \underline{\Omega_{[cd]a}}\Big]$$

$$= \Omega_{[da]c}$$

$$\Rightarrow \omega_{ac}{}^d + \omega^d{}_{ac} = \Omega_{[ba]c}\,\eta^{bd} = \overset{\text{see note below}}{\circleddash}(\partial_b e_a{}^\rho - \partial_a e_b{}^\rho)e_{\rho c}\,\eta^{bd}$$

$$= -\partial^d e_a{}^\nu e_{\nu c} + \partial_a e^{d\nu} e_{\nu c}$$

$$= \partial^d e_{\nu c} e_a{}^\nu - \partial_a e_{\nu c} e^{d\nu}$$

NOTE: $\Omega_{[\mu\nu]\rho} = (\partial_\mu e_\nu{}^d - \partial_\nu e_\mu{}^d)e_{d\rho}$

$$\Omega_{[ab]c} = e_a{}^\mu e_b{}^\nu e_c{}^\rho\,\Omega_{[\mu\nu]\rho} = (\partial_a e_\nu{}^d e_b{}^\nu - \partial_b e_\mu{}^d e_a{}^\mu)\eta_{cd}$$

$$\text{Important} \longleftarrow = -(\partial_a e_b{}^\rho - \partial_b e_a{}^\rho)e_{\rho c}$$

$$(*) = e^{-2\alpha\phi}\left[R_{ac} - \frac{1}{2}\partial_a\phi\,\partial_c\phi - \alpha\,\Box\phi\,\eta_{ac}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^b F_{cb}$$

- $$\hat{R}_{\underline{z}\underline{z}} = \hat{e}_{\underline{z}}{}^N \hat{R}_{N\underline{z}} = \overbrace{\hat{e}_{\underline{z}}{}^0}^{0} \hat{R}_{0\underline{z}} + \hat{e}_{\underline{z}}{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\beta\phi}\hat{R}_{z\underline{z}}$$

$$= -e^{-2\alpha\phi}\left[\partial_b\phi\,\partial^b\phi\left(\beta^2 + (D-2)\alpha\beta\right) + \beta\left(\overbrace{\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi}^{\eta^{ab}\nabla_a\nabla_b\phi = \Box\phi}\right)\right]$$

$$+ \frac{1}{4}e^{(2\beta-4\alpha)\phi}F_c{}^b F^c{}_b$$

$$= e^{-2\alpha\phi}\left[-\underbrace{(\beta^2 + (D-2)\alpha\beta)}_{0\ \text{(see note 4)}}\partial_b\phi\,\partial^b\phi - \beta\,\Box\phi\right] + \frac{1}{4}e^{(2\beta-4\alpha)\phi}F^2$$

▲ Using these two vielbein components of the Ricci tensor it's finally straightforward to compute the Ricci scalar

$$\hat{R} = \hat{R}_{AB}\,\eta^{AB}$$

We find:

$$\hat{R} = \hat{R}_{ac}\,\eta^{ac} + \hat{R}_{\underline{z}\underline{z}} = e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \overbrace{(D\alpha+\beta)}^{D\alpha - (D-2)\alpha = 2\alpha}\Box\phi\right]$$

$$-\frac{1}{2}e^{-2D\alpha\phi}\left(F^2 - \frac{1}{2}F^2\right) =$$

$$= e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \underbrace{2\alpha\,\Box\phi}_{\text{boundary term}} - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]$$

▶ The full (D+1)-dimensional action then reduces to

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2}\int d^{D+1}x\;\hat{e}\;\hat{R}$$

$$= \frac{1}{2\kappa_{D+1}^2}\int_0^{2\pi L} dz \int d^D x\; e^{(\alpha D+\beta)\phi}\, e\;\hat{R}$$

$$= \frac{1}{2\underbrace{\frac{\kappa^2_{D+1}}{2\pi L}}_{\kappa_D^2 = \frac{\kappa_{D+1}^2}{2\pi L}}}\int d^D x\; \underbrace{e^{[(D-2)\alpha+\beta]\phi}}\; e\left[R - \frac{1}{2}(\partial\phi)^2 \underbrace{- \frac{1}{4} e}^{-2(D-1)\alpha\phi} F^2\right]$$

(see note 5)

Canonical E-H if

$$\beta = -(D-2)\alpha$$

(see note 4)

Proper normalisation if

$$\alpha^2 = \frac{1}{2(D-2)(D-1)}$$

$$= \frac{1}{2\kappa_D^2}\int d^D x\; e\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2\right] = S_D$$

Therefore we recover an Einstein - Maxwell - Dilaton theory !!

$$\boxed{S_{D+1} = \frac{1}{2\kappa_D^2}\int d^D x\; e\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2\right]}$$

with $\kappa_D^2 = \dfrac{\kappa_{D+1}^2}{2\pi L}$

<u>Example</u> : If $D=4 \Rightarrow e^{-2(D-1)\alpha\phi} = e^{-\sqrt{3}\,\phi}$

Exercise: Compute the $\hat{R}_{b\underline{z}}$ component of the Ricci tensor

$$\hat{R}_{b\underline{z}} = \hat{e}_b{}^N \hat{R}_{N\underline{z}} = \hat{e}_b{}^0 \hat{R}_{0\underline{z}} + \hat{e}_b{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\alpha\phi} e_b{}^0 \hat{R}_{0\underline{z}} - e^{-\alpha\phi} A_b \hat{R}_{z\underline{z}}$$

$$= - e^{(\beta-3\alpha)\phi}\Big[\beta^2 \partial_c\phi \partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \alpha\frac{D-4}{2}\partial_c\phi F^c{}_b$$

$$+ \beta\left(\partial^2\phi A_b + \tfrac{3}{2}\partial_c\phi F^c{}_b + \omega_c{}^c{}_d \partial^d\phi A_b\right) + \tfrac{1}{2}\partial_c F^c{}_0 e_b{}^0$$

$$- \tfrac{1}{2}\partial_b F^c{}_0 e_c{}^0 + \tfrac{1}{2}\omega_c{}^c{}_d F^d{}_b - \tfrac{1}{2}\omega_b{}^c{}_d F^d{}_c\Big]$$

$$+ \frac{1}{4} e^{(3\beta-5\alpha)\phi} F^2 A_b$$

$$+ e^{(\beta-3\alpha)\phi}\Big[\beta^2 \partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \beta\,\Box\phi\, A_b\Big]$$

$$- \frac{1}{4} e^{(3\beta-5\alpha)\phi} F^2 A_b$$

$$= -\frac{1}{2}\underbrace{e^{(\beta-3\alpha)\phi}}_{e^{-(D+1)\alpha}}\Big[-\big(\overbrace{(D-4)\alpha + 3\beta}^{-2(D-1)\alpha}\big)\partial_c\phi F_b{}^c$$

$$+ \partial_c F^c{}_0 e_b{}^0 - \partial_b F^c{}_0 e_c{}^0$$

$$+ \omega_c{}^c{}_d F^d{}_b + \omega_b{}^c{}_d F_c{}^d\Big]$$

NOTE : $-(D+1)\alpha = (D-3)\alpha - 2(D-1)$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi}\Big[-2(D-1)\alpha\,\partial_c\phi F_b{}^c$$

$$\underbrace{- \partial_c F^c{}_0 e_b{}^0 + \partial_b F^c{}_0 e_c{}^0}_{\partial_c F_b{}^c - F_0{}^c\partial_c e_b{}^0 + F_0{}^c \partial_b e_c{}^0} + \omega_c{}^c{}_d F_b{}^d + \omega_b{}^c{}_d F^d{}_c\Big]$$

NOTE: $\nabla_c F_b{}^c = \partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi} \Big[-2(D-1)\alpha \overbrace{\partial_c \phi}^{\nabla_c \phi} F_b{}^c$$

$$\underbrace{+\partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c}_{\nabla_c F_b{}^c} + \omega_{cdb} F^{dc} - F_0{}^c \partial_c e_b{}^0 + F_0{}^c \partial_b e_c{}^0$$

$$+ \omega_{bcd} F^{dc} \Big] = (*)$$

Remark 3

$$\omega_{cdb} + \omega_{bcd} = \frac{1}{2} \Big[\underline{\Omega_{[cd]b}} - \underline{\Omega_{[db]c}} + \Omega_{[bc]d}$$

$$+ \Omega_{[bc]d} - \underline{\Omega_{[cd]b}} + \underline{\Omega_{[db]c}} \Big]$$

$$= \Omega_{[bc]d} = -(\partial_b e_c{}^0 - \partial_c e_b{}^0) e_{0d}$$

$$\Rightarrow (\omega_{cdb} + \omega_{bcd}) F^{dc} = -\partial_b e_c{}^0 e_{0d} F^{dc} + \partial_c e_b{}^0 e_{0d} F^{dc}$$

$$= F_0{}^c \partial_c e_b{}^0 - F_0{}^c \partial_b e_c{}^0$$

$$(*) = \frac{1}{2} e^{(D-3)\alpha\phi} \nabla_c \Big[e^{-2(D-1)\alpha\phi} F_b{}^c \Big] = \hat{R}_{\underline{z}b}$$

IX. (D+1)-dimensional vs D-dimensional EOMs and symmetries

In this section we discuss the equations of motion (EOMs) that result from S_{D+1} and S_D.

i) (D+1)-dimensional EOMs

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d\overset{D+1}{x}\, \hat{e}\, \hat{R} \implies \hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = 0$$

NOTE: $\hat{g}^{MN}\hat{G}_{MN} = \hat{R} - \frac{1}{2}(D+1)\hat{R} = \left(1 - \frac{1}{2}(D+1)\right)\hat{R} = 0$

$$\implies \hat{R} = 0 \implies \hat{R}_{MN} = 0$$

↳ $D \neq 1$

$$\implies \hat{R}_{AB} = 0$$

$$\hat{R}_{AB} = 0 \begin{cases} \hat{R}_{ab} = e^{-2\alpha\phi}\left[R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\,\Box\phi\,\eta_{ab}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^cF_{bc} = 0 \\ \hat{R}_{a\underline{z}} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla_c\left[e^{-2(D-1)\alpha\phi}F_b{}^c\right] = 0 \\ \hat{R}_{\underline{z}\underline{z}} = (D-2)\alpha\, e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}F^2 = 0 \end{cases}$$

- It is important to notice that

$$\phi = 0 \implies \hat{R}_{\underline{z}\underline{z}} = \frac{1}{4}F^2 = 0 \implies \underline{F = 0}$$

↳ Trivial Maxwell !!

ii) D-dimensional EOMs

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx\; e\left[R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}e^{-2(D-1)\alpha\phi}F_{\mu\nu}F^{\mu\nu}\right]$$

The EOMs that follow from the above action are:

- $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) + \frac{1}{2}e^{-2(D-1)\alpha\phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^2 g_{\mu\nu}\right)$$

- $$\nabla^\mu\left(e^{-2(D-1)\alpha\phi}F_{\mu\nu}\right) = 0$$

- $$\Box\phi = -\frac{1}{2}(D-1)\alpha\, e^{-2(D-1)\alpha\phi}F^2$$

▲ It is important to notice again that

$$\phi = 0 \implies F^2 = 0 \implies \underline{F=0}$$

↳ Trivial Maxwell !!

IMPORTANT: Having set $\phi = 0$ in the Ansatz for the (D+1)-dimensional metric would have been inconsistent !! [common mistake]
[Einstein - Maxwell - DILATON]

iii) (D+1)-dimensional symmetries

The symmetry group is (D+1)-dimensional general coordinate transformations. At the infinitesimal level we have

$$\delta_{\hat\xi}\, \hat g_{MN} = \hat\xi^P \partial_P \hat g_{MN} + \hat g_{PN}\partial_M \hat\xi^P + \hat g_{MP}\partial_N \hat\xi^P$$

with

$$\hat\xi^M(x,z) = \left(\hat\xi^\mu(x,z) \,,\, \hat\xi^z(x,z) \right)$$

⚠ However, in order to preserve the KK Ansatz of the (D+1)-dimensional metric, there are the restrictions:

Diffeom : $\hat\xi^\mu = \xi^\mu(x) \quad , \quad \hat\xi^z = \lambda(x) + \underbrace{c\, z}_{\text{linear dependence on } S^1}$

⚠ On the other hand, the (D+1)-dimensional EOMs have a scaling symmetry [not the E-H action!!] of the form:

$$\left.\begin{array}{l} \hat g_{MN} \to a^2\, \hat g_{MN} \\ \hat e \to a^{(D+1)}\, \hat e \\ \hat R \to a^{-2}\, \hat R \end{array}\right\} \quad \hat e\, \hat R \to \underbrace{a^{(D-1)}}_{a\in\mathbb{R}}\, \hat e\, \hat R \quad \underset{\text{infinitesim.}}{\Longrightarrow} \quad \delta_a \hat g_{MN} = 2a\, \hat g_{MN}$$

iv) D-dimensional symmetries

Starting from (D+1)-dimensional diffeomorphisms we will obtain D-dimensional diff + U(1) gauge symmetry + Global symmetries.

Ex : Using $\left\{ \begin{array}{l} \hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu \\ \hat{g}_{\mu z} = \hat{g}_{z\mu} = e^{2\beta\phi} A_\mu \\ \hat{g}_{zz} = e^{2\beta\phi} \end{array} \right\}$ with $\beta = -(D-2)\alpha$

show that $\delta \hat{g}_{MN} = (\delta_\xi + \delta_a) \hat{g}_{MN}$ gives rise to :

$$\delta\phi = \xi^\rho \partial_\rho \phi - \frac{1}{(D-2)\alpha}(c+a)$$

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda - c A_\mu$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \frac{2}{(D-2)} \left[c + a(D-1) \right] g_{\mu\nu}$$

- Setting $a = -\frac{c}{(D-1)}$ one finds :

$$\delta\phi = \underbrace{\delta_\xi \phi} - \overbrace{\frac{c}{(D-1)\alpha}}^{\text{shift} \leftrightarrow \text{Non-linear action}}$$

$$\delta A_\mu = \underbrace{\delta_\xi A_\mu} + \underbrace{\partial_\mu \lambda} - \overbrace{c A_\mu}^{\text{scaling} \leftrightarrow \text{linear action}}$$

$$\delta g_{\mu\nu} = \underbrace{\delta_\xi g_{\mu\nu}}$$

→ Global symmetry $\equiv \mathbb{R}$ (real parameter)

→ U(1) gauge symmetry

→ D-dimensional diffeomorphisms

- Setting $a = -c$ one finds : $\underbrace{n\text{-legs} \Rightarrow nc}$

$$\delta\phi = \delta_\xi \phi \qquad (0\text{-legs})$$

$$\delta A_\mu = \delta_\xi A_\mu + \partial_\mu \lambda - \underbrace{c A_\mu} \qquad (1\text{-leg})$$

$$\delta g_{\mu\nu} = \delta_\xi g_{\mu\nu} - \underbrace{2c\, g_{\mu\nu}} \qquad (2\text{-legs})$$

→ Real scaling $\mathbb{R}$ symmetry of the D-dimensional EOMs known as "trombone" scaling symmetry.

Important : There are two inequivalent $\mathbb{R}$ global symmetries. One is an actual symmetry of the D-dimens action whereas the other is only of the EOMs.

Important : In modern language, the global symmetries of the action upon dimensional reduction are referred to as "dualities". In this case the duality group is just $G_{\text{global}} = \mathbb{R}$ symmetry and affects scalar and vector fields in the reduced theory.

X. Kaluza-Klein reduction of Maxwell and scalar on S^1

In this section we look at other reductions on S^1. The starting point is a $(D+1)$-dimensional Maxwell field $\hat B_M$ with field strength $\hat F_{MN} = \partial_M \hat B_N - \partial_N \hat B_M$.

- The K-K Ansatz for $\hat B_M$ reads:

$$\boxed{\hat B_M = \left(\hat B_\mu \, , \hat B_z \right) = \left(B_\mu(x) \, , \chi(x) \right)}$$

which yields a field strength of the form:

$$\hat F_{MN} = \begin{bmatrix} \hat F_{\mu\nu} & \hat F_{\mu z} \\ \hat F_{z\nu} & 0 \end{bmatrix} = \begin{bmatrix} F_{\mu\nu} & \partial_\mu \chi \\ -\partial_\nu \chi & 0 \end{bmatrix}$$

with:

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$F_{\mu z} = \partial_\mu \chi$$
$$F_{z\nu} = -\partial_\nu \chi$$

- The Maxwell's action in $(D+1)$-dimensions then reduces to:

$$S_{\hat B} = -\frac{1}{4}\int d^{D+1}x \, \underbrace{\sqrt{|\hat g|}}_{\hat e} \, \hat F_{MN} \, \hat F^{MN} = -\frac{1}{4}\int d^{D+1}x \, \hat e \, \hat F_{AB} \, \hat F^{AB} = (*)$$

NOTE 1 : $\hat{F}_{AB} = \hat{e}_A{}^M \hat{e}_B{}^N \hat{F}_{MN}$

- $$\begin{aligned}\hat{F}_{ab} &= \hat{e}_a{}^M \hat{e}_b{}^N \hat{F}_{MN} \\ &= \hat{e}_a{}^\mu \hat{e}_b{}^\nu \hat{F}_{\mu\nu} + \hat{e}_a{}^z \hat{e}_b{}^\nu \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_b{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_b{}^z \overset{0}{\overbrace{\hat{F}_{zz}}} \\ &= e^{-2\alpha\phi} F_{ab} + e^{-2\alpha\phi} A_a \partial_b \chi - e^{-2\alpha\phi} A_b \partial_a \chi \\ &= e^{-2\alpha\phi} \Big[\underbrace{F_{ab} - (\partial_a \chi A_b - \partial_b \chi A_a)}_{\mathcal{F}_{ab} \equiv F_{ab} - 2\partial_{[a}\chi A_{b]}} \Big] = e^{-2\alpha\phi} \mathcal{F}_{ab}\end{aligned}$$

- $$\begin{aligned}\hat{F}_{a\underline{z}} &= \hat{e}_a{}^M \hat{e}_{\underline{z}}{}^N \hat{F}_{MN} \\ &= \hat{e}_a{}^\mu \underbrace{\hat{e}_{\underline{z}}{}^\nu}_{0} \hat{F}_{\mu\nu} + \hat{e}_a{}^z \underbrace{\hat{e}_{\underline{z}}{}^\nu}_{0} \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_{\underline{z}}{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_{\underline{z}}{}^z \overset{0}{\overbrace{\hat{F}_{zz}}} \\ &= e^{-(\alpha+\beta)\phi} \partial_a \chi = -\hat{F}_{\underline{z}a}\end{aligned}$$

- $\hat{F}_{\underline{z}\underline{z}} = 0$

NOTE 2 : $\hat{e} = e^{(\alpha D + \beta)\phi}\, e$

$$\begin{aligned}(*) &= -\frac{1}{4} e^{(\alpha D+\beta)\phi} (2\pi L) \int d^D x\, e \Big[\hat{F}_{ab} \hat{F}^{ab} + \overbrace{\hat{F}_{a\underline{z}} \hat{F}^{a\underline{z}} + \hat{F}_{\underline{z}b} \hat{F}^{\underline{z}b}}^{2\hat{F}_{a\underline{z}}\hat{F}^{a\underline{z}}} \Big] \\ &= -\frac{1}{4} (2\pi L) e^{(\alpha D+\beta)\phi} \int d^D x\, e \Big[e^{-4\alpha\phi} \mathcal{F}_{ab} \mathcal{F}^{ab} \\ &\qquad + 2 e^{-2(\alpha+\beta)\phi} \partial_a \chi \partial^a \chi \Big]\end{aligned}$$

$$\boxed{S_{\hat{B}} = (2\pi L) \int d^D x\, e \Big[-\frac{1}{4} e^{-2\alpha\phi} \mathcal{F}^2 - \frac{1}{2} e^{2(D-2)\alpha\phi} (\partial\chi)^2 \Big]}$$

- The reduction of the scalar field is straightforward. The KK Ansatz reads:

$$\boxed{\hat{\Phi} = \Phi(x)}$$

So that the action reduces as:

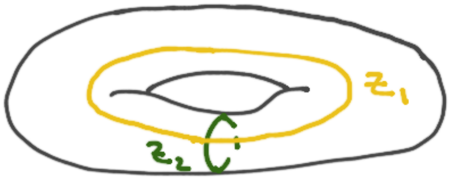
$$\boxed{S_{\hat{\Phi}} = -\frac{1}{2}\int d^{D+1}x \ \hat{e} \ \partial_M \hat{\Phi} \ \partial^M \hat{\Phi} = (2\pi L)\int d^D x \ e \left[-\frac{1}{2}\partial_\mu \Phi \ \partial^\mu \Phi\right]}$$

NOTE 1: $\hat{e} = e^{(D\alpha+\beta)\phi} \ e \overset{\beta=-(D-2)\alpha}{=} e^{2\alpha\phi} e$

NOTE 2: $\partial_A \hat{\Phi} = (\hat{e}_a{}^\mu \partial_\mu \Phi, 0) = e^{-\alpha\phi}(\partial_a \Phi, 0)$

XI. Kaluza-Klein reduction on T^2 and $GL(2)$ duality

In this section we are combining the previous results above and will present the KK reduction of gravity in $(D+2)$ dimensions:



$T^2 \equiv$ 2-torus

coordinates (z_1, z_2)

$$\hat{\hat{g}}_{\mathcal{M}\mathcal{N}} \underset{\substack{\text{step 1}\\ \mathcal{M} = M, z_1}}{\Longrightarrow} \hat{g}_{MN} + \hat{A}_{M1} + \hat{\phi}_1 \underset{\substack{\text{step 2}\\ M = \mu, z_2}}{\Longrightarrow} g_{\mu\nu} + A_{\mu 2} + \phi_2 + A_{\mu 1} + X + \phi_1$$

- Reduction along z_1 :

$$S_{D+2} = \frac{1}{2\kappa_{D+2}^2} \int d^D x \, dz_2 \, dz_1 \; \hat{\hat{e}} \; \hat{\hat{R}}$$

$$= \frac{1}{2\kappa_{D+1}^2} \int d^D x \, dz_2 \; \hat{e} \left[\hat{R} - \frac{1}{2} (\partial \hat{\phi}_1)^2 - \frac{1}{4} e^{-2D\alpha_1 \hat{\phi}_1} \hat{F}_1^{\,2} \right] \equiv S_{D+1}$$

with $\kappa_{D+1}^2 = \dfrac{\kappa_{D+2}^2}{2\pi L_1}$ and $\alpha_1^2 = \dfrac{1}{2(D-1)D}$

- Reduction along z_2 :

$$S_D = \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^{\,2} \right.$$

$$- \frac{1}{2} (\partial \phi_1)^2$$

$$\left. + e^{-2D\alpha_1 \phi_1} \left(- \frac{1}{4} e^{-2\alpha_2 \phi_2} \mathcal{F}_1^{\,2} - \frac{1}{2} e^{2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right) \right]$$

$$= \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{2} e^{-2D\alpha_1 \phi_1 + 2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right.$$

$$\left. - \frac{1}{4} e^{-2D\alpha_1 \phi_1 - 2\alpha_2 \phi_2} \mathcal{F}_1^{\,2} - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^{\,2} \right]$$

with $\alpha_2^2 = \dfrac{1}{2(D-2)(D-1)}$ and $\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2 \partial_{[\mu} \chi A_{\nu]2}$

The action S_D can be enlightening rewritten as

$$S_D = \frac{1}{2\kappa_D^2}\int d^D x \; e \left[R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2} e^{\vec{c}\vec{\phi}} (\partial\chi)^2 - \frac{1}{4} e^{\vec{c}_1\vec{\phi}} \mathcal{F}_1^2 - \frac{1}{4} e^{\vec{c}_2\vec{\phi}} \mathcal{F}_2^2 \right]$$

with

$$\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2\,\partial_{[\mu}\chi\, A_{\nu]2}$$

$$\mathcal{F}_{\mu\nu 2} = F_{\mu\nu 2}$$

and

$$\kappa_D^{\,2} = \frac{\kappa_{D+1}^2}{2\pi L_2} = \frac{\kappa_{D+2}^2}{(2\pi L_1)(2\pi L_2)} = \frac{\kappa_{D+2}^2}{\mathrm{Vol}(T^2)}$$

$$\vec{c} = \left[-\sqrt{\frac{2D}{D-1}} \,,\; \sqrt{\frac{2(D-2)}{D-1}} \right]$$

$$\vec{c}_1 = \left[-\sqrt{\frac{2D}{D-1}} \,,\; -\sqrt{\frac{2}{(D-1)(D-2)}} \right]$$

$$\vec{c}_2 = \left[\;0\; ,\; -\sqrt{\frac{2(D-1)}{D-2}} \right]$$

- Let us rotate the two dilatons $\vec{\phi} = (\phi_1, \phi_2)$ to new ones:

$$\phi = -\frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_1 + \frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_2$$

$$\varphi = -\frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_1 - \frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_2$$

so the D-dimensional action becomes

$$S_D = \frac{1}{2K_D^2}\int d^Dx\, e\, \Big[R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 - \frac{1}{4}e^{q\varphi+\phi}\mathcal{F}_1^{\,2} - \frac{1}{4}e^{q\varphi-\phi}\mathcal{F}_2^{\,2}\Big]$$

with $q^2 = \frac{D}{D-2}$ and the $(D+2)$-dimensional metric reads

$$ds^2_{D+2} = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}\, ds_D^2 + e^{\sqrt{\frac{D-2}{D}}\varphi}\, ds_2^2$$

with

$$ds_2^2 = e^{\phi}(dz_1 + A_{\mu 1}dx^\mu + \chi\, dz_2)^2 + e^{-\phi}(dz_2 + A_{\mu 2}dx^\mu)^2$$

$$\Rightarrow \; ds_2^2\Big|_{\phi=\chi=A_{\mu 1,2}=0} = dz_1^2 + dz_2^2$$

Moduli space: (scalars ≡ "moduli")

- The scalar φ parameterises the volume of volume of T^2 as it appears as a factor in front of ds_2^2.

- The scalar ϕ and χ play different roles. The scalar ϕ parameterises a shape-changing of the torus. It scales the z_1-cycle and the z_2-cycle in opposite manners. The scalar χ varies the angle between the z_1-cycle and the z_2-cycle.

Scalar sector and SL(2) duality

Let's focus on the scalar sector of the above SD action :

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}(\partial\varphi)^2 \underbrace{-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi^2)}_{\mathcal{L}(\phi,\chi)}$$

Global symmetries (or dualities)

i) The scalar φ decouples from the others. It has a global $\mathbb{R}$ shift symmetry

$$\varphi \longrightarrow \varphi + K \qquad \text{with} \quad K \in \mathbb{R}$$

↳ Non-linear action

ii) The symmetry analysis for $\mathcal{L}(\phi,\chi)$ is more interesting. To make the symmetry manifest we define a complex modulus field on T^2 as

$$\tau = \chi + i\,e^{-\phi}$$

in terms of which

$$\mathcal{L}(\phi,\chi) = -\frac{1}{2}\left[(\partial\phi)^2 + e^{2\phi}(\partial\chi)^2\right] = -\frac{\partial_\mu\tau\,\partial^\mu\bar{\tau}}{2\,\mathrm{Im}^2(\tau)}$$

Ex: Show that $\mathcal{L}(\phi,\chi)$ is invariant under the global fractional linear transformation:

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

with $ad - bc = 1$. Show that this transformation acts on (ϕ, χ) as:

$$\left.\begin{aligned} e^{\phi} &\longrightarrow e^{\phi'} = (c\chi + d)^2 e^{\phi} + c^2 e^{-\phi} \\ \chi e^{\phi} &\longrightarrow \chi' e^{\phi'} = (a\chi + b)(c\chi + d) e^{\phi} + ac\, e^{-\phi} \end{aligned}\right\} \begin{array}{l}\text{Non-linear} \\ \text{SL(2) action}\end{array}$$

iii) As scalars couple to vectors, these must also transform. Let us write a constant 2×2 matrix Λ of the form

$$\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with} \qquad |\Lambda| = ad - bc = 1$$

so that $\Lambda \in SL(2)$. Using this matrix Λ, the transformation

$$\begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \longrightarrow (\Lambda^t)^{-1} \begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \quad \Rightarrow \begin{array}{c}\text{Linear} \\ \text{SL(2) action}\end{array}$$

precisely cancells against the transformation of scalars and leaves the action invariant.

As a result, the global symmetry (duality) of the KK reduction of gravity on T^2 turns out to be:

$$G_{\text{global}} = \mathbb{R} \times SL(2) \equiv GL(2)$$

XII. Kaluza-Klein reduction on T^n and $GL(n)$ duality

The discussion above can be generalised to a reduction on a n-dimensional torus T^n.

$$\overset{\overset{\hat{}}{\vdots}\,\{n}{\hat{\hat{g}}}_{\mu\nu} \underset{\text{step } 1}{\Longrightarrow} \cdots\cdots \underset{\text{step } n}{\Longrightarrow} g_{\mu\nu} \oplus \underbrace{A_\mu^{1,2,\dots,n}}_{n \text{ vectors}} \oplus \underbrace{\underbrace{\phi_{1,2,\dots,n}}_{\vec{\phi}} \oplus \underbrace{\chi^n{}_m}_{m>n}}_{\# \text{ scalars} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}}$$

$$\frac{G}{H} = \frac{GL(n)}{SO(n)} \hookleftarrow$$

The $(D+n)$-dim metric

$$ds^2_{D+n} = e^{\frac{1}{3}\vec{g}\vec{\phi}} ds^2_D + \sum_{i=1}^{n} e^{2\vec{\gamma}_i\vec{\phi}} \left(dz^i + A^i_\mu dx^\mu + \chi^i{}_j dz^j\right)^2$$

whose structure is preserved by $(D+n)$-dim diffeomorphisms

$$\hat{\xi}^\mu = \xi^\mu(x) \quad , \quad \hat{\xi}^i(x,z) = \underbrace{\lambda^i(x)}_{U(1)^n} + \underbrace{\Lambda^i{}_j}_{GL(n)} z^j$$

* 11D reduction on T^n and $\frac{E_{n(n)}}{K[E_{n(n)}]}$ coset

- We have seen that when (D+n)-dim gravity is reduced on T^n then the duality group becomes $G_{\text{global}} = \mathbb{R} \times SL(n)$

- If we start from the 11D supergravity theory and reduce it on T^n then the duality group gets enhanced to the exceptional $G_{\text{global}} = E_{n(n)}$

$$S^{\text{SUGRA}}_{11D} = \frac{1}{2\kappa^2_{11D}} \int d^{11}x \, \hat{e} \left[\underset{\substack{\downarrow \\ GL(n) = \mathbb{R} \times SL(n)}}{\hat{R}} - \frac{1}{2 \times 4!} \underset{\substack{\downarrow \\ \text{enhancement to } E_{n(n)}}}{\hat{F}^2_{(4)}} \right] + \dots$$

where $\hat{F}^2_{(4)} = \hat{F}_{MNPQ} \hat{F}^{MNPQ}$ with $\hat{F}_{MNPQ} = \partial_{[M} \hat{A}_{NPQ]}$

- Upon reduction on T^n one finds maximal supergravity in D-dimensions with $\frac{G}{H} = \frac{E_{n(n)}}{K[E_{n(n)}]}$ coset spaces :

↪ maximal compact subgroup K

D	n	$G = E_{n(n)}$	K (max comp)
9	2	$GL(2)$	$SO(2)$
8	3	$SL(2) \times SL(3)$	$SO(2) \times SO(3)$
7	4	$SL(5)$	$SO(5)$
6	5	$SO(5,5)$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$
2	9	$E_{9(9)}$	$K[E_{9(9)}]$

XIII. Prelude to superstrings and D=10,11 supergravity

* From strings to $\mathcal{N}=2$, D=10 Supergravity

Particle evolution in D-dimensions

$$\bullet \leadsto X^M(\underbrace{\tau}_{\text{proper time}})$$

String evolution in D-dimensions

$l_s^2 \sim 2\alpha'$

$$\leadsto X^M(\tau,\sigma)$$

$$+\text{susy} \Rightarrow \left.\begin{array}{l}\theta^1(\tau,\sigma)\\ \theta^2(\tau,\sigma)\end{array}\right\} \text{Grassman variables}$$

Set D=10 and $\theta^{1,2}$ being $\underbrace{\text{M-W}}_{\text{Majorana-Weyl}}$ fermions

→ 2D conformal field theory : $X^M(\tau,\sigma)$, $\theta^{1,2}(\tau,\sigma)$

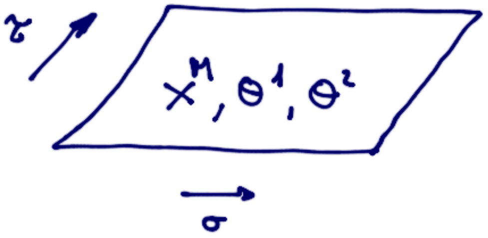
$$S_{2D} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \, \eta^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^M + \text{fermion terms}$$

with $\sigma^\alpha = (\tau,\sigma)$

$\eta^{\alpha\beta} = (-1,1)$

↳ gauge fixing : diff + Weyl in 2D

→ Mode expansion and states



$$\Rightarrow \quad X_M \supset \sum_n \left(a_M^{(n)} e^{-2in(\tau-\sigma)} + \tilde{a}_M^{(n)} e^{-2in(\tau+\sigma)}\right)$$

$$\theta^1 \supset \sum_n b^{(n)} e^{-2in(\tau-\sigma)} \; ; \; \theta^2 \supset \sum_n \tilde{b}^{(n)} e^{-2in(\tau+\sigma)}$$

$$\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Promote a's, ã's, b's, b̃'s to operators with [,] or { , } relations :

$$|\text{state}\rangle = a^{\dagger}_{M}\, a^{\dagger}_{N}\, |0\rangle \Rightarrow \underbrace{G_{MN}}_{\text{metric}} \oplus \underbrace{B_{MN}}_{\text{antisym}} \oplus \underbrace{\overset{\text{"dilaton"}}{\overset{\downarrow}{\Phi}}}_{\substack{\text{scalar}\\ \text{trace}}}$$

D=10 :

→ Mass of a state :

$$M^2 = \frac{1}{l_s^2}\Big[\underbrace{N(a,b) + \tilde{N}(\tilde{a},\tilde{b})}_{\text{ocupation numbers}}\Big] \Rightarrow \overset{l_s \to 0}{\underset{M^2\to\infty}{\bigcirc \to \bullet}} \Rightarrow \overbrace{\text{Keep only massless states !!}}^{\text{"low energy"}}$$

→ $\mathcal{N}=2$, D=10 massles spectrum : $\overbrace{G_{MN},\ B_{(2)},\ \Phi,\ C_{(p)}}^{\text{Bosons}}$; $\overbrace{\chi^{1,2}_{\alpha},\ \Psi^{1,2}_{M\alpha}}^{\text{Fermions}}$

$(\text{ch}\,\Psi^1 \neq \text{ch}\,\Psi^2)$ IIA : $p=1,3 \Rightarrow C_M$, C_{MNP}

$(\text{ch}\,\Psi^1 = \text{ch}\,\Psi^2)$ IIB : $p=0,2,4 \Rightarrow C_{(0)}$, C_{MN} , C_{MNPQ}

NOTE : A p-form $C_{(p)}$ has p antisymmetric indices $C_{(p)} = C_{[M_1 \cdots M_p]}$

- Lagrangian : a candidate

$$\mathcal{L}_{10D} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\, \sqrt{G}\,\Big[R - \frac{1}{2}\partial_M\Phi\,\partial^M\Phi - \frac{1}{2\times 3!}\, e^{-\Phi}\, \underbrace{H_{MNP}\, H^{MNP}} + \dots + \text{fermi}$$

$$H_{(3)} \equiv H_{MNP} = \partial_{[M} B_{NP]} = dB_{(2)}$$

with $2\kappa_{10}^2 = \frac{1}{2\pi}(2\pi l_s)^8$

→ We can also study a probe string propagating in a background $\{ G_{MN}, B_{MN}, \Phi, C_{(p)} \}$ generated by other strings around:

$$S_{\substack{\text{probe}\\ \text{string}}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \Big[(\partial_\alpha X^M)(\partial^\alpha X^N) \underbrace{G_{MN}(X)}_{} + \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \underbrace{B_{MN}(X)}_{} + \dots$$

G_{MN}, $B_{(2)}$, etc can be viewed as couplings in the 2D field theory !!

$$\text{Conformal invariance} \Rightarrow \underbrace{\beta_G^{MN} = \beta_B^{MN} = \dots = 0}_{}$$

At lowest order in $\frac{\sqrt{\alpha'}}{L}$ ~ system scale

$\Rightarrow$ E.O.M of an action !!

→ $\mathcal{N}=2$, $D=10$ Supergravity action:

$$S_{\substack{\text{SUGRA}\\ \mathcal{N}=2,\, D=10}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \Big[R - \frac{1}{2} \partial^M \Phi \partial_M \Phi - \frac{1}{2\times 3!} e^{-\Phi} H_{MNP} H^{MNP}$$

$$- \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{G} \begin{cases} \text{IIA}: \frac{1}{2!} e^{3/2\,\Phi} \tilde{F}_{MN} \tilde{F}^{MN} + \frac{1}{4!} e^{1/2\,\Phi} \tilde{F}_{M_1 \dots M_4} \tilde{F}^{M_1 \dots M_4} \\ \text{IIB}: e^{2\Phi} \partial_M C_{(0)} \partial^M C_{(0)} + \frac{1}{3!} e^{\Phi} \tilde{F}_{MNP} \tilde{F}^{MNP} + \frac{1}{5!} \tilde{F}_{M_1 \dots M_5} \tilde{F}^{M_1 \dots M_5} \end{cases}$$

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x \left\{ \begin{array}{l} \text{IIA: } \overbrace{\epsilon^{M_1\dots M_{10}} \; B_{M_1M_2} \; F_{M_3\dots M_6} \; F_{M_7\dots M_{10}}}^{B_{(2)}\wedge F_{(4)}\wedge F_{(4)} \; \Rightarrow \text{ wedge products}} \\ \text{IIB: } \underbrace{\epsilon^{M_1\dots M_{10}} \; C_{M_1\dots M_4} \; H_{M_5M_6M_7} \; F_{M_8M_9M_{10}}}_{C_{(4)}\wedge H_{(3)}\wedge F_{(3)}} \end{array} \right.$$

$$+ \; S_{Fermi}\,(\chi^{1,2}, \Psi^{1,2})$$

where the gauge invariant field strengths are given by:

IIA : $\tilde{F}_{(2)} = F_{(2)} = dC_{(1)}$

$$\tilde{F}_{(4)} = \underbrace{F_{(4)}}_{dC_{(3)}} + C_{(1)}\wedge H_{(3)}$$

NOTE: There is a massive IIA theory with $F_{(0)} = cte$

IIB: $\tilde{F}_{(3)} = \underbrace{F_{(3)}}_{dC_{(2)}} - H_{(3)}\wedge C_{(0)}$

$$\tilde{F}_{(5)} = \underbrace{F_{(5)}}_{dC_{(4)}} + \frac{1}{2}\left[B_{(2)}\wedge F_{(3)} - C_{(2)}\wedge H_{(3)}\right]$$

⇒ Self-dual : $\boxed{\tilde{F}_{(5)} = *\,\tilde{F}_{(5)}}$

Math : $F_{(n)} = F_{M_1\cdots M_N} = \partial_{[M_1} C_{M_2\cdots M_n]} \equiv d\,C_{(n)}$

⇒ Starting from closed superstrings we have obtained $\mathcal{N}=2$, $D=10$ Supergravities as the low-energy limit !!

⇒ Superstrings live in a ten-dimensional space-time ...

... so what about $10-4=6$ extra dimensions ?

→ The type IIA supergravity can be connected to the one and unique $\mathcal{N}=1$, $D=11$ Supergravity conjectured to be the low-energy limit of a mysterious theory of membranes called "M-theory"

$$\boxed{\begin{aligned} S_{\substack{\text{SUGRA}\\ \mathcal{N}=1,\, D=11}} &= \frac{1}{2K_{11}^2}\int d^{11}x \sqrt{G}\left[R - \frac{1}{2\times 4!} F_{\hat{M}_1\ldots\hat{M}_4} F^{\hat{M}_1\ldots\hat{M}_4}\right] \\ &- \frac{1}{12K_{11}^2}\int d^{11}x \underbrace{\epsilon^{\hat{M}_1\ldots\ldots\hat{M}_{11}} A_{\hat{M}_1\hat{M}_2\hat{M}_3} F_{\hat{M}_4\ldots\hat{M}_7} F_{\hat{M}_8\ldots\hat{M}_{11}}}_{A_{(3)}\wedge F_{(4)}\wedge F_{(4)}} \\ &+ S_{\text{Fermi}}(\Psi) \end{aligned}}$$

with $2K_{11}^2 = \frac{1}{2\pi}(2\pi \ell_p)^9$

↳ Planck scale

* The field content of the theory is $G_{\hat{M}\hat{N}} \oplus A_{\hat{M}\hat{N}\hat{P}} \oplus \Psi_{\hat{M}\alpha}$

with $F_{(4)} \equiv F_{\hat{M}_1\ldots\hat{M}_4} = \partial_{[\hat{M}_1} A_{\hat{M}_2\hat{M}_3\hat{M}_4]} \equiv dA_{(3)}$. It is invariant under local supersymmetry transformations

$$\delta_\epsilon e_{\hat{M}}{}^{\hat{A}} = \bar{\epsilon}\,\Gamma^{\hat{A}}\Psi_{\hat{M}}$$

$$\delta_\epsilon A_{\hat{M}\hat{N}\hat{P}} = -3\,\bar{\epsilon}\,\Gamma_{[\hat{M}\hat{N}}\Psi_{\hat{P}]}$$

$$\delta_\epsilon \Psi_{\hat{M}} = D_{\hat{M}}\epsilon + \frac{1}{12}\left[\Gamma_{\hat{M}}\frac{1}{4!}F_{\hat{Q}\hat{R}\hat{S}\hat{T}}\Gamma^{\hat{Q}\hat{R}\hat{S}\hat{T}} - 3\,\frac{1}{3!}F_{\hat{M}\hat{N}\hat{P}\hat{Q}}\Gamma^{\hat{N}\hat{P}\hat{Q}}\right]\epsilon$$

Important: Note that there is no coupling to be tuned!!

* M-theory $\Rightarrow$ IIA $\begin{cases} G_{\hat{M}\hat{N}} \Rightarrow G_{MN} \oplus G_{M10} \equiv C_M \oplus G_{10\,10} \equiv \Phi \\ A_{\hat{M}\hat{N}\hat{P}} \Rightarrow A_{MNP} \equiv C_{MNP} \oplus A_{MN10} \equiv B_{MN} \end{cases}$

$\hat{M} = (M, 10)$

↳ $M = 0, \ldots, 9$

↳ $\hat{M} = 0, \ldots.\, 10$

Then one finds that

$$\underbrace{G_{\hat{M}\hat{N}} \, , \, A_{\hat{M}\hat{N}\hat{P}}}_{\mathcal{N}=1,\ D=11 \text{ SUPERGRAVITY}} \Longrightarrow \underbrace{G_{MN} \, , \, B_{MN} \, , \, \Phi \, , \, C_M \, , \, C_{MNP}}_{\mathcal{N}=2\ ,\ D=10 \text{ Type IIA SUPERGRAVITY}}$$

Important: The 11D action also reduces consistently to the Type IIA one (not only the field content)

XIV. 11D and Type II reductions on $T^{7,6}$

Let us consider 11D SUGRA and perform a KK decomposition

- $G_{\hat{M}\hat{N}}$: $G_{\mu\nu}$, $\underline{G_{\mu\tilde{m}}}$ (7) , $\underline{G_{\tilde{m}\tilde{n}}}$ (28)
- $A_{\hat{M}\hat{N}\hat{P}}$: $\underbrace{A_{\mu\nu\rho}}_{\text{not indep (dual to } V)}$, $\underline{A_{\mu\nu\tilde{m}}}$ (7) , $\underline{A_{\mu\tilde{m}\tilde{n}}}$ (21) , $A_{\tilde{m}\tilde{n}\tilde{p}}$ (35)

$\Rightarrow$

scalars = 70

vectors = 28

metric = 1

NOTE : 11D index splitting $\hat{M} = (\overset{4d}{\mu}, \overset{7d}{\tilde{m}})$

Let us now consider Type II SUGRAS in 10D and perform a KK decomposition

NOTE : 10D index splitting $M = (\overset{4D}{\mu}, \overset{6D}{m})$

Universal sector

- G_{MN} : $G_{\mu\nu}$, $\underline{G_{\mu m}}$ (6) , $\underline{G_{mn}}$ (21)
- B_{MN} : $\underline{B_{\mu\nu}}$ (1) , $\underline{B_{\mu m}}$ (6) , $\underline{B_{mn}}$ (15) $\Rightarrow$ scalars = 38, vectors = 12, metric = 1
- Φ : $\underline{\Phi}$ (1)

IIA : odd p-forms $p = 1, 3$

- C_M : C_μ (1) , $\underline{C_m}$ (6) $\Rightarrow$ scalars = 32, vectors = 16
- C_{MNP} : $\underbrace{C_{\mu\nu\rho}}_{\text{not-independent (dual to V)}}$, $\underline{C_{\mu\nu m}}$ (6) , $\underline{C_{\mu mn}}$ (15) , $\underline{C_{mnp}}$ (20)

IIB : even p-forms $p = 0, 2, 4$ (self-dual)

- $C_{(0)}$: $\underline{C_{(0)}}$ (1)
- $C_{(2)}$: $\underbrace{\text{non-dyn}}$ $\overset{\text{not-indep}}{\underbrace{\text{(dual to V)}}}$, $\underline{C_{\mu\nu}}$ (1) , $\underline{C_{\mu m}}$ (6) , $\underline{C_{mn}}$ (15)
- $C_{(4)}$: $C_{\mu\nu\rho\sigma}$, $C_{\mu\nu\rho m}$, $\underline{C_{\mu\nu mn}}$ (15) , $\underline{C_{\mu mnp}}$ (20) , $\underline{C_{mnpq}}$ (15)

≋> scalars = 32

vectors = 16

Important: Upon suitable dualisations (2-form ↔ scalars) and Kaluza-Klein inspired field redefinitions, the dimensionally reduced theory in 4D is :

Field content :

56 × 56 matrix

$\Uparrow$

* $\text{scalars} = \underbrace{70}_{\phi^{i=1,\dots,70}}$ ≋> $M_{MN}(\phi) \in \underbrace{\frac{E_{7(7)}}{SU(8)}}_{\text{coset space } \frac{G}{H}}$

[like $\frac{SL(2)}{SO(2)}$]

NOTE: $E_{7(7)}$ irreps : $\underbrace{56}_{\text{fundam } M}$, $\underbrace{133}_{\text{adjoint } \alpha}$, ...

* $\text{vectors} = \underbrace{28}_{A_\mu^{\Lambda=1,\dots,28}}$ ≋> Abelian vector fields

"ungauged theory"

* metric : $g_{\mu\nu}$

≋> Bosonic sector of $\mathcal{N}=8$ SUGRA !!

NOTE: Reducing 10D fermions ⇒ $\mathcal{N}=8$ SUGRA

XV. Gauged supergravities from Type IIB flux compactifications

We will now consider reductions in the presence of fluxes and sources : $F_{(p)}$, D-branes , ...

The charges of these sources are quantised in string theory (not in supergravity) and so fluxes :
[quantum]

$$\frac{1}{(2\pi\sqrt{\alpha'})^{p-1}} \int_{\underbrace{\Sigma_p}_{p\text{-cycle within } T^6}} F_{(p)} \in \mathbb{Z} \quad \Rrightarrow \quad \text{Dirac quantisation}$$

Ex : Type IIB on T^6 : $F_{(3)} = dC_{(2)} + F^{(bg)}_{(3)}$; $H_{(3)} = dB_{(2)} + H^{(bg)}_{(3)}$

- $F_{(3)}$: $F^{(bg)}_{mnp} \rightsquigarrow \binom{6}{3} = 20$ indep. flux parameters
- $H_{(3)}$: $\underbrace{H^{(bg)}_{mnp}}_{\text{along } T^6} \rightsquigarrow \binom{6}{3} = 20$ indep. flux parameters

$\rightsquigarrow$ Type IIB action :

$$S_{IIB} \supset \int_{\mathcal{M}_4 \times T^6} H_{(3)} \wedge F_{(3)} \wedge C_{(4)}$$

$$\Rrightarrow \quad \int_{T^6} H_{(3)}^{(bg)} \wedge F_{(3)}^{(bg)} = \underset{\Downarrow}{N_3}$$

Net charge of
D3-branes / O3-planes
(pos charge) (neg charge)

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}$

" Tadpole cancellation conditions "

[Gauss law]

Upon dimensional reduction in the presence of background fluxes one obtains general gauged supergravities in 4D. The action is given by :

$$S_{\substack{\text{gauged}\\ \text{SUGRA}}} = \int d^4x \sqrt{-|g|}\, \Big\{ \frac{R}{2}$$

$$+ \frac{1}{96}\, \mathrm{Tr}\big[\overset{\text{charged scalars}}{\nabla_\mu M}\, \nabla^\mu M^{-1} \big] - \overbrace{V(\phi, \underbrace{\text{fluxes}}_{\text{couplings}})}^{\text{Scalar potential}}$$

$$+ \frac{1}{4}\, \mathcal{I}_{\Lambda\Sigma}(\phi)\, H_{\mu\nu}{}^{\Lambda}\, \underset{\hookrightarrow \text{ non-abelian vectors}}{H^{\mu\nu\,\Sigma}}$$

$$+ \frac{1}{4}\, \frac{1}{2\sqrt{-|g|}}\, \mathcal{R}_{\Lambda\Sigma}(\phi)\, \varepsilon^{\mu\nu\rho\sigma}\, H_{\mu\nu}{}^{\Lambda}\, H_{\rho\sigma}{}^{\Sigma}$$

$$+ \underset{\hookrightarrow \text{ topological terms}}{\mathcal{L}_{top}} + \text{fermi-terms} + \underset{\hookrightarrow \text{ to restore susy}}{\text{fermi masses}} \Big\}$$

Strategy : Use gauged supergravities as an effective 4D description of flux compactifications.

⇛ $G = E_{7(7)}$ symmetry as a guiding principle !!

* **Scalar potential :** This is probably the most distinctive feature of a gauged supergravity.

Recalling its expression in $\mathcal{N}=8$ gauged supergravity

$$V(M,X) = \frac{g^2}{672}\Big[X_{MN}{}^{R}\, X_{PQ}{}^{S}\, M^{MP}\, M^{NQ}\, M_{RS} + 7\, X_{MN}{}^{Q}\, X_{PQ}{}^{N}\, M^{MP} \Big]$$

⇛ $\boxed{V(M,X) \quad \text{vs} \quad V(M,\text{fluxes})}$

⤋

$$\boxed{\text{Embedding Tensor} \iff \text{Type II fluxes}}$$

⤋

"CHARTING THE LANDSCAPE OF TYPE II FLUX COMPACTIFICATIONS"

As we saw before, background fluxes $H^{(bg)}_{mnp}$, etc carry internal space-time indices $m,n=1,\dots,6$ that transform under $SL(6)$ internal diffeomorphisms.

In order to establish a neat dictionary between fluxes and components of the embedding tensor it will prove useful to perform a group-theoretical branching of $X_{MN}{}^{P} \in 912$ of $E_{7(7)}$:

$$E_{7(7)} \supset SL(2)\times SO(6,6) \supset SL(2)\times SL(6)$$

$$912 \equiv X_{MN}{}^{P} \rightarrow (2,12) \equiv \xi_{\alpha M} \longrightarrow (2,6) + (2,6')$$

$$(2,220) \equiv f_{\alpha MNP} \left\{ \begin{array}{l} \overset{(H_{mnp},\, F_{mnp})}{\boxed{(2,20)}} + (2,6+84) \\ + (2,20) + (2,6'+84') \end{array} \right.$$

$$(3,32) \equiv \Xi_{\alpha\beta\mu} \longrightarrow \overset{F_m}{\boxed{(3,6')}} + (3,20) + (3,6)$$

$$(1,352') \equiv F_{M\dot{\mu}} \left\{ \begin{array}{l} \overset{F_{mnpqr}}{\boxed{(1,6)}} + \overset{\omega_{mn}{}^{p}}{\boxed{(1,6'+84')}} \\ + (1,70+20+70') \\ + \underset{H_m}{\boxed{(1,6)}} + (1,6+84) \end{array} \right.$$

NOTE: $\alpha=1,2$ of $SL(2)$
$M=1,\dots,12$ of $SO(6,6)$
$\mu,\dot{\mu}=1,\dots,32$ of $SO(6,6)$ [M-W]

In this manner we encounter the following Type IIB background fluxes :

- $(F_{(3)}, H_{(3)}) \in (2, 20)$
- $F_{(1)} = d C_{(0)} \in (3, 6')$
- $H_{(1)} = d \Phi \in (1, 6)$
- $F_{(5)} \in (1, 6)$

“Gauge background fluxes”

- $\omega_{mn}{}^{p} \in (1, 84')$ $\Rightarrow$ “ Metric fluxes ”

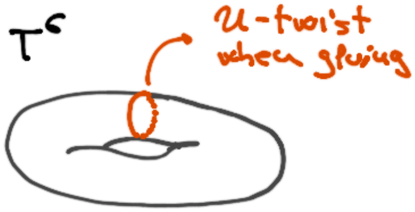
as components of the embedding tensor $X_{MN}{}^{R} \in 912$ of $E_{7(7)}$.

* **Metric fluxes and twisted tori** : Introduce a twist on the T^6 one-form basis

$$[\mathcal{U}(y)]^{m}{}_{n} \in G_T$$

so that

$$e^{m} = [\mathcal{U}(y)]^{m}{}_{n} \, dy^{n}$$



and

$$ds_6^2 = \delta_{mn}\, e^m e^n$$

The twist is based on a twist group G_T with algebra

$$[E_m, E_n] = \overbrace{\omega_{mn}{}^p}^{\text{structure constants of } \mathfrak{g}_T} E_p$$

The left-invariant one-forms $e^p(y)$ satisfy the "Maurer-Cartan" equation

$$de^p + \frac{1}{2}\,\omega_{mn}{}^p\, e^m \wedge e^n \overset{\text{vanishing curvature}}{=} 0$$

with $\omega_{mn}{}^p \equiv$ torsion on the original torus given by

$$\omega_{mn}{}^p = [U^{-1}]_m{}^{m'}\,[U^{-1}]_n{}^{n'}\left(\partial_{m'}[U]^p{}_{n'} - \partial_{n'}[U]^p{}_{m'}\right)$$

Introducing a twisted exterior derivative $D \equiv d + \omega$ and demanding $D^2 = 0$ one gets

$$\omega_{[mn}{}^p\,\omega_{r]s}{}^q = 0 \quad \Rightarrow$$

Jacobi identity
the algebra $\mathfrak{g}_T$

$\Downarrow$

Quadratic constraints
on the metric fluxes

NOTE: Twisted torus $\equiv$ Group manifold (locally)

*** Quadratic constraints and sources :** Let us start from the Quadratic Constraints (QC) on the embedding tensor $X_{MN}{}^{P}$ with non-zero components

$$\underbrace{H_{mnp}\,,\; F_{mnp}\,,\; \omega_{mn}{}^{p}}_{\text{Background fluxes}} \subset \underbrace{X_{MN}{}^{P}}_{\text{Embedding tensor}}$$

- $QC_{\mathcal{N}=8}$ in 4D :

$$\Omega^{MN} X_{MP}{}^{Q} X_{NR}{}^{S} = 0 \quad \Rightarrow \quad \begin{array}{ll} \text{i)} & H_{[mnp} F_{qrs]} = 0 \\ \text{ii)} & \omega_{[mn}{}^{p}\, \omega_{q]p}{}^{r} = 0 \\ \text{iii)} & \omega_{[mn}{}^{p}\, H_{qr]p} = 0 \\ \text{iv)} & \omega_{[mn}{}^{p}\, F_{qr]p} = 0 \end{array}$$

- Sources in 10D : [Twisted derivative $D \equiv d + \omega$]

$[N_3 = N_{O3} - N_{D3}]$

i) $H_3 \wedge F_3 = N_3 = 0$ $\Rightarrow$ Absence of D3/O3 sources

ii) $D^2 = 0$ $\Rightarrow$ Absence of KK5-branes

iii) $D H_{(3)} = 0 \Rightarrow$ Absence of NS5-branes

iv) $\underbrace{\omega\, F_{(3)}}_{4\text{-form}} = 0 \Rightarrow$ Absence of D5-branes

$4\text{-form} \wedge C_{(6)} \Rightarrow N_5 = 0$

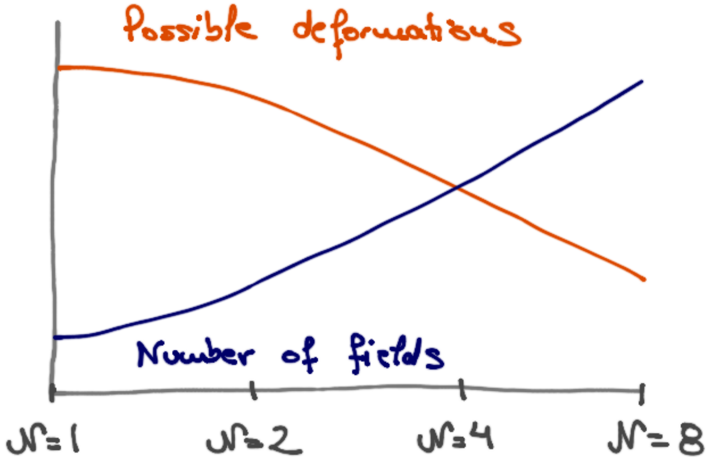
Message : No net charge is allowed for any type of sources in order to preserve $\mathcal{N}=8$ supersymmetry in the 4D compactified theory.

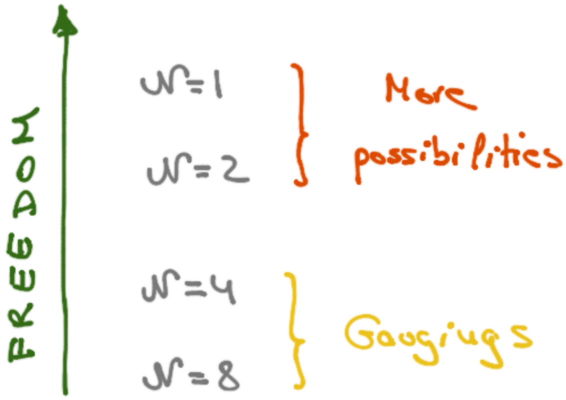
If we start adding sources in 10D we will break (some) supersymmetries

$QC_{\mathcal{N}=1}$	$\subset$	$QC_{\mathcal{N}=2}$	$\subset$	$QC_{\mathcal{N}=4}$	$\subset$	$QC_{\mathcal{N}=8}$
[String Pheno]		[Black holes]		[DFT]		[AdS/CFT]

- Moduli stabilisation
- String Cosmology

$\approx\!>$





XVI. Type IIB moduli stabilisation

Type IIB dimensional reduction from 10D to 4D produces a large set of scalar fields $\phi^{i=1,\dots,70}$ spanning the coset space

$$M(\phi) \in \frac{E_{7(7)}}{SU(8)} \quad \Rightarrow \quad \overbrace{133}^{E_{7(7)}} - \overbrace{63}^{SU(8)} = 70 \text{ scalars}$$

$$\Downarrow$$

$$\boxed{\text{Moduli fields}}$$

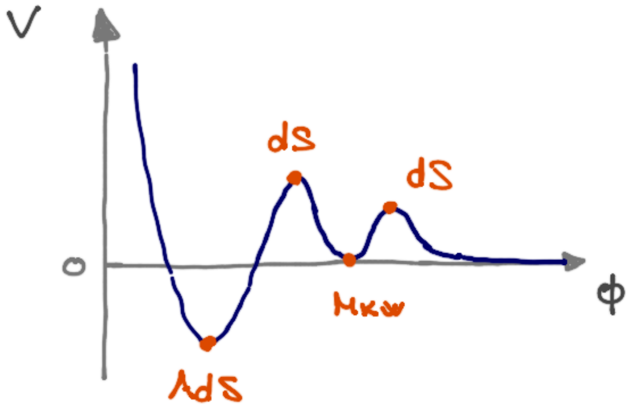
Problem: Massless scalars = Long range interactions

$$\Downarrow$$

(Precision tests of GR)

"Moduli problem"

Solution (?): Fluxes $\Rightarrow V(\phi, \text{fluxes}) = \underbrace{m^2_{ij}}_{\text{masses} \,=\, \text{fluxes}} \phi_i \phi_j + \dots$



$\Rrightarrow$

E.O.M : $\Box\phi = \dfrac{\partial V}{\partial \phi}$

$$\langle\phi\rangle = \phi_0 \Rightarrow \left.\frac{\partial V}{\partial \phi}\right|_{\phi=\phi_0} = 0$$

$$\Rightarrow \boxed{\Lambda_{cc} \equiv V(\phi_0)}$$

Question : Do fluxes suffice to $\overbrace{\text{stabilise}}^{m_{ij}^2>0}$ moduli in a de Sitter $\underbrace{(dS)}_{\Lambda_{cc}>0}$ [quasi Minkowski (Mkw)] vacuum ?

Technical difficulty : 70 scalars are way too many to extremise $V(\phi)$

$\Rightarrow$ Set most of them to zero consistently by virtue of symmetry argument :

$$\underbrace{\mathbb{Z}_2^*}_{\substack{\text{Orientifold}\\ \text{involution}\\ \Omega_p(-1)^{F_L}\sigma^*}} \times \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\frac{T^6}{\mathbb{Z}_2\times\mathbb{Z}_2}\ \text{orbifold}} \subset E_{7(7)} \approx> \underbrace{\frac{E_{7(7)}}{SU(8)}}_{70\text{ scalars}} \supset \underbrace{\left[\frac{SL(2)}{SO(2)}\right]^7}_{14\text{ scalars}}$$

$$T^6 = \underbrace{T^2 \times T^2 \times T^2}_{\substack{\text{exchange}\\ \text{symmetry}}}$$

$$\Downarrow$$

$$\underbrace{\left[\frac{SL(2)}{SO(2)}\right]^7}_{14\text{ scalars}} \supset \underbrace{\left[\frac{SL(2)}{SO(2)}\right]^3}_{\boxed{6\text{ scalars}}}\ \checkmark$$

Sources : We will allow for sources preserving at least $\mathcal{N}=1$ supersymmetry.

Equivalently

$$QC_{\mathcal{N}=1} = 0 \quad \text{but} \quad QC_{\mathcal{N}>1} = N_{\text{sources}} \neq 0$$

✓

and the resulting 4D supergravity is then $\mathcal{N}=1$ supersymmetric. The potential $V(\phi, \text{fluxes})$ takes the schematic form

$$V(\phi) = V_{\text{fluxes}}(\phi) + \underbrace{V_{\text{sources}}(\phi)}_{}$$

Terms proportional to $QC_{\mathcal{N}>1} \neq 0$

* **The $\mathcal{N}=1$ SUGRA model** : We have reduced the model to an $\mathcal{N}=1$ SUGRA coupled to 3 chiral superfields whose (complex) scalar component we denote :

- $S = \chi_S + i\, e^{-\phi_S} \in \frac{SL(2)}{SO(2)}$

- $T = \chi_T + i\, e^{-\phi_T} \in \frac{SL(2)}{SO(2)}$ $\Rrightarrow$ " STU models "

  [6 (real) scalars]

- $U = \chi_U + i\, e^{-\phi_U} \in \frac{SL(2)}{SO(2)}$

The 10D type IIB origin of these scalar fields is given by

- $S = C_{(0)} + i\, e^{-\Phi}$ $\Rrightarrow$ Axion-dilaton

- U $\Rrightarrow$ Complex structure modulus [shape of T^6]

$$G_{mn} = \frac{\mathrm{Im}\,T}{\mathrm{Im}\,U}\begin{bmatrix} |U|^2 & -\mathrm{Re}\,U \\ -\mathrm{Re}\,U & 1 \end{bmatrix} \otimes \mathbb{I}_3$$

$\hookrightarrow T^6 = T^2 \times T^2 \times T^2$

- $T = \frac{1}{\mathrm{Vol}_6} \int_{T^6} \overbrace{C_{(4)}}^{\text{purely internal}} \wedge \underbrace{\omega}_{\text{2-cycle on } T^6} + i\, e^{-\Phi} A^2_{T^2}$

  $\hookrightarrow A_{T^2} \equiv \mathrm{Vol}_{T^2}$

$\Rrightarrow$ Kähler modulus [size of T^6]

In order to generalise the results here to more general SU(3)-structure manifolds (like CY_3 manifolds), let us introduce a set of SU(3)-structure forms:

$$J \equiv \text{2-form} \in \mathbb{R} \quad , \quad \Omega \equiv \text{Holomorphic 3-form} \in \mathbb{C}$$

in terms of which

$$\Omega(U) = (e^1+Ue^2)\wedge(e^3+Ue^4)\wedge(e^5+Ue^6)$$

$$T = \frac{1}{vol_6} \int_{M_6} \underbrace{\left(C_{(4)} + \frac{i}{2} e^{-\Phi} \overset{\curvearrowright A_{T^2}}{J\wedge J} \right)}_{\mathcal{J} \equiv \text{complexified Kähler 4-form}} \wedge \underbrace{\omega}_{\text{2-cycle}} \quad , \quad vol_6 = \int_{M_6} \Omega \wedge \bar{\Omega}$$

As an $\mathcal{N}=1$ theory, the full Lagrangian is encoded in a Kähler potential $K(S,T,U) \in \mathbb{R}$ and a holomorphic superpotential $W(S,T,U) \in \mathbb{C}$.

The Kähler potential for this model is given by

$$K(S,T,U) = -\operatorname{Log}\left[-i(S-\bar{S})\right] - 3\operatorname{Log}\left[-i(T-\bar{T})\right] - 3\operatorname{Log}\left[-i(U-\bar{U})\right]$$

The superpotential depends on the IIB fluxes F_{mnp}, H_{mnp}, etc. that are being considered. Including only gauge background fluxes (F_{mnp}, H_{mnp}) one gets

$$W(S,U) = \int_{M_6 = T^6} \left(F_{(3)} - S\, H_{(3)} \right) \wedge \Omega(U)$$

$$\hookrightarrow T^6 = T^2 \times T^2 \times T^2$$

NOTE:

$$\left.\begin{aligned} F_{(3)} &= a_0\,\beta^0 + a_1\,\beta^1 + a_2\,\alpha_1 + a_3\,\alpha_0 \\ H_{(3)} &= b_0\,\beta^0 + b_1\,\beta^1 + b_2\,\alpha_1 + b_3\,\alpha_0 \end{aligned}\right\} \underbrace{(\alpha_0, \alpha_1, \beta^1, \beta^0)}_{\text{3-cycles on } T^6}$$

$$= P_F(U) - S\,P_H(U)$$

with

$$P_F(U) = a_0 - 3\,a_1\,U + 3\,a_2\,U^2 - a_3\,U^3$$

$$P_H(U) = b_0 - 3\,b_1\,U + 3\,b_2\,U^2 - b_3\,U^3$$

The $\mathcal{N}=1$ scalar potential for $\Phi^i = \{S, T, U\}$

$$V = e^K \left[K^{i\bar{j}}\, D_i W\, D_{\bar{j}} \overline{W} - 3\, W\, \overline{W} \right]$$

with $K_{i\bar{j}} = \dfrac{\partial K}{\partial \Phi^i\, \partial \overline{\Phi}^{\bar{j}}}$ and $D_i W = \dfrac{\partial W}{\partial \Phi^i} + \dfrac{\partial K}{\partial \Phi^i}\, W$ being the "Kähler derivative".

Important: This scalar potential V suffices to stabilise (S, U) by solving $D_S W = 0$, $D_U W = 0$

... but leaves T unstabilised !!

$\Rightarrow$ How to stabilise T ??

Important : Stabilising (S,U) requires background fluxes for which

$$H_{(3)} \wedge F_{(3)} = N_3 = \underbrace{N_{O3}}_{\text{O3-planes must be present !!}} - N_{D3} > 0$$

≈> $QC_{\mathcal{N}=4} = 0 \quad \subset \quad QC_{\mathcal{N}=8} \neq 0$

✓ ✗

≈> SUSY reduced to $\mathcal{N}=4$ due to the presence of sources.

XVII. Stabilising the Kähler modulus T

We will present now two possible mechanisms to stabilise the Kähler modulus

a) Non-perturbative effects : They introduce exponentials in the superpotential

$$W(S,T,U) = W_{fluxes}(S,U) + W_{np}(T)$$

with

$$W_{np} = A\, e^{-aT}$$

and $(A, a) \equiv$ Model dependent quantities

- Gaugino condensation on D7-branes
- D-brane instantons

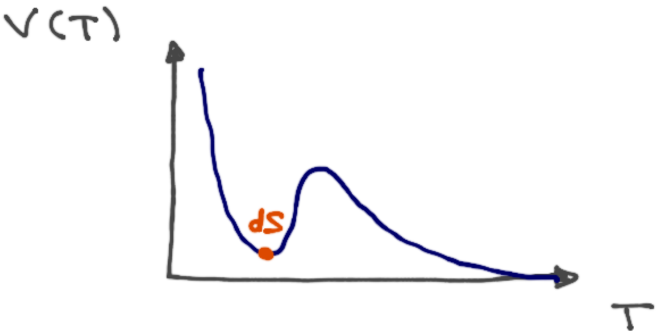
!

(anti D3)

* Application : dS vacua
- Two-step procedure [KKLT]
- Single-step procedure

$A = A(S,U)$; $A = A(M)$

squarks condensates ≡ open string sector



b) Non-geometric fluxes : Conjectured on the basis of $E_{7(7)}$ covariance (stringy dualities)

Recalling the embedding tensor group theoretical decompositions

$$E_{7(7)} \supset SL(2) \times SO(6,6) \supset SL(2) \times SL(6)$$

$$912 \equiv X_{MN}{}^{P} \rightarrow (2,220) \equiv f_{\alpha MNP} \left\{ \begin{array}{l} \overset{(F_{mnp},\, H_{mnp})}{\boxed{(2,20)}} + \overset{(Q^{mn}{}_{p},\, P^{mn}{}_{p})}{\boxed{(2,6+84)}} \\ + (2,20) + (2,6'+84') \end{array} \right.$$

with $(Q^{mn}{}_{p}\,,\ P^{mn}{}_{p}) \equiv$ " Non-geometric fluxes "

$\rightarrow D^2=0$ with $D \equiv d + Q\cdot + P\cdot$

$$\approx> \ QC_{\mathcal{N}=8} \text{ in } D=4 \ => \left\{ \begin{array}{l} \cdot \ Q\,Q = P\,P = QP + PQ = 0 \\ \cdot \ \underbrace{Q \cdot F_{(3)}}_{2\text{-form}} = P \cdot H_{(3)} = Q \cdot H_{(3)} + P \cdot F_{(3)} = 0 \end{array} \right.$$

$\wedge\, C_{(8)} \Rightarrow N_7 = 0$

The superpotential is given by

specific index contraction

$$W(S,U,T) = \int_{M_6=T^6} \Big[\big(F_{(3)} - S\, H_{(3)}\big) + \big(Q - S\, P\big) \cdot \mathcal{J}(T) \Big] \wedge \Omega(U)$$

$$= P_F(U) - S\, P_H(U) + \underbrace{\big(P_Q(U) - S\, P_P(U) \big)\, T}_{}$$

New flux couplings from $E_{7(7)}$-covariance !!

with

$$\mathcal{P}_Q(U) = c_0 + 3\, c_1\, U - 3\, c_2\, U^2 - c_3\, U^3$$

$$\mathcal{P}_P(U) = d_0 + 3\, d_1\, U - 3\, d_2\, U^2 - d_3\, U^3$$

* Application : dS vacua with $N_3 \neq 0$ & $N_7 = 0$ ✓

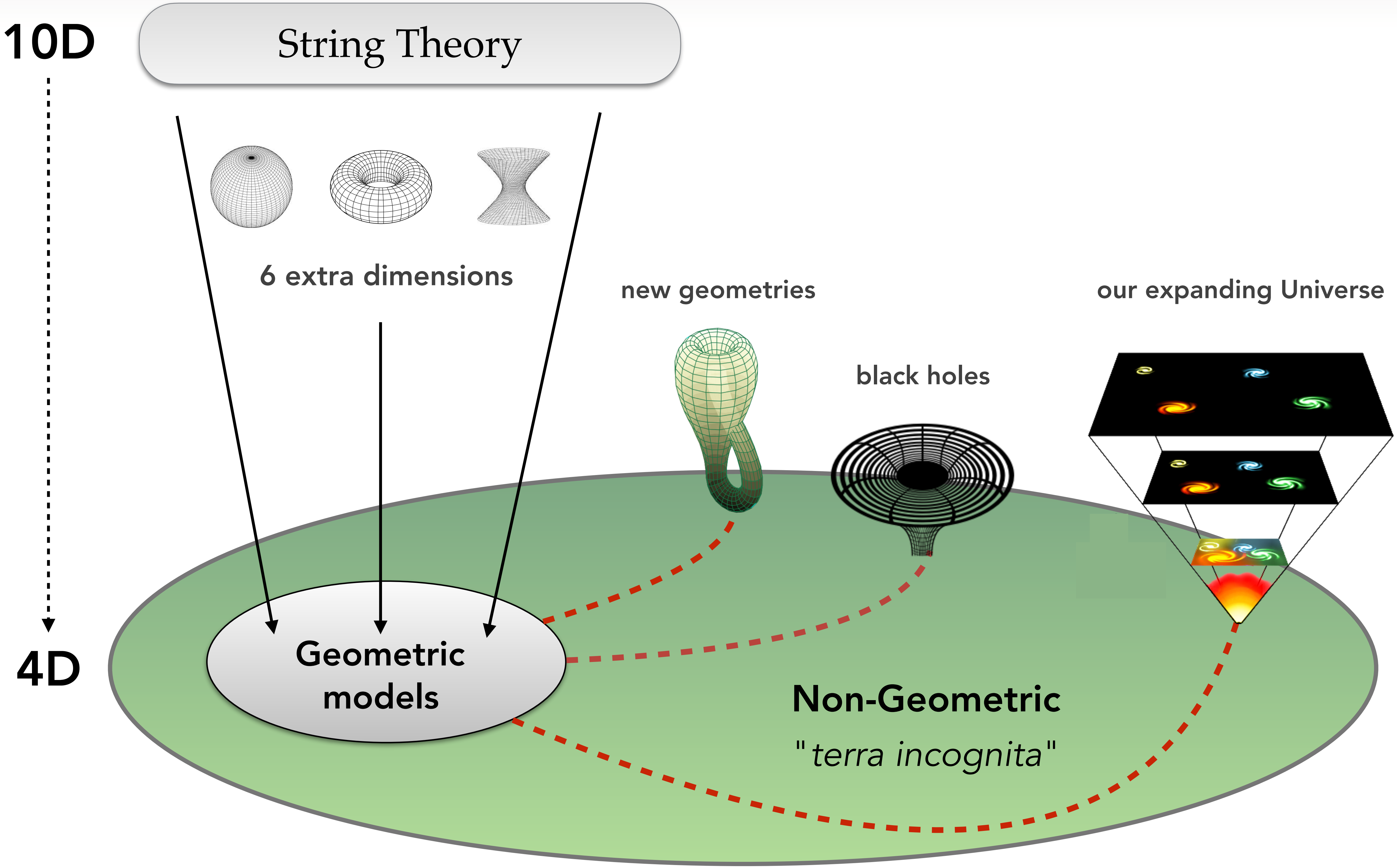
... but ... higher-dimensional origin ?

XVIII. Some final considerations

- Poincaré vs duality covariance
- Generalised geometries :

$$D = d + \omega + Q\cdot + \dots$$

- DFT and EFT : Duality-covariant reformulation of Type IIB and 11D supergravities
- Phenomenological implications : moduli stabilisation, string cosmology, ...
- Holography



10D

String Theory

6 extra dimensions

new geometries

black holes

our expanding Universe

4D

Geometric models

Non-Geometric

"*terra incognita*"