2. From random walk to probability distributions

The aim of this chapter is to understand why the Gaussian probability distribution appears so often in Physics.

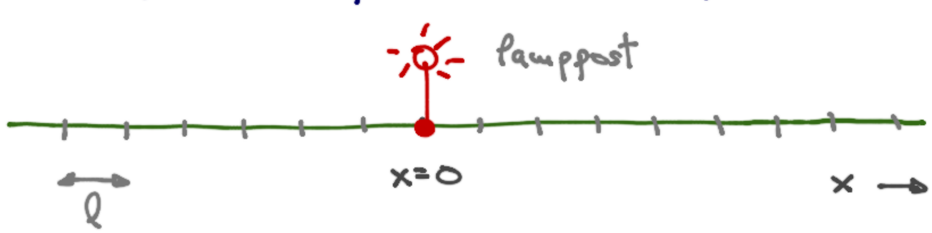
(∞)

Statistical ensemble : It is an idealisation consisting of a large number of virtual copies of a system, considered all at once, each of which represents a possible state that the real system might be in. In other words, a statistical ensemble is a probability distribution for the state of the system.

I. A simple probabilistic system : Random Walk

The problem is stated as follows :

* A drunk starts out from a lamppost ($x=0$) and each step is of equal length ℓ.
* Step to right with probability p and to left with probability $q = 1 - p$. Each step is independent of the preceding step.



* After N steps the drunk will be at $x = m\ell$
* Compute the probability (distribution) $P_N(m)$ of finding the drunk at position $x = m\ell$ of N steps.

Language: One can replace the drunk by a particle.

Denoting

$$n_1 \equiv \text{steps to the right}$$

$$n_2 \equiv \text{steps to the left}$$

one has

$$\left.\begin{aligned} N &= n_1 + n_2 \\ m &= n_1 - n_2 \end{aligned}\right\} \quad \begin{aligned} n_1 &= \tfrac{1}{2}(N+m) \\ n_2 &= \tfrac{1}{2}(N-m) \end{aligned}$$

Notice that

$$m = n_1 - n_2 = 2n_1 - N \quad \Rightarrow \quad \begin{cases} N \text{ odd} \Rightarrow m \text{ odd} \\ N \text{ even} \Rightarrow m \text{ even} \end{cases}$$

Probability of a sequence: Since there is no memory, the probability of any <u>one</u> given sequence of n_1 steps to the right and n_2 steps to the left is

$$\underbrace{p \cdots p}_{n_1}\ \underbrace{q \cdots q}_{n_2} = p^{n_1} q^{n_2}$$

But a given pair (n_1, n_2) can have multiple realisations so that the particle ends up at the same $x = m\ell$ after the N steps

$$\text{Multiplicity} = \frac{N!}{n_1!\, n_2!}$$

Ex: $N = 3$

		n_1	n_2	m	$\frac{N!}{n_1!\, n_2!}$
1)	→ → →	3	0	3	1
2)	← → →				
3)	→ ← →	2	1	1	3
4)	→ → ←				
5)	← ← →				
6)	← → ←	1	2	−1	3
7)	→ ← ←				
8)	← ← ←	0	3	−3	1

The probability of taking (in a total of N step) n_1 steps to the right and $n_2 = N - n_1$ steps to the left is

$$\boxed{W_N(n_1) = \underbrace{\frac{N!}{n_1!\,(N-n_1)!}}_{\text{multiplicity}}\; p^{n_1}\,(1-p)^{N-n_1}}$$

Using $n_1 = \frac{1}{2}(N+m)$ yields

$$P_N(m) = W_N(n_1) = \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\,\left[\frac{1}{2}(N-m)\right]!}\; p^{\frac{1}{2}(N+m)}\,(1-p)^{\frac{1}{2}(N-m)}$$

Ex: In the case $N = 3$ with $p = q = 1/2$ one finds

$$P_3(m) = W_3(n_1) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$$

The probability distribution above is called "binomial dist" by virtue of the binomial expansion

$$(p+q)^N = \sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!}\, p^n q^{N-n}$$

II. Random walk and mean values

★ Mean values in Mathematics : Let u be a variable that can take any of the M discrete values

$$u_1, u_2, \dots, u_M$$

with respective probabilities (stochastic variable)

$$P(u_1),\ P(u_2), \dots, P(u_M)$$

The mean or average value of u is defined by

$$\bar{u} \equiv \frac{P(u_1)\,u_1 + P(u_2)\,u_2 + \dots + P(u_M)\,u_M}{P(u_1) + P(u_2) + \dots + P(u_M)} = \frac{\sum_{i=1}^{M} P(u_i)\,u_i}{\underbrace{\sum_{i=1}^{M} P(u_i)}_{}}$$

$$= \sum_{i=1}^{M} P(u_i)\,u_i \ .$$

$= 1$ as $P(u_i)$ is defined as a probability

"normalisation cond"

More generically on has

$$\overline{f(u)} \equiv \sum_{i=1}^{M} P(u_i)\, f(u_i)$$

for a function $f(u)$.

• Property 1 : $\overline{f(u) + g(u)} = \sum_{i=1}^{M} P(u_i)\left[f(u_i) + g(u_i)\right]$

$$= \sum_{i=1}^{M} P(u_i) f(u_i) + \sum_{i=1}^{M} P(u_i)\, g(u_i)$$

$$= \overline{f(u)} + \overline{g(u)} .$$

• Property 2 : $\overline{c f(u)} = c\, \overline{f(u)}$ for $c \equiv$ cte.

There are some specific mean values which are particularly relevant in statistical physics

0) mean value $\bar{u} = \sum_{i=1}^{M} P(u_i)\, u_i$

1) first moment of u about its mean : $\overline{\Delta u} \equiv \overline{(u - \bar{u})} = \bar{u} - \bar{u} = 0$

2) second moment of u about its mean or "dispersion"

$$\overline{(\Delta u)^2} \equiv \sum_{i=1}^{M} P(u_i)(u_i - \bar{u})^2 \geq 0$$

$\Delta^{*} u \equiv \sqrt{\overline{(\Delta u)^2}} \equiv$ "Standard deviation"

Note that

$$\overline{(\Delta u)^2} = \overline{(u - \bar{u})^2} = \overline{(u^2 - 2u\bar{u} + \bar{u}^2)} = \overline{u^2} - 2\bar{u}^2 + \bar{u}^2 = \overline{u^2} - \bar{u}^2$$

$$\Rightarrow \overline{(\Delta u)^2} = \overline{u^2} - \bar{u}^2 \geq 0 \Rightarrow \overline{u^2} \geq \bar{u}^2$$

n) n^{th} moment of u about its mean : $\overline{(\Delta u)^n}$

NOTE: Probability generating function $\Rightarrow g(t, u) \equiv \overline{e^{tu}}$

$$\Rightarrow \overline{u^k} = \left.\frac{d^k g}{dt^k}\right|_{t=0}$$

* Mean values for the random walk system : Let us focus on the binomial probability distribution

$$W(n_1) = \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1}\, q^{N-n_1}$$

• Normalisation :

$$\sum_{n_1=0}^{N} W(n_1) = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1}\,(1-p)^{N-n_1} = \underbrace{(p+q)}_{1}{}^{N} = 1^N = 1$$

o) Mean number of steps to the right :

$$\overline{n_1} \equiv \sum_{n_1=0}^{N} W(n_1)\, n_1 = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1}\, q^{N-n_1}\, n_1 = (*)$$

NOTE : $n_1\, p^{n_1} = p\, \frac{d}{dp}\left(p^{n_1}\right)$

$$(*) = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, q^{N-n_1}\, p\, \frac{d}{dp}\left(p^{n_1}\right)$$

$$= p\, \frac{d}{dp}\left[\sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1}\, q^{N-n_1}\right] = p\, \frac{d}{dp}\left[(p+q)^N\right]$$

$$= Np\, \underbrace{(p+q)}_{1}{}^{N-1} = Np \implies \boxed{\overline{n_1} = Np}$$

NOTE: Similarly one finds that $\overline{n_2} = Nq \Rightarrow \overline{n_1} + \overline{n_2} = N\overbrace{(p+q)}^{1} = N.$

2) Dispersion :

$$\overline{(\Delta n_1)^2} = \overline{n_1^2} - \overline{n_1}^2 = \underbrace{\overline{n_1^2}}_{} - (Np)^2$$

This is what we have to compute

$$\overline{n_1^2} \equiv \sum_{n_1=0}^{N} W(n_1)\, n_1^2 = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1} q^{N-n_1}\, n_1^2 = (*)$$

NOTE : $n_1^2\, p^{n_1} = \left[p \frac{d}{dp} \right]^2 (p^{n_1})$

$$(*) = \left(p \frac{d}{dp} \right)^2 \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1} q^{N-n_1} =$$

$$= \left(p \frac{d}{dp} \right)^2 (p+q)^N = p \frac{d}{dp} \left(pN (p+q)^{N-1} \right)$$

$$= pN \underbrace{(p+q)}_{1}{}^{N-1} + p^2 N (N-1) \underbrace{(p+q)}_{1}{}^{N-2}$$

$$= pN \left[1 + p(N-1) \right] = pN \left[\overbrace{(1-p)}^{q} + pN \right]$$

$$= Npq + (Np)^2.$$

So one finds

$$\overline{(\Delta n_1)^2} = \overline{n_1^2} - (Np)^2 = Npq \implies \boxed{\overline{(\Delta n_1)^2} = Npq}$$

It is also meaningful to introduce the relative standard deviation

$$\frac{\Delta^* n_1}{\overline{n_1}} = \frac{\sqrt{\overline{(\Delta n_1)^2}}}{\overline{n_1}} = \frac{\sqrt{Npq}}{Np} = \sqrt{\frac{q}{p}}\, \frac{1}{\sqrt{N}} \quad \xRightarrow[p=q]{} \quad \frac{\Delta^* n_1}{\overline{n_1}} = \frac{1}{\sqrt{N}}$$

In terms of $m = n_1 - n_2 = 2n_1 - N$ one has

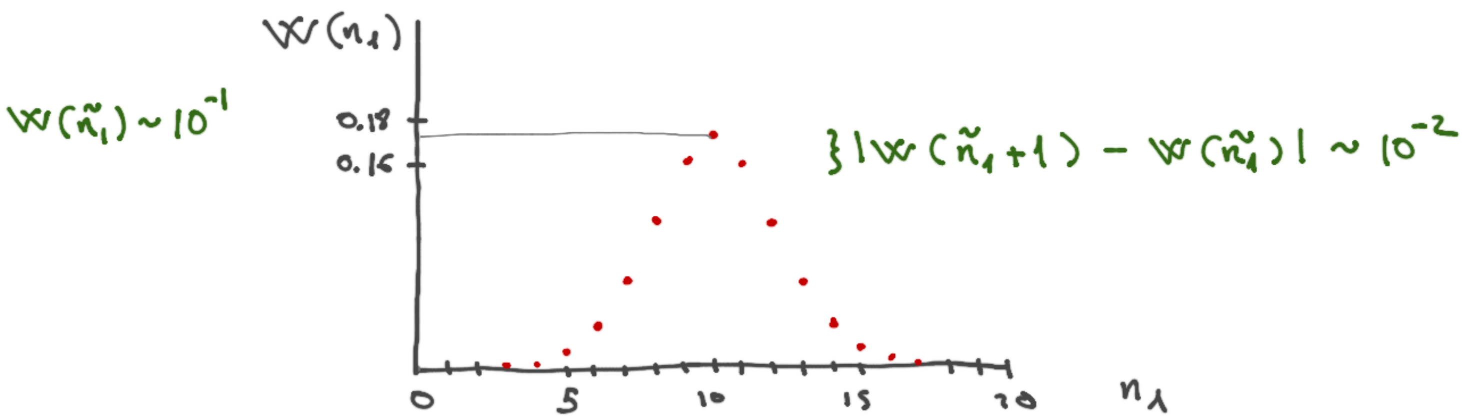
0) $\overline{m} = 2\overline{n}_1 - N = 2Np - N = Np + N(p-1) = N(p-q)$

1) $\Delta m \equiv m - \overline{m} = (2n_1 - N) - (2\overline{n}_1 - N) = 2(n_1 - \overline{n}_1) = 2\Delta n_1 \Rightarrow (\Delta m)^2 = 4(\Delta n_1)^2$

2) $\overline{(\Delta m)^2} \equiv 4\overline{(\Delta n_1)^2} = 4Npq$

III. Probability distribution at large N

When $N \gg 1$ the probability distribution $W(n_1)$ has a sharp maximum around a value $n_1 = \tilde{n}_1$.



$N = 20$

If N is large and one considers regions around the maximum $n_1 = \tilde{n}_1$, one has that

$$|W(n_1+1) - W(n_1)| \ll W(n_1)$$

Idea: W can, to a good aproximation, be considered as a continuous function of the continuous variable n_1 although only $n_1 \in \mathbb{N}$ are of physical relevance.

The maximum of $W(n_1)$ is located at

$$\frac{dW}{dn_1} = 0 \qquad \text{or equivalently} \qquad \frac{d \ln W}{dn_1} = 0$$

NOTE: $\ln W$ is a much more slowly varying function.

Taylor expanding around $n_1 = \tilde{n}_1$ gives

$$\ln W(n_1) = \ln W(\tilde{n}_1) + B_1 (n_1 - \tilde{n}_1) + \frac{1}{2!} B_2 (n_1 - \tilde{n}_1)^2 + \ldots$$

with

$$B_k = \left. \frac{d^k \ln W}{dn_1^k} \right|_{n_1 = \tilde{n}_1}$$

Since $n_1 = \tilde{n}_1$ is a maximum then $B_1 = 0$ and $B_2 < 0$ so

$$\ln W = \ln W(\tilde{n}_1) - \frac{1}{2} |B_2| (n_1 - \tilde{n}_1)^2 + \text{higher-order terms}$$

To first approximation one finds

$$W = \underbrace{\tilde{W}}_{W(\tilde{n}_1)} \cdot e^{-\frac{|B_2|}{2} (n_1 - \tilde{n}_1)^2}$$

Math: Starting from $W(n) = \frac{N!}{n! \, (N-n)!} \, p^n q^{N-n}$ then

$$\ln W = \ln N! - \operatorname{Log} n! - \ln (N-n)! + n \ln p + (N-n) \ln q$$

Trick: If n is a large integer $n \gg 1$ then $\ln n!$ can be considered an almost continuous function of n since $\ln n!$ changes by a small fraction of itself if n is changed by a small integer. Hence

$$\frac{d \ln n!}{dn} \simeq \frac{\ln (n+1)! - \ln n!}{1} = \ln \frac{(n+1)!}{n!} = \ln (n+1)$$

and for $n \gg 1$

$$\frac{d \ln n!}{dn} \simeq \ln n$$

Then one has that

$$\frac{d}{dn_1} \ln W = -\ln n_1 + \ln (N-n_1) + \ln p - \ln q = \ln \left[\left(\frac{N-n_1}{n_1} \right) \frac{p}{q} \right]$$

NOTE 1: $\frac{d \ln (\overbrace{N-n_1}^{Y})!}{d n_1} = - \frac{d \ln Y!}{dY} \simeq -\ln Y = -\ln (N-n_1)$

So the maximum is located at

$$\left. \frac{d}{dn_1} \ln W \right|_{n_1 = \tilde{n}_1} = 0 \Rightarrow \frac{N-\tilde{n}_1}{\tilde{n}_1} = \frac{q}{p}$$

$$\Rightarrow (N-\tilde{n}_1) p = \tilde{n}_1 q$$

$$\Rightarrow Np = \tilde{n}_1 \overbrace{(p+q)}^{1}$$

$$\Rightarrow \tilde{n}_1 = Np = \overline{n_1} \equiv\!> $$ Maximum at the mean value

Taking a further derivative

$$\frac{d^2}{dn_1} \ln w = \frac{n_1}{N-n_1} \frac{q}{p} \left(-\frac{N}{n_1^2} \frac{p}{q} \right) = -\frac{N}{n_1 (N-n_1)}$$

and evaluating it at the maximum one gets

$$\left. \frac{d^2}{dn_1^2} \ln w \right|_{n_1 = \tilde{n}_1} = - \frac{N}{\tilde{n}_1 (N-\tilde{n}_1)} = - \frac{N}{Np(N-Np)}$$

$$= - \frac{N}{N^2 p \underbrace{(1-p)}_{q}} = - \frac{1}{Npq}$$

$$\Rightarrow \quad B_2 = -\frac{1}{Npq} < 0$$

Finally the factor $\tilde{w} = w(\tilde{n}_1)$ can be determined from the normalisation condition

$$\sum_{n_1=0}^{N} w(n_1) \simeq \int w(n_1)\, dn_1 \underset{\eta = n_1 - \tilde{n}_1}{=} \int_{-\infty}^{\infty} w(\underbrace{\tilde{n}_1 + \eta}_{n_1})\, d\eta \overset{!}{=} 1$$

NOTE: The integration over $-\infty < \eta < \infty$ is an excellent aproximation since the contribution far from $\eta = 0$ is negligible.

$$\int_{-\infty}^{\infty} w(n_1)\, d\eta \simeq \int_{-\infty}^{\infty} \tilde{w}\, e^{-\frac{|B_2|}{2}\eta^2} d\eta = \tilde{w} \sqrt{\frac{2\pi}{|B_2|}} \overset{!}{=} 1$$

see Math

$$\Rightarrow \quad \tilde{w} = \sqrt{\frac{|B_2|}{2\pi}} = (2\pi N p q)^{-1/2}$$

Math: Relevant integrals :

- $$\int_0^\infty e^{-\eta} \eta^p \, d\eta = \Gamma(p+1) = p!$$

- $$\int_{-\infty}^\infty e^{-\eta^2} \, d\eta = \sqrt{\pi}$$

- $$I(p) \equiv \int_0^\infty e^{-\alpha \eta^2} \eta^p \, d\eta = \frac{1}{2} \Gamma\left(\frac{p+1}{2}\right) \alpha^{-\frac{p+1}{2}}$$

so that

$$I(0) = \frac{1}{2} \sqrt{\pi} \, \alpha^{-1/2}$$

$$I(1) = \frac{1}{2} \alpha^{-1}$$

$$I(2) = \frac{1}{4} \sqrt{\pi} \, \alpha^{-3/2}$$

$$I(3) = \frac{1}{2} \alpha^{-2}$$

. . . .

Usin these formulae we obtain

$$\int_{-\infty}^\infty e^{-\frac{1}{2}|B_2| \eta^2} d\eta = 2 \times I(0) = \sqrt{\pi} \frac{\sqrt{2}}{\sqrt{|B_2|}} = \sqrt{\frac{2\pi}{|B_2|}}$$

$$\alpha = \frac{|B_2|}{2}$$

The previous computations show that

$$W(n_1) = (2\pi N p q)^{-1/2} \, e^{\left[-\frac{(n_1 - Np)^2}{2Npq}\right]}$$

Recalling that $\overline{n_1} = \tilde{n}_1 = Np$ and $\overline{(\Delta n_1)^2} = Npq$ then

$$\boxed{W(n_1) = \left(2\pi \, \overline{(\Delta n_1)^2}\right)^{-1/2} e^{-\frac{(n_1 - \overline{n}_1)^2}{2\,\overline{(\Delta n_1)^2}}}}$$

"Gaussian distribution"

$(N \gg 1, \quad n_1 \gg 1)$

IV. Gaussian probability distribution

Let us move to the more physical variable

$$x \equiv m \cdot \ell = (2n_1 - N)\,\ell \quad \Rightarrow \quad dx = \ell\, dm = 2\ell\, dn_1$$

$m = n_1 - n_2$

so that

$$W(x) = [2\pi N p q]^{-1/2} \, e^{-\frac{\left[\frac{x}{\ell} - N(p-q)\right]^2}{8Npq}}$$

NOTE: $(n_1 - Np) = \frac{1}{2}(N + m - 2Np) = \frac{1}{2}[m - N\underbrace{(2p-1)}_{(p-q)}] = \frac{1}{2}[m - N(p-q)]$

$= \frac{1}{2}\left[\frac{x}{\ell} - N(p-q)\right]$

Equating the cumulative probability $\mathcal{P}(x)\,dx = W(n_1)\,dn_1 = W(x)\,\frac{dx}{2\ell}$

$$\mathcal{P}(x) = \frac{1}{2\ell} W(x) = \frac{1}{2\ell}\left[2\pi N p q\right]^{-1/2} e^{-\frac{\left[\frac{x}{\ell} - N(p-q)\right]^2}{8Npq}}$$

$$= \frac{1}{2\ell}\left[2\pi N p q\right]^{-1/2} e^{-\frac{\left[x - N(p-q)\ell\right]^2}{\ell^2\, 8Npq}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = P(m)\,\frac{dx}{2\ell}$$

with $\mu \equiv (p-q)N\ell$ and $\sigma \equiv 2\sqrt{Npq}\,\ell$.

* Normalisation :

$$\int_{-\infty}^{\infty} \mathcal{P}(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$y = x-\mu$
$dy = dx$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$2 \times I(0)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\, 2\,\frac{1}{2}\sqrt{\pi}\,\sqrt{2}\,\sigma = 1 \quad ✓$$

* Mean value of x :

$$\overline{x} = \int_{-\infty}^{\infty} \mathcal{P}(x)\, x\, dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (y+\mu)\, e^{-\frac{y^2}{2\sigma^2}}\, dy$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y\, e^{-\frac{y^2}{2\sigma^2}}\, dy}_{\text{zero} \equiv \text{odd function}} + \mu \frac{1}{\sqrt{2\pi}\,\sigma} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\, dy}_{2 \times I(0)}$$

$$= \mu \frac{1}{\sqrt{2\pi}\,\sigma}\, 2\, \frac{1}{2} \sqrt{\pi}\, \sqrt{2}\, \sigma = \mu = (p-q) N \ell$$

* Dispersion :

$$\overline{(\Delta x)^2} = \overline{(x-\mu)^2} = \int_{-\infty}^{\infty} (x-\mu)^2\, \mathcal{P}(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y^2\, e^{-\frac{y^2}{2\sigma^2}}\, dy$$

$$2 \times I(2) \leftarrow = \frac{1}{\sqrt{2\pi}\,\sigma}\, \underbrace{2\, \frac{1}{4}}_{\frac{1}{2}} \sqrt{\pi}\, \underbrace{(2\sigma^2)^{3/2}}_{\sqrt{2}\,\sigma\; 2\sigma^2} = \sigma^2 = 4Npq\,\ell^2$$

TAKE HOME MESSAGE : Starting from the simplest probabilistic system (random walk) and taking $N \gg 1$ and $n_1 \gg 1$ limits we found that a Gaussian distribution emerges !!

v. Poisson probability distribution

It appears also as a limit of the binomial distribution

$$N \gg 1 \quad , \quad p \ll 1 \quad \text{so} \quad n_1 \ll N$$

Starting from

$$W(n_1) = \underbrace{\frac{N!}{n_1!\,(N-n_1)!}}_{\frac{N(N-1)(N-2)\cdots(N-n_1+1)}{n_1!} \;\simeq\; \frac{N^{n_1}}{n_1!}} p^{n_1} \underbrace{q^{N-n_1}}_{e^{-Np}} \simeq \frac{N^{n_1}}{n_1!} p^{n_1} e^{-Np} = \underbrace{\frac{\lambda^{n_1}}{n_1!} e^{-\lambda}}_{\text{Poisson distribution}}$$

$N \gg n_1$

$N \gg 1$

with $\lambda \equiv Np$.

NOTE : $\log q^{N-n_1} = (N-n_1) \log q = (N-n_1) \log (1-p)$

$= N \; (-p + \text{higher order}) = -Np$

$p \ll 1$

$N \gg n_1$

$\Rightarrow q^{N-n_1} = e^{-Np}$

NOTE : $\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

* Normalisation : $\displaystyle\sum_{n_1=0}^{N} \frac{\lambda^{n_1}}{n_1!} e^{-\lambda} \underset{N \gg 1}{=} e^{-\lambda} \sum_{n_1=0}^{\infty} \frac{\lambda^{n_1}}{n_1!} = e^{-\lambda} e^{\lambda} = 1$ ✓

* Mean value of n_1 :

$$\overline{n_1} = \sum_{n_1=0}^{N} n_1 \frac{\lambda^{n_1}}{n_1!} e^{-\lambda} = \sum_{n_1=1}^{N} n_1 \frac{\lambda^{n_1}}{n_1!} e^{-\lambda}$$

$$= e^{-\lambda} \lambda \sum_{n_1=1}^{N} \frac{\lambda^{(n_1-1)}}{(n_1-1)!} = e^{-\lambda} \lambda \overbrace{\sum_{m=0}^{\infty} \frac{\lambda^m}{m!}}^{e^{\lambda}} = \lambda$$

$n_1 - 1 = m$

$N >> 1$

* Dispersion :

$$\overline{(\Delta n_1)^2} = \underbrace{\overline{n_1^2}}_{(*)} - \overline{n_1}^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

$$(*) \quad \overline{n_1^2} = \sum_{n_1=0}^{N} n_1^2 \frac{\lambda^{n_1}}{n_1!} e^{-\lambda} = e^{-\lambda} \sum_{n_1=1}^{N} n_1 \frac{\lambda^{n_1}}{(n_1-1)!}$$

$$= e^{-\lambda} \lambda \frac{d}{d\lambda} \sum_{n_1=1}^{N} \frac{1}{(n_1-1)!} \lambda^{n_1-1} \lambda = e^{-\lambda} \lambda \frac{d}{d\lambda} \left(e^{\lambda} \cdot \lambda \right)$$

$n_1 - 1 = m$

$N >> 1$

NOTE: $\lambda \dfrac{d \lambda^{n_1}}{d\lambda} = n_1 \lambda^{n_1}$

$$= e^{-\lambda} \lambda \left(e^{\lambda} \lambda + e^{\lambda} \right) = \lambda(\lambda+1) .$$

TAKE HOME MESSAGE : The Poisson distribution has to do with very unlikely $(p << 1)$ events that happen to occur because of a very large population $(N >> 1)$.