[November 2018]

Lectures ULB

Content Part 1 :

I. Particles in 4D

- Minkowski space-time $\mathbb{R}^{1,3}$ with coordinates x^μ and $\mu = 0,1,2,3$

and a flat metric $\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ $\Rightarrow ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$

- Isometries = Poincaré group $\begin{cases} \text{Lorentz } SO(1,3) \text{ rotations } \Lambda^\mu{}_\nu \\ \text{Translations} \end{cases}$

NOTE

Proper : $|\Lambda| = +1$

Orthochronous : $\Lambda^0{}_0 \geq 1$

Preserve direction of time and are connected to the identity e.

- Transformation : $\boxed{x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{cte} x^\nu + \underbrace{c^\nu}_{cte}}$

NOTE: $\Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu \eta_{\mu'\nu'} = \eta_{\mu\nu}$

- Generators :

$SO(1,3)$: $(L_{\alpha\beta})^\mu{}_\nu = -(L_{\beta\alpha})^\mu{}_\nu = 2\,\delta^\mu_{[\alpha}\eta_{\beta]\nu}$ $\begin{cases} J_i = \frac{1}{2}\epsilon_{ijk} L_{jk} \\ K_i = L_{0i} \end{cases}$

Translations: P_μ (abelian)

NOTE: $J = (\text{orbital} + \underbrace{\text{intrinsic}}_{\text{spin}})$ angular momentum

- Casimir operators commute with all the generators $\Rightarrow$ labels for states

$$C_1 = P^\mu P_\mu = -m^2 \quad (i)$$

$$C_2 = W^\mu W_\mu = ??$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu L_{\rho\sigma}$ is the Pauli-Lubanski vector

(i) Massive particle $(m \neq 0)$

First we go to the rest frame where $p^\mu = (p^0, \vec{p}) = (m, \vec{0})$ and then

$$W^\mu = \frac{1}{2} \epsilon^{\mu 0 \rho\sigma} m L_{\rho\sigma} \begin{cases} W^0 = 0 \\ W^i = -\frac{1}{2} m \epsilon^{ijk} L_{jk} = -m J^i \end{cases}$$

$$\Longrightarrow C_2 = W^\mu W_\mu = m^2 \underbrace{J^2}_{J = \text{spin in rest frame}} = m^2 s(s+1) \quad (ii)$$

J = spin in rest frame : $J_- J_+ |\alpha, \beta_{max}\rangle = 0$
$\Rightarrow \alpha = \beta_{max}(\beta_{max}+1)$

(ii) Massless particle $(m=0)$

Now it is not possible to go to a rest frame. This time we compute

$$W^\mu P_\mu = \frac{1}{2} \epsilon^{\overbrace{\mu\nu}^{\text{antisym}}\rho\sigma} P_\nu L_{\rho\sigma} P_\mu = 0 \quad \forall m$$

$$C_2 = W^\mu W_\mu \underset{\substack{m\to 0 \\ (ii)}}{=} 0 \quad , \quad C_1 = p^\mu p_\mu \underset{\substack{m\to 0 \\ (i)}}{=} 0$$

As a result both W^μ and p^μ are light-like and orthogonal. This means that, if we go to a frame where

$$p^\mu = (E, 0, 0, E)$$

then

$$\boxed{W^\mu = \lambda \cdot p^\mu}$$

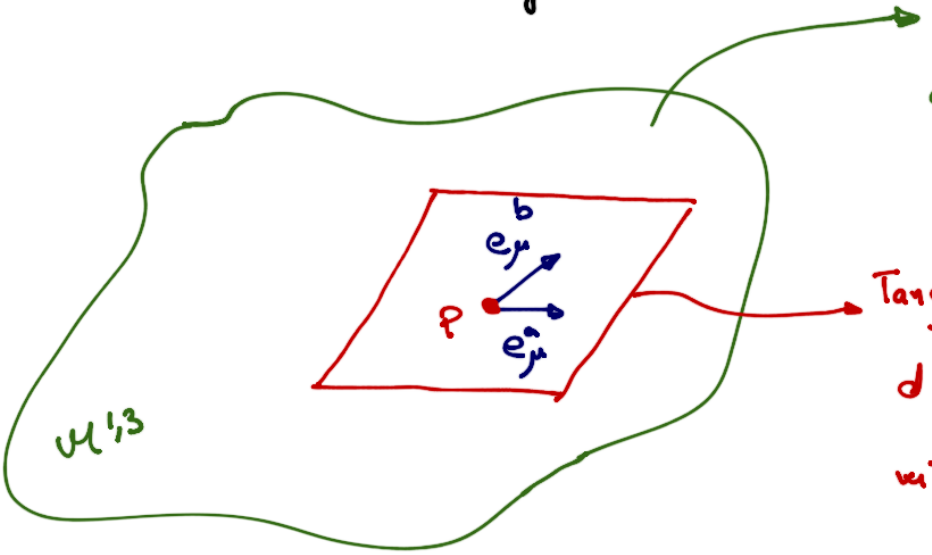
with λ being the helicity. One then has

Projection of $\vec{J}$ along direction of motion $\vec{p}$

$$\left.\begin{aligned} W^0 &= \frac{1}{2} \epsilon^{ijk} P_i L_{jk} = P_i J^i = \vec{P}\cdot\vec{J} \\ -(p^0)^2 + |\vec{p}|^2 = 0 &\Rightarrow p^0 = |\vec{p}| \overset{!}{=} \frac{1}{\lambda} W^0 \end{aligned}\right\} \boxed{\lambda = \frac{W^0}{|\vec{p}|} = \frac{\vec{P}\cdot\vec{J}}{|\vec{p}|}}$$

Under parity $\begin{cases} \vec{x} \to -\vec{x} \\ t \to +t \end{cases} \Rightarrow \begin{matrix} \vec{p} \to -\vec{p} \\ \vec{J} \to \vec{J} \end{matrix} \Rightarrow \lambda \to -\lambda \Rightarrow$ Parity invariance requires states with $\pm\lambda$

- Space-time : metric and frames



space-time $\mathcal{M}^{1,3}$

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

Tangent space (at point p)

$ds^2 = \eta_{ab}\, e^a e^b$

with $e^a = \underbrace{e_\mu{}^a}_{\text{frame}} dx^\mu$

* Locally one can define a "tetrad" $e_\mu{}^a$ such that

$$\boxed{e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}}$$

* GCT vs Local Lorentz transf.
$$\begin{cases} \text{GCT}: & T'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu} T^\nu(x) \\ \text{Local Lorentz}: & T'^a(x) = \Lambda^a{}_b(x)\, T^b(x) \end{cases}$$

with $\Lambda^{a'}{}_a \Lambda^{b'}{}_b \eta_{a'b'} = \eta_{ab}$

$\Rightarrow \Lambda \in SO(1,3)$

Important : Note that frames are not unique (redundancy) as

$$e'_\mu{}^a \equiv \Lambda^a{}_b(x)\, e_\mu{}^b \Rightarrow g'_{\mu\nu} = e'_\mu{}^a e'_\nu{}^b \eta_{ab}$$
$$= e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}$$

This local (gauge) symmetry will introduce a gauge field for the particles (spinors) transforming under the local Lorentz group

=> spin connection $\omega_\mu{}^{ab}$!!

* Connections and covariant derivatives

- World tensors : Christoffel : $\Gamma_{\mu\nu}{}^{\rho}(g) = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \overbrace{\Gamma_{\mu\rho}{}^{\nu}}^{} V^\rho$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^{\rho} V_\rho$$

=> $\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$

(no torsion)

- Tangent-space tensors : Lorentz SO(1,3) generators $(M_{cd})^a{}_b = 2\delta_{[c}^a \eta_{d]b}$

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^{cd} \overbrace{(M_{cd})^a{}_b}^{} V^b = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b$$

$$D_\mu V_a = \partial_\mu V_a - \omega_\mu{}^{cd} (M_{cd})^b{}_a V_b = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b$$

} Vector

$$D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \omega_\mu{}^{cd} \underbrace{(\gamma_{cd})_\alpha{}^\beta}_{M_{cd} = \frac{1}{4}\gamma_{cd}} \psi_\alpha \quad \Big\} \text{ Spinor}$$

- Vielbein postulate : $\nabla_\mu e_\nu{}^a = 0 \Rightarrow D_\mu V^a = e_\nu{}^a \nabla_\mu V^\nu$

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \qquad (\times e_a{}^\lambda)$$

$$\Rightarrow \quad e_a{}^\lambda \left(\partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_{\nu b} \right) = \Gamma_{\mu\nu}{}^\lambda$$

=> One indep connection !!

- The spin connection is a composite field

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$$

(no torsion)

$$T_{\mu\nu}{}^a \equiv D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b\, e_\nu{}^b - \omega_\nu{}^a{}_b\, e_\mu{}^b$$

Torsion

Important: $T_{\mu\nu}{}^a \overset{!}{=} 0 \Rightarrow \underbrace{\omega_\mu{}^{ab}(e)}_{\text{Levi-Civita connection (torsion free)}} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c$

II. Actions and symmetries

$$\text{Symmetries}\begin{cases}\text{space-time symmetries}\\ \text{gauge }\underline{\text{internal}}\text{ symmetries}\end{cases}\begin{cases}\text{GCT} \longrightarrow \text{Diff}: x'^\mu = x^\mu - \xi^\mu(x)\\ \text{Local Lorentz} \longrightarrow \psi'_\alpha = \psi_\alpha - \lambda^{ab}[\gamma_{ab}]_\alpha{}^\beta\psi_\beta\end{cases}$$

$\boxed{s=0}$ Scalar field ϕ [no world or tangent space index]

- G.C.T : $\phi'(x') = \phi(x) \Rightarrow \delta\phi = \mathcal{L}_\xi\phi = \xi^\mu\partial_\mu\phi$
- gauge : No gauge symmetry (matter field)
- action : $S_\phi = \int d^4x\,\sqrt{-|g|}\left(-\frac{1}{2}g^{\mu\nu}\underbrace{\partial_\mu\phi}_{\nabla_\mu\phi = \partial_\mu\phi}\partial_\nu\phi\right)$
- EOM : $\Box\phi = 0$

$\boxed{s=1/2}$ Fermion field ψ_α [$\alpha \equiv$ tangent space spinorial index]

- Local Lorentz : $\psi'_\alpha(x) = \Lambda_\alpha{}^\beta\psi_\beta(x) \Rightarrow \delta_\lambda\psi_\alpha = \lambda^{ab}[\gamma_{ab}]_\alpha{}^\beta\psi_\beta$
- gauge : No gauge symmetry (matter field)
- action : $S_\psi = \int d^4x\,\sqrt{-|g|}\left(+\frac{1}{2}\bar{\psi}\gamma^\mu\nabla_\mu\psi - \frac{1}{2}\bar{\psi}\overleftarrow{\nabla}_\mu\gamma^\mu\psi\right)$

where $\nabla_\mu\psi_\alpha = \partial_\mu\psi_\alpha + \frac{1}{4}\underbrace{\omega_\mu{}^{ab}}_{\text{spin connection}}[\gamma_{ab}]_\alpha{}^\beta\psi_\beta \equiv D_\mu\psi_\alpha$

$\bar{\psi} \equiv \psi^t C \overset{!}{=} i\psi^\dagger\gamma^0$

↳ Majorana condition with $(\gamma^\mu)^t = t_0 t_1 C\gamma^\mu C^{-1}$

NOTE: In the absence of torsion one can integrate by parts S_ψ to write it as

$$S_\psi = \int d^4x \sqrt{-|g|}\, \bar{\Psi}\, \gamma^\mu D_\mu \Psi \qquad \text{[Dirac]}$$

- EOM : $\gamma^\mu D_\mu \Psi_\alpha = 0$

$\boxed{s = 1}$ Abelian [Maxwell] vector field A_μ [$\mu \equiv$ world index]

- GCT : $A'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \Rightarrow \delta_\xi A_\mu = \mathcal{L}_\xi A_\mu = \xi^\rho \partial_\rho A_\mu + (\partial_\mu \xi^\rho) A_\rho$
- gauge : $A'_\mu(x) = A_\mu(x) + \nabla_\mu \theta(x) \Rightarrow \delta_\theta A_\mu = \nabla_\mu \theta(x) = \partial_\mu \theta(x)$

  ↳ scalar function $\theta(x)$
- action : $S_A = \int d^4x \sqrt{-|g|} \left(-\frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu}} \right)$

  $\underbrace{F_{\mu\nu}}_{\text{gauge invariant}} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

  ↳ $\Gamma^\rho_{[\mu\nu]} = 0$ [No torsion]
- EOM : $\nabla_\mu F^{\mu\nu} = 0$

$\boxed{s = 3/2}$ Gravitino (vector-spinor) field $\Psi_{\mu\alpha}$ [$\mu \equiv$ world index, $\alpha \equiv$ tangent space index]

- GCT : $\Psi'_{\mu\alpha}(x') = \dfrac{\partial x^\nu}{\partial x^\mu} \Psi_{\nu\alpha}(x) \Rightarrow \delta_\xi \Psi_{\mu\alpha} = \mathcal{L}_\xi \Psi_{\mu\alpha} = \xi^\rho \partial_\rho \Psi_{\mu\alpha} + (\partial_\mu \xi^\rho) \Psi_{\rho\alpha}$
- Local Lorentz : $\Psi'_{\mu\alpha}(x) = \Lambda_\alpha{}^\beta \Psi_{\mu\beta}(x) \Rightarrow \delta_\Lambda \Psi_{\mu\alpha} = \lambda^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\mu\beta}$
- gauge : $\Psi'_{\mu\alpha}(x) = \Psi_{\mu\alpha}(x) + \nabla_\mu \epsilon_\alpha(x) \Rightarrow \delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$
- action : $S_\Psi = \int d^4x \sqrt{-|g|}\, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \underbrace{\nabla_\nu \Psi_\rho}$ [Rarita-Schwinger]

$$\nabla_\nu \Psi_{\rho\alpha} = \partial_\nu \Psi_{\rho\alpha} + \frac{1}{4} \omega_\nu{}^{ab} [\gamma_{ab}]_\alpha{}^\beta \Psi_{\rho\beta} - \Gamma^\lambda_{\nu\rho} \Psi_{\lambda\alpha} = D_\nu \Psi_{\rho\alpha} - \underbrace{\Gamma^\lambda_{\nu\rho} \Psi_{\lambda\alpha}}$$

No torsion ⇒ Irrelevant due to $\gamma^{\mu\nu\rho}$

variation w.r.t $\bar{\Psi}$

Important: $$\delta_\epsilon S_{\Psi} = \int d^4x \sqrt{-|g|} \left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \delta_\epsilon \Psi_\rho \right) \times 2$$

$$= \int d^4x \sqrt{-|g|} \left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\rho \epsilon \right) \times 2$$

$$= \int d^4x \sqrt{-|g|} \Big[\underbrace{\nabla_\rho \left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \epsilon \right)}_{\text{boundary term}} + \nabla_\rho \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \epsilon \Big] \times 2$$

(no torsion)

$$= - \int d^4x \sqrt{-|g|}\, \bar{\epsilon} \gamma^{\mu\nu\rho} \nabla_{[\nu} \nabla_{\rho]} \Psi_\mu \times 2$$

$$= - \int d^4x \sqrt{-|g|}\, \bar{\epsilon} \gamma^{\mu\nu\rho} \frac{1}{2} \left(\frac{1}{4} R_{\nu\rho ab} [\gamma^{ab}] \right) \Psi_\mu \times 2$$

$$= - \frac{1}{4} \int d^4x \sqrt{-|g|}\, \bar{\epsilon} \gamma^{\mu\nu\rho} \gamma_{ab} R_{\mu\nu}{}^{ab} \Psi_\rho = (*)$$

NOTE: $\gamma^{\mu\nu\rho}\gamma_{ab} = \gamma^{\mu\nu\rho}{}_{ab} + 6\, \gamma^{[\mu\nu}{}_b\, \delta^{\rho]}_a + 6\, \gamma^{[\mu}\, \delta^{\nu\rho]}_{ba}$

- $\gamma^{\mu\nu\rho}{}_{ab} R_{\mu\nu}{}^{ab} = 0$ (only in $D=4$)
- $6\, \gamma^{[\mu\nu}{}_b\, \delta^{\rho]}_a R_{\mu\nu}{}^{ab} = 2\, \gamma^{\mu\nu}{}_b\, \delta^\rho_a R_{\mu\nu}{}^{ab} + 4\, \gamma^{\nu\rho}{}_b\, \delta^\mu_a R_{\mu\nu}{}^{ab}$

$$= 2\, \gamma^{\mu\nu b} R_{\mu\nu}{}^\rho{}_b + 4\, \gamma^{\nu\rho b} R_{\mu\nu}{}^\mu{}_b$$

Torsion free Bianchi id $R_{[\mu\nu\rho]}{}^a = 0$

$$= \boxed{2\, \gamma^{\mu\nu b} R_{\mu\nu}{}^\rho{}_b} + \boxed{4\, \gamma^{\nu\rho b} R_{\nu b}} \to 0 \text{ (sym)}$$

- $6\, \gamma^{[\mu}\, \delta^{\nu\rho]}_{ba} R_{\mu\nu}{}^{ab} = 4\, \gamma^\mu \delta^{\nu\rho}_{ba} R_{\mu\nu}{}^{ab} + 2\, \gamma^\rho \delta^{\mu\nu}_{ba} R_{\mu\nu}{}^{ab}$

$$= 4\, \gamma^\mu R_{\mu\nu}{}^{\rho\nu} + 2\, \gamma^\rho R_{\mu\nu}{}^{\nu\mu}$$

$$= 4\, \gamma^\mu R_\mu{}^\rho - 2\, \gamma^\rho R$$

$$= 4\, \gamma^\mu \left(R_\mu{}^\rho - \frac{1}{2} \delta^\rho_\mu R \right)$$

NOTE: $\bar{\chi} \gamma_{\mu_1 \dots \mu_n} \lambda = t_n \bar{\lambda} \gamma_{\mu_1 \dots \mu_n} \chi$ with $t_0 = t_3 = +1$, $t_1 = t_2 = -1$
(4D)

$$(*) = -\int d^4x\, \sqrt{-|g|}\, \left(R_\mu{}^\rho - \frac{1}{2}\, \delta_\mu^\rho\, R \right) \bar{\epsilon}\, \gamma^\mu \Psi_\rho$$

$$= -\int d^4x\, \sqrt{-|g|}\, \underbrace{\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \right)}_{\text{Einstein tensor } G_{\mu\nu}\ !!} \bar{\epsilon}\, \gamma^\mu \Psi^\nu$$

Therefore:

i) $\delta_\epsilon S_\Psi = 0$ in Minkowski space-time where $\nabla_\mu \Psi_\rho = \partial_\mu \Psi_\rho$ and $\nabla_\mu \epsilon_\alpha = \partial_\mu \epsilon_\alpha$. This is the work by Rarita-Schwinger (free $s=3/2$ field has ϵ_α-gauge invariance)

ii) $\delta_\epsilon S_\Psi \propto \underbrace{G_{\mu\nu}}_{\text{gravity}} \bar{\epsilon}\, \gamma^\mu \Psi^\nu \neq 0 \Rightarrow$ This suggests that gravity has something to say about general ϵ_α-gauge invariance !!

- EOM : $\gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho = 0$

  or $\gamma^{\mu\nu\rho} D_\nu \Psi_\rho = 0$ (no torsion)

$\boxed{s=2}$ Metric field $g_{\mu\nu}$ [$\mu,\nu \equiv$ world indices)

- GCT : $g'_{\mu\nu}(x') = \dfrac{\partial x^\rho}{\partial x'^\mu} \dfrac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x) \Rightarrow \delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2\, \partial_{(\mu} \xi^\rho\, g_{\nu)\rho}$
- gauge : No gauge symmetry in GR, but ... ϵ_α-gauge symmetry?
- action : $S_g = \dfrac{1}{2\kappa^2} \displaystyle\int d^4x\, \sqrt{-|g|}\, g^{\mu\nu} R_{\mu\nu}$ [Einstein - Hilbert] (2nd order formalism)
- EOM : $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0$

$\kappa^2 = 8\pi G_N$ = grav coupling cte

Important: Unlike for $s=0, 1/2, 1, 3/2$, the theory of $s=2$ (GR) is a highly interacting theory with a dimensionfull coupling constant $[\kappa] = [L] = [E^{-1}]$

Remark 1 : Torsion and integration by parts

- In a space with torsion : $\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho}(g) - K_{\mu\nu}{}^{\rho}$
  with $K_{\mu\nu\rho} = K_{\mu[\nu\rho]} = e_{\nu a}\, e_{\rho b}\, K_\mu{}^{ab} = -\frac{1}{2}\left(T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu}\right)$
  and $T_{[\mu\nu]\rho} = \underbrace{T_{\mu\nu}{}^{a}}_{\text{Torsion}}\, e_{\rho a}$

- Similarly : $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + K_\mu{}^{ab}$

- When there is torsion in space-time :

$$\int d^Dx \sqrt{-|g|}\, \nabla_\mu V^\mu = \int d^Dx \sqrt{-|g|}\, \left(\partial_\mu V^\mu + \Gamma_{\mu\nu}{}^{\mu} V^\nu\right) = (*)$$

NOTE : $\partial_\mu \sqrt{-|g|} = \sqrt{-|g|}\, \Gamma_{\rho\mu}{}^{\rho}(g)$

$$= \int d^Dx \left[\partial_\mu\left(\sqrt{-|g|}\, V^\mu\right) - \left(\partial_\mu \sqrt{-|g|}\right) V^\mu + \sqrt{-|g|}\, \Gamma_{\mu\nu}{}^{\mu} V^\nu\right]$$

$$= \int d^Dx \left[\partial_\mu\left(\sqrt{-|g|}\, V^\mu\right) + \sqrt{-|g|}\, \underbrace{\left(\Gamma_{\mu\nu}{}^{\mu} - \Gamma_{\mu\nu}{}^{\mu}(g)\right)}_{-K_{\mu\nu}{}^{\mu}} V^\nu\right]$$

$$= \underbrace{\int d^Dx\, \partial_\mu\left(\sqrt{-|g|}\, V^\mu\right)}_{\text{boundary term}} - \underbrace{\int d^Dx \sqrt{-|g|}\, K_{\mu\nu}{}^{\mu} V^\nu}_{\text{Contorsion piece spoiling the integration by parts}}$$

Remark 2 : Can gravity compensate for the lack of ϵ_α-invariance of S'_Ψ ?

(i) The fact that $(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$ appears in $\delta_\epsilon S_\Psi$ is highly remarkable.

(ii) We know from supersymmetry that $\overset{(\varphi)}{s=0}$ and $\overset{(\text{Weyl } \psi)}{s=1/2}$ can furnish a susy (chiral) multiplet transforming as

$$\left.\begin{aligned} \delta_\epsilon \varphi &\propto \bar{\epsilon} \psi \\ \delta_\epsilon \psi &\propto \bar{\epsilon} \sigma^\mu (\partial_\mu \varphi) \end{aligned}\right\} \text{Free theory or Wess-Zumino model in flat space} \left\{ \begin{aligned} &\bullet\ \delta\mathcal{L} = \text{boundary terms} \\ &\bullet\ \text{SUSY algebra on-shell} \end{aligned}\right.$$

(iii) We also know from supersymmetry that $\overset{(\text{Majorana } \lambda)}{s=1/2}$ and $\overset{(A_\mu)}{s=1}$ can furnish a susy (vector) multiplet transforming as

$$\left.\begin{aligned} \delta_\epsilon A_\mu &\propto \bar{\epsilon} \gamma_\mu \lambda \\ \delta_\epsilon \lambda &\propto \bar{\epsilon} \gamma_{\mu\nu} F^{\mu\nu} \end{aligned}\right\} \text{Super Yang-Mills theory in flat space}$$

(iv) Could there be a similar story involving $\overset{(\Psi)}{s=3/2}$ and $\overset{(e)}{s=2}$??

Beautiful idea!!

But do degrees of freedom match : Ψ vs e ?

III. DOF : Massless particles in Minkowski space-time

1) Helicity states in 4D

- massive particles : $2s+1$ helicity states (s = spin)
- massless particles : $s=0$: # helicity states $= 1$
  $s \geq 1/2$: # helicity states $= 2$: $\underbrace{\lambda = \pm s}_{\text{parity inv}}$

2) Degrees of freedom :

2.1) Quantum d.o.f = helicity states = on-shell d.o.f

2.2) Classical d.o.f = Cauchy initial value problem for the EOMs after gauge-fixing

3.3) Off-shell d.o.f = # components − gauge d.o.f
= # of indep EOMs

$$\boxed{\text{Classical d.o.f} = 2 \times \text{on-shell d.o.f.}}$$

⊛ Scalar field (real) : $s=0$ $(m=0)$

2.1) on-shell => 1 state

2.2) classical => 2 states $[\overbrace{\Box\phi = 0}^{2^{nd}\text{ order}}$ & no gauge freedom$]$

3.3) off-shell => 1 state $\overset{!}{=}$ # of indep EOMs

⊛ Spinor field (Majorana) : $s = 1/2$ $(m=0)$

2.1) on-shell => 2 states

2.2) classical => 4 states $[\overbrace{\gamma^\mu \partial_\mu \Psi_\alpha = 0}^{1^{st}\text{ order}}$ & no gauge sym$]$

3.3) off-shell => 4 states $\overset{!}{=}$ # of indep EOMs

⊛ Vector field : $s=1$ $(m=0)$

2.1) on-shell $\Rightarrow$ 2 states

2.2) Classical $\Rightarrow$ 4 states $\Big[\overbrace{\partial_\mu F^{\mu\nu}=0}^{2^{nd}\text{ order}}$ & $A_\mu \to A_\mu + \partial_\mu\theta\Big]$

2.3) off-shell $\Rightarrow$ $4-\underset{\hookrightarrow\,\theta\text{-transf}}{1} \overset{!}{=} 3$ d.o.f $=$ # of indep EOMs as $\partial_\mu\partial_\nu F^{\mu\nu}=0$

→ Derivation of 2.2)

$$\text{EOM}: \partial_\mu F^{\mu\nu}=0 \xrightarrow[\substack{\text{gauge}\\ \text{fixing}\\ \text{(Coulomb)}}]{\partial_i A^i=0}
\begin{array}{l}
(\nu=0)\ \partial_i\partial^i A^0 - \overbrace{\partial_i\partial^0 A^i}^{0} = 0\\
(\nu=i)\ \partial_0\partial^0 A^i - \partial_0\partial^i A^0 + \partial_j\partial^j A^i - \underbrace{\partial_j\partial^i A^j}_{0}\\
\qquad = \Box A^i - \partial^i\partial_0 A^0
\end{array}$$

Compatibility $\Leftarrow$ gauge fixing

$\nabla^2\theta = 0$

$\Rightarrow$ no d.o.f

$\Rightarrow$ $(\nu=0)$ $\nabla^2 A^0 = 0$ $\Rightarrow$ no d.o.f

(momentum space)

$\nabla^2\phi = k^2\phi(k)=0$

$\Rightarrow \phi(k)=0$

$(\nu=i)$ $\Box A^i = 0$

NOTE: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

So, after the gauge-fixing, the EOMs reduce to

$$\Box A^i = 0 \quad \underline{\underline{\text{subject}}} \text{ to the gauge fixing } \partial_i A^i = 0$$

$\Rightarrow$ classical d.o.f $\Rightarrow 6-2=4$

⊛ Gravitino field : $s = 3/2$ ($m=0$ and $\underline{\underline{\text{real}}} \equiv$ Majorana)

2.1) On-shell $\Rightarrow$ 2 states

2.2) Classical $\Rightarrow$ 4 states $\Big[\overbrace{\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha}}^{1^{st}\text{ order}}$ & $\Psi_{\mu\alpha} \to \Psi_{\mu\alpha} + \partial_\mu\epsilon_\alpha\Big]$

2.3) Off-shell $\Rightarrow$ $16 - \underset{\substack{\downarrow\\ \epsilon_\alpha\text{-transf}}}{4} = 12 \overset{!}{=}$ # of indep EOM as $\partial_\mu\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha} = 0$

→ Derivation of 2.2)

$$\text{EOM}: \gamma^{\mu\nu\rho}\partial_\nu\psi_{\rho\alpha}=0 \xrightarrow[\text{gauge fixing}]{\gamma^i\psi_{i\alpha}=0} (\mu=0)\ \left(\gamma^0\gamma^{jk}-\overbrace{\eta^{0j}\gamma^k}^{0}+\overbrace{\eta^{0k}\gamma^j}^{0}\right)\partial_j\psi_{k\alpha}$$

$$\Rightarrow \gamma^{jk}\partial_j\psi_{k\alpha}=0$$

$$(\mu=i)\ \gamma^{i\nu\rho}\partial_\nu\psi_{\rho\alpha}=\gamma^{ijk}\partial_j\psi_{k\alpha}+\underbrace{\gamma^{i0k}}_{-\gamma^{0ik}=-\gamma^0\gamma^{ik}}\partial_0\psi_{k\alpha}+\underbrace{\gamma^{ij0}}_{\gamma^{0ij}=\gamma^0\gamma^{ij}}\partial_j\psi_{0\alpha}$$

$$=\gamma^{ijk}\partial_j\psi_{k\alpha}+\gamma^0\gamma^{ij}(\partial_j\psi_{0\alpha}-\partial_0\psi_{j\alpha})$$

NOTE : $\gamma^{\mu\nu\rho}=\gamma^\mu\gamma^{\nu\rho}-\eta^{\mu\nu}\gamma^\rho+\eta^{\mu\rho}\gamma^\nu$

Therefore, the EOMs read:
$$\begin{cases}\gamma^{jk}\partial_j\psi_{k\alpha}=0 & (i.1)\\ \gamma^{ijk}\partial_j\psi_{k\alpha}+\gamma^0\gamma^{ij}(\partial_j\psi_{0\alpha}-\partial_0\psi_{j\alpha})=0 & (ii.1)\end{cases}$$

- Using $\gamma^{jk}=\gamma^j\gamma^k-\delta^{jk}$ and the gauge condition on (i.1), then

(i.2) $\partial^i\psi_{i\alpha}=0$

- Multiplyind (ii.1) from the left with γ_i gives:

$$\underbrace{\gamma_i\gamma^{ijk}}_{\gamma_i(\gamma^i\gamma^{jk}-\underbrace{\delta^{ij}\gamma^k}_{0}+\delta^{ik}\gamma^j)}\partial_j\psi_{k\alpha}+\underbrace{\gamma_i\gamma^0}_{-\gamma^0\gamma_i}\underbrace{\gamma^{ij}}_{(\gamma^i\gamma^j-\delta^{ij})}(\partial_j\psi_{0\alpha}-\partial_0\psi_{j\alpha})=$$

$$=(3\gamma^{jk}+\underbrace{\gamma^k\gamma^j}_{\gamma^{kj}+\delta^{kj}})\partial_j\psi_{k\alpha}-\gamma^0\underbrace{(3\gamma^j-\gamma^j)}_{2\gamma^j}(\partial_j\psi_{0\alpha}-\underbrace{\partial_0\psi_{j\alpha}}_{0})$$

$$=\underbrace{2\gamma^{jk}\partial_j\psi_{k\alpha}}_{0\ (i.1)}+\underbrace{\partial^k\psi_{k\alpha}}_{0\ (i.2)}-2\gamma^0\gamma^j\partial_j\psi_{0\alpha}=-2\gamma^0\gamma^j\partial_j\psi_{0\alpha}=0 \quad (ii.2)$$

- Multiplying (ii.2) by $\gamma^i\partial_i$ one finds

$$-2\gamma^i\gamma^0\gamma^j\partial_i\partial_j\psi_{0\alpha} = +2\gamma^0\underbrace{\gamma^i\gamma^j}_{\gamma^{ij}+\delta^{ij}}\partial_i\partial_j\psi_{0\alpha} = 2\gamma^0\nabla^2\psi_{0\alpha}$$

$$\Rightarrow \nabla^2\psi_{0\alpha} = 0 \xrightarrow[\text{no d.o.f}]{} \psi_{0\alpha} = 0$$

Now we can go back to (ii.1) and set $\psi_{0\alpha} = 0$

$$\underbrace{\gamma^{ijk}}_{\gamma^i\underbrace{\gamma^{jk}}_{\underbrace{\gamma^j\gamma^k}_{0}-\underbrace{\delta^{jk}}_{0}} - \underbrace{\delta^{ij}\gamma^k}_{0} + \delta^{ik}\gamma^j}\partial_j\psi_{k\alpha} - \gamma^0\underbrace{\gamma^{ij}}_{\underbrace{\gamma^i\gamma^j}_{0}-\delta^{ij}}\partial_0\psi_{j\alpha} = \gamma^j\partial_j\psi^i{}_\alpha + \gamma^0\partial_0\psi^i{}_\alpha = \gamma^\mu\partial_\mu\psi_{i\alpha} = 0 \quad (ii.3)$$

So, after the gauge-fixing, the EOMs reduce to

$$\gamma^\mu\partial_\mu\psi_{i\alpha} = 0 \ \ \underline{\underline{\text{subject}}} \text{ to the gauge fixing } \begin{cases} \gamma^i\psi_{i\alpha} = 0 \\ \partial^i\psi_{i\alpha} = 0 \end{cases}$$

$\Rightarrow$ Classical d.o.f $\Rightarrow 12 - 4 - 4 = 4$

ⓧ Linearised graviton : $s = 2$ $(m=0)$ → $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}$

2.1) On-shell ⇒ 2 states

2.2) Classical ⇒ 4 states $\Big[\overbrace{R_{\mu\nu} = 0}^{2^{nd}\text{ order}}$ & $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}\Big]$

2.3) Off-shell $\Rightarrow 10 - \underset{\substack{\downarrow \\ \xi^\mu\text{- transf}}}{4} = 6 \overset{!}{=} \#$ of indep EOM as

→ Derivation of 2.2 : Consider the fluctuation $h_{\mu\nu}$ as a gauge field

EOM : $R_{\mu\nu} = 0$ ⇒ Let's look a bit at the linearised level.

- First we compute the linearised Christoffel connection $\Gamma_{\mu\nu}^{\rho\,\text{lin}}(h)$

$$\Gamma_{\mu\nu}{}^{\rho}(g) = \frac{1}{2} g^{\rho\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right]$$

$$\xrightarrow[\text{linearise}]{} \kappa\, \frac{1}{2} \eta^{\rho\sigma}\left[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right]$$

- Second we compute the linearised $R_{\mu\nu}$

$$R_{\mu\nu}(\Gamma) = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \overbrace{\Gamma\Gamma - \Gamma\Gamma}^{\kappa^2}$$

$$\xrightarrow[\text{linearise}]{} \kappa\, \frac{1}{2}\Big[\partial^\sigma\partial_\mu h_{\sigma\nu} + \cancel{\partial^\sigma\partial_\nu h_{\mu\sigma}} - \Box h_{\mu\nu}$$

$$- \partial_\nu\partial_\mu h_\sigma{}^\sigma - \cancel{\partial_\nu\partial^\sigma h_{\mu\sigma}} + \partial_\nu\partial_\sigma h_\mu{}^\sigma\Big]$$

$$= \kappa\, \frac{1}{2}\left[\partial^\sigma\partial_\mu h_{\sigma\nu} + \partial^\sigma\partial_\nu h_{\sigma\mu} - \partial_\mu\partial_\nu h_\sigma{}^\sigma - \Box h_{\mu\nu}\right]$$

Therefore:

$$R_{\mu\nu}^{\text{Linear}} = -\frac{1}{2}\kappa\left[\Box h_{\mu\nu} - \partial^\sigma(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma}) + \partial_\mu\partial_\nu h_\sigma{}^\sigma\right]$$

Let's look at a linearised GCT : $x'^\mu = x^\mu - \xi^\mu(x)$ → ξ^μ of order κ

$$\delta g_{\mu\nu} = \kappa\, \mathcal{L}_\xi g_{\mu\nu} = \left[\xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}\right]\kappa =$$

NOTE: $\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}{}^\lambda g_{\lambda\nu} - \Gamma_{\rho\nu}{}^\lambda g_{\mu\lambda} \Rightarrow \partial_\rho g_{\mu\nu} = \underbrace{\nabla_\rho g_{\mu\nu}}_{0\,=\,\text{Generalised Vielbein postul.}} + \Gamma_{\rho\mu}{}^\lambda g_{\lambda\nu} + \Gamma_{\rho\nu}{}^\lambda g_{\mu\lambda}$

$$\kappa\left[\overbrace{\xi^\rho \Gamma_{\rho\mu}{}^{\lambda} g_{\lambda\nu}}^{} + \overbrace{\xi^\rho \Gamma_{\rho\nu}{}^{\lambda} g_{\mu\lambda}}^{} + \overbrace{\partial_\mu \xi^\rho g_{\rho\nu}}^{} + \overbrace{\partial_\nu \xi^\rho g_{\mu\rho}}^{}\right]$$

$$=\kappa\left[\underbrace{\left(\partial_\mu \xi^\rho + \Gamma^{\rho}_{\mu\lambda}\xi^\lambda\right)}_{\nabla_\mu \xi^\rho} g_{\rho\nu} + \underbrace{\left(\partial_\nu \xi^\rho + \Gamma_{\nu\lambda}{}^{\rho}\xi^\lambda\right)}_{\nabla_\nu \xi^\rho} g_{\mu\rho}\right]$$

$$\Rightarrow \quad \delta g_{\mu\nu} = \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right)\kappa$$

At the linearised level only ∂_μ inside ∇_μ contributes so that:

(i) $\quad \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \equiv$ linearised gauge transf.

An analogous computation for $\delta R_{\mu\nu}$ yields

$$\delta R_{\mu\nu} = \left(\underbrace{\xi^\rho \nabla_\rho R_{\mu\nu}}_{} + \nabla_\mu \xi^\rho R_{\rho\nu} + \nabla_\nu \xi^\rho R_{\mu\rho}\right)\kappa$$

This time the transport term is not zero for $R_{\mu\nu}$

Since $R_{\mu\nu}^{\text{Linear}} \sim O(\kappa)$, it follows that:

(ii) $\quad \delta R_{\mu\nu}^{\text{Linear}} = 0 \quad \Rightarrow$ Linear fluctuations are gauge-invariant!!

Important: The gravitational fluctuation $h_{\mu\nu}$ is a gauge field with gauge transformation (i) and that satisfies the gauge invariant wave equation:

$$R_{\mu\nu}^{\text{Linear}} = 0$$

In order to determine the d.o.f we will **not** perform a Lorentz covariant gauge fixing:

$$\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h_\lambda{}^\lambda = 0$$

See Marc's notes

Lorentz-covariant gauge fixing

Pros: $\Box h_{\mu\nu} = 0$ E.O.M

Massless field at the speed c

-> Gravitational wave

Cons: Residual gauge d.o.f ξ_μ

$$\delta(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h^\lambda_\lambda) = \underbrace{\Box \xi_\mu}_{\Box \xi_\mu = 0 \text{ residual d.o.f}}$$

Following the same logic as for the A_μ and $\Psi_{\mu\alpha}$ fields we will perform a non-covariant gauge fixing

$$\partial^i h_{i\nu} = 0 \quad \equiv \text{ Gauge fixing}$$

By computing $\delta(\partial^i h_{i\nu}) = 0$, one can see what are the residual gauge d.o.f:

$$\delta(\partial^i h_{i\nu}) = \nabla^2 \xi_\nu + \partial^i \partial_\nu \xi_i = 0 \qquad (iii)$$

- Let's focus on ($\nu = j$) and contract with ∂^j

$$\nabla^2(\partial^j \xi_j + \partial^i \xi_i) = 2\nabla^2 \underbrace{(\partial^i \xi_i)}_{\partial^i \xi_i = 0} = 0$$

Plugging it back into (iii) we see that

$$\nabla^2 \xi_\mu = 0 \Rightarrow \xi_\mu = 0 \Rightarrow \text{No residual gauge d.o.f !!}$$

Let's now plug the gauge fixing $\partial^i h_{i\nu} = 0$ into the EOM

$$R_{\mu\nu}^{\text{Linear}} = -\frac{1}{2}\kappa\left[\Box h_{\mu\nu} - \partial^0\left(\partial_\mu h_{0\nu} + \partial_\nu h_{\mu 0}\right) + \partial_\mu\partial_\nu\left(\overset{\text{summed}}{\overbrace{h_{ii}}} - h_{00}\right)\right] = 0$$

$$(\mu=\nu=0) \;:\; \Box h_{00} - \partial^0\partial_0 h_{00} - \partial^0\partial_0 h_{ii} = \nabla^2 h_{00} - \partial^0\partial_0 h_{ii} = 0 \quad (iv)$$

NOTE: Remember that indices are rised and lowered with $\eta_{\mu\nu}$

$$\underbrace{(\mu=\nu=i)}_{\text{summed}} \;:\; \Box h_{ii} - 2\partial^0 \underbrace{\partial_i h_{0i}}_{0} + \nabla^2 h_{ii} - \nabla^2 h_{00} =$$

$$2\nabla^2 h_{ii} + \partial^0\partial_0 h_{ii} - \nabla^2 h_{00} = 0 \quad (v)$$

Combining the two equations one has

$$(iv) + (v) = \nabla^2 h_{ii} = 0 \Rightarrow h_{ii} = 0 \quad (vi)$$

Plugging (vi) into (iv) yields

$$\nabla^2 h_{00} = 0 \Rightarrow h_{00} = 0 \quad (vii)$$

$$(\mu=0, \nu=i) \;:\; \Box h_{0i} - \partial^0\partial_0 h_{0i} - \partial^0 \underbrace{\partial_i h_{00}}_{0} + \partial_0\partial_i\left(\underbrace{h_{ii}}_{0} - \underbrace{h_{00}}_{0}\right)$$

$$= \nabla^2 h_{0i} = 0 \Rightarrow h_{0i} = 0 \quad (viii)$$

$$(\mu=i, \nu=j) \;:\; \Box h_{ij} = 0$$

after having used the previous results.

So, after the gauge-fixing, the EOMs reduce to

$$\Box h_{ij} = 0 \quad \underline{\underline{\text{subject}}} \text{ to the gauge fixing } \left\{ \begin{array}{l} \partial^i h_{ij} = 0 \\ \underbrace{h_{ii}}_{\text{summed}} = 0 \end{array} \right.$$

$\Rightarrow$ classical d.o.f $\Rightarrow 12 - 6 - 2 = 4$

3) Matching off-shell degrees of freedom:

$s=0$ $(\phi \in \mathbb{R})$ → 1 d.o.f

$s=1/2$ (ψ = Majorana or Weyl) → 4 d.o.f

$s=1$ (A_μ) → 3 d.o.f

$s=3/2$ (Ψ) → 12 d.o.f

$s=2$ (e) → 6 d.o.f

Then:

i) Chiral multiplet $\equiv \overbrace{(\phi_1, \phi_2)}^{(1+1)_B} \oplus \overbrace{\psi}^{4_F} \oplus \overbrace{F \in \mathbb{C}}^{\text{Auxiliar } 2_B}$

ii) Vector multiplet $\equiv \overbrace{A_\mu}^{3_B} \oplus \overbrace{\psi}^{4_F} \oplus \overbrace{D \in \mathbb{R}}^{\text{Auxiliar } 1_B}$

iii) Gravity multiplet $\equiv \overbrace{e_\mu{}^a}^{6_B} \oplus \overbrace{\Psi}^{12_F} \oplus \overbrace{(\mathcal{F} \in \mathbb{C}, \mathcal{A}_\mu \in \mathbb{R})}^{\text{Auxiliar } 2_B + 4_B}$

IV. $\mathcal{N}=1$ Supergravity with $\Lambda=0$

We have collected already quite some clues for the local ϵ_α-invariance to actually be local supersymmetry involving not only the $s=3/2$ field but also the $s=2$ metric field. We are going to work this idea out now.

Let's start from:

$$S_g = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\,g^{\mu\nu}R_{\mu\nu} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-|g|}\,e_a{}^{\mu}\,e_b{}^{\nu}\,R_{\mu\nu}{}^{ab}(e)$$

$$S_\Psi = -\underbrace{\frac{1}{2\kappa^2}}_{\text{we have introduced this constant}}\int d^4x\,\sqrt{-|g|}\;\bar\Psi_\mu\,\gamma^{\mu\nu\rho}\,\underbrace{\nabla_\nu\Psi_\rho}_{D_\nu[\omega]\Psi_\rho}$$

Important: We are considering a space-time without torsion

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$$

so integration by parts works normally: $\int dx\,\sqrt{-|g|}\,\nabla_\mu V^\mu = \int dx\,\partial_\mu\left(\sqrt{-|g|}\,V^\mu\right)$

We already know that $\delta_\epsilon \Psi_{\mu\alpha} = \nabla_\mu \epsilon_\alpha = D_\mu \epsilon_\alpha$. But the question is: what should be $\delta_\epsilon e_\mu{}^a$?

$$\delta_\epsilon\, e_\mu{}^a = c\,\bar\epsilon\,\gamma^a\,\Psi_\mu$$ (what else could it be?)

with c being a constant to be fixed. Note that this combination already appeared in $\delta_\epsilon S_\Psi \Rightarrow$ ✓

Let's compute things explicitely

$$S = S_g(e) + S_\psi(e,\psi)$$

$$\delta_e S = \underbrace{\Big(\underbrace{\frac{\delta S_g}{\delta e}}_{A.1} + \underbrace{\frac{\delta S_\psi}{\delta e}}_{A.2}\Big)}_{A}\delta_e e + \underbrace{\frac{\delta S_\psi}{\delta \Psi}\delta_e \Psi}_{B \to \text{This we already computed}}$$

$$\underline{\underline{B}}: \quad \frac{\delta S_\psi}{\delta \Psi}\delta_e\Psi = \frac{1}{2K^2}\int d^4x \underbrace{e}_{\sqrt{|-g|}}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\bar{\varepsilon}\gamma^\mu\Psi^\nu$$

A.1: This is, by definition, the computation of the Einstein equations in terms of frames $e_\mu{}^a$:

$$\frac{\delta S_g}{\delta e_\mu{}^a}\delta e_\mu{}^a = \frac{1}{2K^2}\int d^4x\left[\delta(e)R + 2e\,\delta(e_a{}^\mu)e_b{}^\nu R_{\mu\nu}{}^{ab} + e\,e_a{}^\mu e_b{}^\nu\,\delta(R_{\mu\nu}{}^{ab})\right] = (*)$$

- $\delta(|M|) = |M|\,\mathrm{Tr}(M^{-1}\delta M) \Rightarrow \delta(e) = e\,e_a{}^\mu\delta e_\mu{}^a$
- $e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu \Rightarrow \delta e_\mu{}^a e_a{}^\nu + e_\mu{}^a\delta e_a{}^\nu = 0$

$$\Rightarrow \delta e_a{}^\nu e_\mu{}^a = -e_a{}^\nu\delta e_\mu{}^a \qquad (\times e_b{}^\mu)$$

$$\Rightarrow \delta e_b{}^\nu = -e_a{}^\nu e_b{}^\mu\,\delta e_\mu{}^a$$

- $\delta(R_{\mu\nu}{}^{ab}) = D_\mu\delta\omega_\nu{}^{ab} - D_\nu\delta\omega_\mu{}^{ab} \underset{\hookrightarrow\text{no torsion}}{=} \nabla_\mu\delta\omega_\nu{}^{ab} - \nabla_\nu\omega_\mu{}^{ab} \underset{\hookrightarrow\text{no torsion}}{\Longrightarrow}$ Total derivative

$$(*) = \frac{1}{2K^2}\int d^4x\, e\left(e_a{}^\mu R\,\delta e_\mu{}^a - 2\,e_b{}^\nu e_c{}^\mu e_a{}^\rho R_{\mu\nu}{}^{ab}\,\delta e_\rho{}^c\right)$$

$$= \frac{1}{2K^2}\int d^4x\, e\left(e_a{}^\mu R\,\delta e_\mu{}^a - 2\,R_\mu{}^\rho e_c{}^\mu\,\delta e_\rho{}^c\right) = (\text{relabeling})$$

$$= \frac{1}{2K^2}\int d^4x \; e \left(e_a{}^\rho R - 2 \, R_\mu{}^\rho \, e_a{}^\mu \right) \delta e_\rho{}^a = (*)$$

Now we plug: $\delta_\epsilon e_\rho{}^a = c \; \bar{\epsilon} \gamma^a \Psi_\rho$

$$(*) = -\frac{c}{K^2}\int d^4x \; e \left(R_\mu{}^\rho e_a{}^\mu - \frac{1}{2} e_a{}^\rho R \right) \bar{\epsilon} \gamma^a \Psi_\rho$$

$$= -\frac{c}{K^2}\int d^4x \; e \left(R_\mu{}^\rho - \frac{1}{2} \delta_\mu^\rho R \right) \bar{\epsilon} \gamma^\mu \Psi_\rho$$

$$= -\frac{c}{K^2}\int d^4x \; e \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \bar{\epsilon} \gamma^\mu \Psi^\nu$$

A.2: $$\frac{\delta S_\Psi}{\delta e_\rho{}^a} \delta e_\rho{}^a = -\frac{1}{2K^2}\int d^4x \; \delta \left[e \, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho \right] \overset{\text{lowest order } \Psi}{\overset{\downarrow}{=}} 0$$

NOTE: $\delta e = e \, e_a{}^\mu \delta e_\mu{}^a = c \; e \, e_a{}^\mu \bar{\epsilon} \gamma^a \Psi_\mu \Rightarrow \epsilon \Psi^3\text{-terms}$

$$\delta(\gamma^{\mu\nu\rho}) = \delta(e_a{}^\mu e_b{}^\nu e_c{}^\rho) \gamma^{abc}$$
$$= \delta e_a{}^\mu e_b{}^\nu e_c{}^\rho \gamma^{abc} + \ldots$$
$$= - e_d{}^\mu e_a{}^\lambda \delta e_\lambda{}^d \; e_b{}^\nu e_c{}^\rho \gamma^{abc} + \ldots$$
$$= - e_d{}^\mu \delta e_\lambda{}^d \; \gamma^{\lambda\nu\rho} + \ldots$$
$$= - e_d{}^\mu \gamma^{\lambda\nu\rho} (\bar{\epsilon} \gamma^d \Psi_\lambda) \Rightarrow \epsilon \Psi^3\text{- terms}$$

$$\delta(\nabla_\nu \Psi_\rho) = \delta(D_\nu \Psi_\rho) = \delta(\partial_\nu \Psi_\rho + \tfrac{1}{4} \omega_\nu{}^{ab} [\gamma_{ab}] \psi_\rho)$$
$$= \frac{1}{4} \delta(\omega_\nu{}^{ab}) [\gamma_{ab}] \Psi_\rho$$
$$\Rightarrow \epsilon \Psi^3\text{- terms}$$
$$\downarrow$$

schematically: $\omega = e^{-1}\partial e - e^{-1} e^{-1} e \partial e$

Wrapping up B, (A.1) and (A.2) we obtain at lowest order ($\bar\epsilon\Psi$-terms) in fermions the following result:

$$\delta_\epsilon S = \delta_\epsilon(S_g + S_\Psi) = \underbrace{\left(\frac{1}{2}-c\right)}_{c=\frac{1}{2}\,!!}\frac{1}{\kappa^2}\int d^4x\, e\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\bar\epsilon\gamma^\mu\Psi^\nu$$

Therefore we have an ϵ_α-invariant action at lowest order in fermionic fields with transformation rules:

$$(i)\quad \boxed{\begin{aligned}\delta_\epsilon e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon\gamma^a\Psi_\mu\\ \delta_\epsilon\Psi_\mu &= D_\mu\epsilon\end{aligned}} \Rightarrow \text{SUGRA}\;!!$$

Remark : It was crucial that $\nabla_\nu\Psi_\rho$ in the action R-S action appears with a $\gamma^{\mu\nu\rho}$. This allowed us to replace $\nabla_\nu\Psi_\rho = D_\nu\Psi_\rho$ fitting well with $D_\mu\epsilon$ and

$$D_{[\mu}D_{\nu]} = R_{\mu\nu} \Rightarrow \text{GR}\;!!$$

However, there is an easier way to see how GR emerges from the SUSY algebra of supersymmetry:

$$\begin{aligned}[\delta_{\epsilon_1},\delta_{\epsilon_2}]e_\mu{}^a &= \delta_{\epsilon_1}\left(\frac{1}{2}\bar\epsilon_2\gamma^a\Psi_\mu\right) - (1\leftrightarrow 2)\\
&= \frac{1}{2}\bar\epsilon_2\gamma^a\nabla_\mu\epsilon_1 - (1\leftrightarrow 2) = \frac{1}{2}\bar\epsilon_2\gamma^a D_\mu\epsilon_1 - (1\leftrightarrow 2)\\
&= \frac{1}{2}\bar\epsilon_2\gamma^a D_\mu\epsilon_1 - \frac{1}{2}\bar\epsilon_1\gamma^a D_\mu\epsilon_2\\
t_1=-1 \quad &= \frac{1}{2}\left(\bar\epsilon_2\gamma^a D_\mu\epsilon_1 + D_\mu\bar\epsilon_2\gamma^a\epsilon_1\right) = D_\mu\left(\overbrace{\frac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1}^{\equiv\xi^a}\right)\\
&= D_\mu\xi^a \qquad \text{with } \xi^a = \frac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1\end{aligned}$$

Therefore we have found that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = D_\mu \xi^a$

Moreover $D_\mu \xi^a = \nabla_\mu \xi^a$ and

$$\delta_\xi e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a = \xi^\rho \partial_\rho e_\mu{}^a + \partial_\mu \xi^\rho e_\rho{}^a = \text{(explicit covariantisation)}$$

$$= \xi^\rho \underbrace{\nabla_\rho e_\mu{}^a}_{0} + \cancel{\xi^\rho \Gamma_{\rho\mu}{}^\lambda} e_\lambda{}^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b + \underbrace{\nabla_\mu \xi^\rho e_\rho{}^a}_{\nabla_\mu \xi^a} - \cancel{\Gamma_{\mu\lambda}{}^\rho \xi^\lambda} e_\rho{}^a$$

$\equiv$ no torsion

$$= \partial_\mu \xi^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b$$

Then : $$\nabla_\mu \xi^a = \delta_\xi e_\mu{}^a + \underbrace{\xi^\rho \omega_\rho{}^a{}_b e_\mu{}^b}_{\delta_\Lambda e_\mu{}^a \text{ with } \Lambda^a{}_b = \xi^\rho \omega_\rho{}^a{}_b} = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$$

So we have shown that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = \delta_\xi e_\mu{}^a + \delta_\Lambda e_\mu{}^a$

$\Rightarrow$ Local SUSY algebra implies $\begin{cases} \text{GCT} \\ \text{Local Lorentz} \end{cases} \Rightarrow$ General Relativity !!

→ Next question : Are the transformations (i) a symmetry of the action to all orders in fermions ?

Unfortunately the answer is no . . . ☹ . . .
It had been surprising otherwise as the Rarita-Schwinger action describes a free $s = 3/2$ fiel Ψ_ρ whereas the E-H action describing gravity is highly interacting !!

IMPORTANT : The computation we have performed did not rely on the dimension D or any special feature but having a real Ψ. Therefore, it works in the same way in any D. What makes certain dimensions special is that local SUSY can be stated at all orders in fermions. For example : $\mathcal{N}=1$ D=4 or D=11

To have $\mathcal{N}=1$ and $D=4$ sugra to all orders in fermions one has to introduce Ψ^4-terms both in the R-S action and in the transformation rules :

$$S_{\Psi} = -\frac{1}{2\kappa^2}\int d^4x\, e\left\{ \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{16}\Big[(\bar{\Psi}^\rho \gamma^\mu \Psi^\nu)(\bar{\Psi}_\rho \gamma_\mu \Psi_\nu + 2\bar{\Psi}_\rho \gamma_\nu \Psi_\mu) - 4(\bar{\Psi}_\mu \gamma^\rho \Psi_\rho)(\bar{\Psi}^\mu \gamma^\rho \Psi_\rho) \Big] \right\}$$

with a torsion-free $D_\mu \Psi_\rho = \partial_\mu \Psi_\rho + \frac{1}{4}\omega_\mu{}^{ab}(e)\,[\gamma_{ab}]\,\Psi_\rho$

The SUSY transformation rules (i) must be also modified with higher-order fermion terms :

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\epsilon}\gamma^a \Psi_\mu$$
$$\delta \Psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\left(\omega_\mu{}^{ab}(e) + K_\mu{}^{ab}\right)[\gamma_{ab}]\,\epsilon$$

with $K_{\mu\nu\rho} = -\frac{1}{4}\left(\bar{\Psi}_\mu \gamma_\rho \Psi_\nu - \bar{\Psi}_\nu \gamma_\mu \Psi_\rho + \bar{\Psi}_\rho \gamma_\nu \Psi_\mu\right)$

IMPORTANT : We see that local SUSY at the full fermion label can be very conveniently described using gravitino torsion. This is an example of how torsion appears in Physics. Some 1^{st} and 1.5 order formulation of supergravity make all these structures manifest and render the problem of full local supersymmetry tractable !!

Some involved fermionic manipulations "Fierzing" are also required in the process.

Appendix : Supergravity with extended ($\mathcal{N}>1$) supersymmetry.

- There are theories of supergravity in $D=4$ with more than one ($\mathcal{N}=1$) gravitino fields : the so called "extended SUGRA's"

- The field content of the SUGRA multiplet includes :

$$\underbrace{e_\mu{}^a}_{\substack{\text{metric}\\ (s=2)}} \oplus \underbrace{\begin{matrix}\Psi_{\mu\alpha}{}^{1}\\ \vdots \\ \Psi_{\mu\alpha}{}^{\mathcal{N}}\end{matrix}}_{\substack{\mathcal{N}\text{ gravitini}\\ (s=3/2)}} \oplus \text{EXTRA FIELDS (bosonic and fermionic)}$$

- The most studied cases are $\mathcal{N} = \overbrace{2}^{\text{Minimally extended}} \; , \; \overbrace{4}^{\text{Half maximal}} \; , \; \overbrace{8}^{\text{Maximally extended}}$

  Number of supercharges in the susy algebra : 8 16 32

$$\ast \quad \mathcal{N}=2 \; : \quad \underbrace{e_\mu{}^a}_{s=2} , \underbrace{A_\mu}_{\substack{s=1\\ \text{(graviphoton)}}} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,2}}_{s=3/2}$$

$$\ast \quad \mathcal{N}=4 \; : \quad \underbrace{e_\mu{}^a}_{s=2} , \underbrace{A_\mu{}^{1,\dots,6}}_{s=1} , \underbrace{\overset{\in\mathbb{C}}{\tau}}_{s=0} \in \frac{SL(2)}{SO(2)} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,\dots,4}}_{s=3/2} , \underbrace{\psi_\alpha{}^{1,\dots,4}}_{s=1/2}$$

$$\ast \quad \mathcal{N}=8 \; : \quad \underbrace{e_\mu{}^a}_{s=2} , \underbrace{A_\mu{}^{1,\dots,28}}_{s=1} , \underbrace{\phi^{1,\dots,70}}_{\substack{s=0\\ \in\mathbb{R}}} \in \frac{E_{7(7)}}{SU(8)} \oplus \underbrace{\Psi_{\mu\alpha}{}^{1,\dots,8}}_{s=3/2} , \underbrace{\psi_\alpha{}^{1,\dots,56}}_{s=1/2}$$

V. $\mathcal{N}=1$ Supergravity with $\Lambda \neq 0$

Question : Is local supersymmetry compatible with a cosmological constant Λ ? What modifications are required ?

$$S_g = S_{E-H} + S_\Lambda = \frac{1}{2\kappa^2}\int d^4x \, e \left(e_a{}^\mu e_a{}^\nu R_{\mu\nu}{}^{ab} - \Lambda \right)$$

$$S_\Psi = S_{R-S} + S_{mass} = -\frac{1}{2\kappa^2}\int d^4x \, e \left(\bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho + m \, \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu \right)$$

NOTE : A mass term of the form $g^{\mu\nu} \bar{\Psi}_\mu \Psi_\nu$ does not describe the right number of d.o.f.

SUSY transformation rules :

$$(ii) \quad \boxed{\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \Psi_\mu \\ \delta \Psi_\mu &= D_\mu \epsilon - g \gamma_\mu \epsilon \end{aligned}} \implies \text{SUGRA} + \Lambda \; !!$$

Let's repeat the computation we performed in the case $\Lambda = m = g = 0$

The new pieces to compute are :

- $\dfrac{\delta S_{EH}}{\delta e_\mu{}^a}$ = same computation $= -\dfrac{1}{2\kappa^2}\displaystyle\int d^4x \, e \, G_{\mu\nu} \, \bar{\epsilon} \gamma^\mu \Psi_\nu$ ▲

- $\dfrac{\delta S_\Lambda}{\partial e_\mu{}^a} = \dfrac{-1}{2\kappa^2}\displaystyle\int d^4x \, \Lambda \, \delta(e) = \dfrac{-1}{2\kappa^2}\displaystyle\int d^4x \, e \Lambda e_a{}^\mu \delta e_\mu{}^a =$

  $= \dfrac{-\Lambda}{4\kappa^2}\displaystyle\int d^4x \, e \, \bar{\epsilon} \gamma^\mu \Psi_\mu$ ■

- $\dfrac{\delta S_{R-S}}{\delta e_\mu{}^a} = 0$ (same computation at lowest order in fermions)

• $$\frac{\delta S_{R-S}}{\delta \Psi_\mu} = -\frac{1}{2\kappa^2}\int d^4x\, e \left(-\nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\, \overbrace{\delta_\epsilon \Psi_\rho}^{\text{new transf (ii)}}\right)\times 2 = \text{same computation}$$

$$= \underbrace{\frac{1}{2\kappa^2}\int d^4x\, e\, G_{\mu\nu}\, \bar{\epsilon}\gamma^\mu \Psi^\nu}_{\text{Piece from } D_\rho \epsilon} - \underbrace{\frac{g}{\kappa^2}\int d^4x\, e\, \nabla_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\gamma_\rho \epsilon}_{A.3} = (*)$$

<u>A. 3</u> : $$-\frac{g}{\kappa^2}\int d^4x\, e\, \nabla_\nu \bar{\Psi}_\mu \underbrace{\gamma^{\mu\nu\rho}\gamma_\rho}_{2\gamma^{\mu\nu}} \epsilon = +\frac{2g}{\kappa^2}\int d^4x\, e\, \nabla_\mu \bar{\Psi}_\nu \gamma^{\mu\nu}\epsilon$$

$$\overset{t_2=-1}{=} -\frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu \Psi_\nu = -\frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\gamma^{\mu\nu} D_\mu \Psi_\nu$$

$$(*) = \frac{1}{2\kappa^2}\int d^4x\, e\, \underset{▲}{G_{\mu\nu}}\, \bar{\epsilon}\gamma^\mu \Psi^\nu - \frac{2g}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\underset{●}{\gamma^{\mu\nu}} D_\mu \Psi_\nu$$

• $$\frac{\delta S'_{mass}}{\delta e_a{}^\mu} = -\frac{m}{2\kappa^2}\int d^4x\, \delta\left[e\, e_a{}^\mu e_b{}^\nu \bar{\Psi}_\mu \gamma^{ab}\Psi_\nu\right] \Rightarrow \epsilon\Psi^3-\text{terms}$$

• $$\frac{\delta S_{mass}}{\delta \Psi_\nu} = -\frac{m}{2\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \gamma^{\mu\nu}\delta\Psi_\nu \times 2 =$$

$$= -\frac{m}{\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \gamma^{\mu\nu} D_\nu \epsilon + \frac{mg}{\kappa^2}\int d^4x\, e\, \bar{\Psi}_\mu \overbrace{\gamma^{\mu\nu}\gamma_\nu}^{3\gamma^\mu}\epsilon$$

$$\overset{t_1=t_2=-1}{=} +\frac{m}{\kappa^2}\int d^4x\, e\, \bar{\epsilon}\underset{●}{\gamma^{\mu\nu}} D_\mu \Psi_\nu - \frac{3mg}{\kappa^2}\int d^4x\, e\, \underset{■}{\bar{\epsilon}}\gamma^\mu \Psi_\mu$$

For the various terms to cancel one needs:

$m = 2g$, $\frac{\Lambda}{4} = -3mg \Rightarrow \Lambda = -12mg = -24g^2 < 0$

<u>IMPORTANT</u>: Invariance to all order in fermions is achieved by adding the torsion terms to $\delta\Psi_\mu$ as in the $\Lambda = 0$ case.

VI. Coupling $\mathcal{N}=1$ Supergravity to SYM and Matter fields

We are now going to couple the $\mathcal{N}=1$ supergravity multiplet to other multiplets of $\mathcal{N}=1$ supersymmetry: $\underbrace{\text{vector mult.}}_{\text{Super Yang-Mills (SYM)}}$ & $\underbrace{\text{chiral mult.}}_{\text{Matter}}$

MULTIPLET		-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2
gravity	$g_{\mu\nu}$	1								1
	Ψ_μ		1						1	
vector	A_μ			1				1		
chiral	ψ_α				1		1			
	(ϕ_1, ϕ_2)					11				

Table: $\mathcal{N}=1$ Multiplets, fields and helicity states

STRATEGY : Gravity as a conformal gauge theory !!

Our starting point is Eistein gravity: $S_g = \int d^4x \sqrt{-|g|}\, g^{\mu\nu} R_{\mu\nu}$

for which $\delta S_g = -\int d^4x \sqrt{-|g|}\, G^{\mu\nu} \delta g_{\mu\nu}$ with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

S_g is not invariant under local (gauge) scale transfos with parameter $\lambda_D(x)$

$$\delta_{\lambda_D} g_{\mu\nu} = -2 \lambda_D g_{\mu\nu}$$

as $\delta_{\lambda_D} S_g = +2 \lambda_D \int d^4x \sqrt{-|g|}\, G$ with $G = g^{\mu\nu} G_{\mu\nu} = R - \frac{1}{2} 4 R = -R$

Remarkably, it can be made scale inv. if coupled to a scalar ϕ :

$$\mathcal{S} = \int d^4x \sqrt{-|g|} \left[+ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \frac{1}{12} R \phi^2 \right]$$

$\hookrightarrow$ wrong kinetic term => Unphysical (non-dynamicall)

with $\delta\phi = \lambda_D \phi$

$$\delta \mathcal{S} = \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta \mathcal{S}}{\delta \phi} \delta\phi = 0$$

$$\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} = \delta \left(\int d^4x \sqrt{-|g|} \, \frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi \right) + \delta \left(\int d^4x \sqrt{-|g|} \, \frac{1}{12} R \phi^2 \right)$$

$$= \int d^4x \sqrt{-|g|} \left[\frac{1}{4} \partial_\rho \phi \, \partial^\rho \phi \, g^{\mu\nu} - \frac{1}{2} \partial^\mu \phi \, \partial^\nu \phi - \frac{1}{12} G^{\mu\nu} \phi^2 \right] \delta g_{\mu\nu}$$

$$= \lambda_D \int d^4x \sqrt{-|g|} \left[-2 \partial_\rho \phi \, \partial^\rho \phi + \partial^\rho \phi \, \partial_\rho \phi + \frac{1}{6} \overset{-R}{\overbrace{G}} \phi^2 \right]$$

$$= \lambda_D \int d^4x \sqrt{-|g|} \left[- \partial_\rho \phi \, \partial^\rho \phi - \frac{1}{6} R \phi^2 \right]$$

$$\frac{\delta \mathcal{S}}{\delta \phi} = \int d^4x \sqrt{-|g|} \left[2 \times \left(-\frac{1}{2}\right) \Box \phi + \frac{1}{6} R \phi \right] \delta\phi$$

$$= \lambda_D \int d^4x \sqrt{-|g|} \left[- \Box \phi \, \phi + \frac{1}{6} R \phi^2 \right]$$

$$= \lambda_D \int d^4x \sqrt{-|g|} \left[\partial_\rho \phi \, \partial^\rho \phi + \frac{1}{6} R \phi^2 \right] \qquad [\phi \equiv \text{compensator}]$$

Therefore the above action is not only Poincarè but scale (Weyl) invariant. We can use the local $\lambda_D(x)$ transformations to gauge-fix the compensating field ϕ as

$$\phi = \frac{\sqrt{6}}{\kappa} \implies S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-|g|} \, R \implies \text{GR !!}$$

IMPORTANT : Weyl invariance + gauge fixing compensator => Poincarè

Gauged (local) Weyl gravity
+
Scalar compensator

$$\xRightarrow[\text{gauge fixing}]{} \quad \text{General Relativity}$$

- Gravity can also be reformulated in a fully conformal covariant (not only Weyl) setup => Conformal SO(4,2) Gravity

Gauged (local) conformal gravity
+
Scalar compensator

$$\xRightarrow[\text{gauge fixing}]{\substack{R_{\mu\nu}(P)=0 \\ R_{\mu\nu}(M)=0}} \quad \text{General Relativity}$$

- Supergravity can be reformulated in a fully superconformal covariant setup => Superconformal SU(2,2|1) Gravity.

Gauged (local) superconformal gravity
+
Chiral multiplet compensator

$$\xRightarrow[\text{gauge fixing}]{\substack{R_{\mu\nu}(P)=0 \\ R_{\mu\nu}(M)=0 \\ R_{\mu\nu}(Q)=0}} \quad \text{Pure } \mathcal{N}=1 \text{ Supergravity}$$

Q : Why to bother with (super) conformal gravity + gauge fixings instead of constructing supergravity directly ?

A : Systematicity ≡ symmetry principle

Coleman - Mandula : Maximal space-time symmetry of an interacting field theory is conformal symmetry

Haag - Łopuszański - Sohnius : Supersymmetric extension of the conformal algebra ⇒ Superconformal algebra

- Therefore, the most efficient and systematic way of constructing matter-coupled supergravities is the "superconformal method". This method can also be combined with superspace techniques.

Gauged (local) superconformal gravity
+
Chiral multiplet compensator
+
Chiral multiplets
+
Vector multiplets
+
Real multiplet

$R_{\mu\nu}(P) = 0$
$R_{\mu\nu}(M) = 0$
$R_{\mu\nu}(Q) = 0$

$$\Longrightarrow$$

gauge fixing

$\mathcal{N}=1$ Supergravity
+
Chiral multiplets
+
Vector multiplets

✓

- Therefore, the most efficient and systematic way of constructing matter-coupled supergravities is the "superconformal method". This method can also be combined with superspace techniques.

The bosonic Lagrangian

We focus on the bosonic terms in the action S as they also fix all the fermionic couplings by demanding local supersymmetry:

$$S = \frac{1}{2\kappa^2}\int d^4x\, e\Big[R - \frac{1}{4}\overbrace{\mathrm{Re}\, f_{ab}(\phi)}^{\text{“}g^{-2}\delta_{ab}\text{-like”}} F_{\mu\nu}{}^a F^{\mu\nu b} - \frac{1}{4}\frac{1}{e}\overbrace{\mathrm{Im}\, f_{ab}(\phi)}^{\text{“}\frac{1}{8\pi^2}\Theta\,\delta_{ab}\text{-like”}}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^a F_{\rho\sigma}{}^b$$
$$- G_{i\bar{j}}(\phi,\bar{\phi})\,\nabla_\mu\phi^i\,\nabla^\mu\bar{\phi}^{\bar{j}} - V(\phi,\bar{\phi}) + \text{fermion terms.}$$

with $\phi^i \in \mathbb{C}$ and $A_\mu^a \in \mathbb{R}$ spanning an internal gauge symmetry G

- $a = 1, \dots, n_v$ (vector multiplets : $n_v = \dim(G)$)
- $i = 1, \dots, n_c$ (chiral multiplets)

We will consider matter charged under G :

$$\nabla_\mu\phi^i = \partial_\mu\phi^i - A_\mu{}^a \underbrace{\delta_a\phi^i}_{\text{linear sym} \Rightarrow \delta_a\phi^i = \overbrace{(t_a)^i{}_j}^{\text{generators of } R[G]}\phi^j} = \partial_\mu\phi^i - A_\mu{}^a k_a{}^i(\phi)$$

$$\nabla_\mu\bar{\phi}^{\bar{i}} = (\nabla_\mu\phi^i)^*$$

NOTE: Demanding gauge invariance of $G_{i\bar{j}}(\phi,\bar{\phi})$ under a a gauge transformation

$$\delta_\Lambda\phi^i = \Lambda^a(x)\,(t_a)^i{}_j\,\phi^j = \Lambda^a\, k_a{}^i(\phi)$$
$$\delta_\Lambda\bar{\phi}^{\bar{i}} = \Lambda^a(x)\,(t_a^*)^{\bar{i}}{}_{\bar{j}}\,\bar{\phi}^{\bar{j}} = \Lambda^a\, \bar{k}_a{}^{\bar{i}}(\bar{\phi})$$

requires :

$$\delta_\Lambda G_{i\bar{j}} = 0 \Rightarrow \begin{cases} \bar{k}_j{}^a(\phi,\bar\phi) \equiv G_{j\bar\imath}(\phi,\bar\phi)\,\bar{k}^{a\bar\imath}(\bar\phi) = -i\,\dfrac{\partial P^a(\phi,\bar\phi)}{\partial\phi^j} \\[2ex] k_{\bar{j}}{}^a(\phi,\bar\phi) \equiv G_{\bar{j}i}(\phi,\bar\phi)\,k^{ai}(\phi) = -i\,\dfrac{\partial P^a(\phi,\bar\phi)}{\partial\bar\phi^{\bar{j}}} \end{cases}$$

where the functions $P^a(\phi,\bar\phi)$ are called "moment maps" or "Killing prepotentials".

* Scalar geometry and $V(\phi,\bar\phi)$

$\mathcal{N}=1$ supersymmetry requires the space of fields ϕ^i to be a Kähler manifold, so that

$$G_{i\bar{j}} = \frac{\partial^2 K(\phi,\bar\phi)}{\partial\phi^i\,\partial\bar\phi^{\bar{j}}}$$

in terms of a "Kähler potential" $K(\phi,\bar\phi)\in\mathbb{R}$. In terms of the Kähler potential $K(\phi,\bar\phi)$ and its derivatives

$$K_i \equiv \frac{\partial K}{\partial\phi^i} \quad , \quad K_{\bar{j}} = \frac{\partial K}{\partial\bar\phi^{\bar{j}}}$$

the moment maps are expressed as

$$P^a = -\frac{i}{2}\left[k^{ai}\,\partial_i K - \bar{k}^{a\bar{j}}\,\partial_{\bar{j}} K\right] - r^a$$

with $r^a = \delta^{a\,U(1)}\,\underbrace{\xi_{FI}}$ (only for Abelian factors in G)

Fayet-Iliopoulos ≡ arbitrary parameters

Lastly the scalar potential $V(\phi,\bar\phi)$ is given by

$$V = e^{\kappa^2 K}\Big[\underbrace{D_i W\,\overbrace{G^{i\bar j}\, D_{\bar j}\overline{W}}^{e^{-\frac{1}{2}\kappa^2 K}\,F^i}}_{\text{F-terms}>0} - 3\kappa^2|W|^2\Big] + \underbrace{\tfrac{1}{2}\,P^a\,\overbrace{\mathrm{Re}\,f^{-1}_{ab}\,P^b}^{D^a}}_{\text{D-terms}>0}$$

in terms of an arbitrary holomorphic "Superpotential" $W(\phi)$ where the Kähler derivatives read:

$$D_i W = \partial_i W + \kappa^2\,(\partial_i K)\,W$$

$$D_{\bar i}\overline{W} = \partial_{\bar i}\overline{W} + \kappa^2\,(\partial_{\bar i} K)\,\overline{W}$$

Important: Theory defined by: $\underbrace{G}_{\text{gauge group + representation } R[G] \text{ for } \phi.} \oplus f_{ab}(\phi)\,,\ K(\phi,\bar\phi)\,,\ W(\phi) \oplus P^a(\phi,\bar\phi)$

* Local susy transformations & susy breaking.

The local susy transformations take the generic form

δ_ϵ Fermion $\sim \bar\epsilon$ Boson , δ_ϵ Boson $\sim \bar\epsilon$ Fermion

=> Lorentz invariance at the vacuum requires $\langle \text{Fermion} \rangle = 0$ and consequently $\langle \delta_\epsilon \text{ Boson} \rangle = 0$ always.

=> Lorentz invariance at the vacuum permits $\langle \text{Boson} \rangle \neq 0$ and consequently $\langle \delta_\epsilon \text{ Fermion} \rangle \begin{cases} = 0 \to \text{susy preserved} \\ \neq 0 \to \text{susy broken (spontaneously)} \end{cases}$

Let us look at the δ_ϵ Fermion :

Gravitino : $\delta_\epsilon \Psi_\mu \sim D_\mu \epsilon + i\, e^{\frac{1}{2}\kappa^2 K}\, W \;\bar\epsilon \gamma_\mu$

Chiralini : $\delta_\epsilon \psi^i \sim \bar\epsilon F^i$

Gaugini : $\delta_\epsilon \lambda^a \sim g\, \bar\epsilon D^a$

with

- $F^i = e^{\frac{1}{2}\kappa^2 K}\, G^{i\bar{j}} D_{\bar{j}} \bar{W}$ => F-term in $V(\phi, \bar\phi)$
- $D^a = \mathrm{Re}\, f^{-1}_{ab}\, P^b$ => D-term in $V(\phi, \bar\phi)$

As a result one has that

$$\text{SUSY} \left\{ \begin{array}{l} \delta_\epsilon \psi^i = 0 \\ \delta_\epsilon \lambda^a = 0 \end{array} \right. \Rightarrow \begin{array}{l} \langle F^i \rangle = 0 \\ \langle D^a \rangle = 0 \end{array} \Rightarrow V = -3\, e^{\kappa^2 K} \kappa^2 |W|^2 < 0$$

AdS vacuum !!

NOTE : If SUSY is broken the gravitino $\Psi_{\mu\alpha}$ gets a mass and acquires a longitudinal mode $\partial_\mu \eta_\alpha$ by eating up the goldstino associated to the direction of ~~SUSY~~.

Ex: F-term breaking => $\eta^\alpha F_i \psi^i$ and $m^2_{3/2} = \kappa^4 e^{\kappa^2 K} |W|^2$.

NOTE: The existence of a de Sitter vacuum requires susy to be broken => ϕ^i scalars relevant for cosmology !!

[late-time cosmic acceleration, inflation, ...]

* Global susy : Switching off Gravity [$\kappa^2 \to 0$]

In order to go from local susy (supergravity) to local susy (supersymmetric field theory) one switches off gravity by setting $\kappa^2 \to 0$. As a result :

- $g_{\mu\nu} \longrightarrow \eta_{\mu\nu}$ (Minkowski space-time)
- $\Psi_{\mu\alpha} \longrightarrow 0$ (no gauge field for a global symmetry)
- $F_i \longrightarrow F_i = \partial_i W$
- $D^a \longrightarrow D^a$ (gauge structure unaffected)

$$\Rightarrow \quad V = G^{i\bar{j}} \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{j}}} + \frac{1}{2} \operatorname{Re} f_{ab} D^a D^b \geq 0$$

Therefore $V > 0$ => Susy breaking !!

NOTE: Susy field theories are very interesting playgrounds where to discover universality classes of phenomena both classically and also at the quantum level.

VII. Prelude to superstrings and D=10,11 Supergravities

* From strings to $\mathcal{N}=2$, D=10 Supergravity

Particle evolution in D-dimensions

$$\bullet \approx> X^M \underbrace{(\tau)}_{\text{proper time}}$$

String evolution in D-dimensions

$l_s^2 \sim 2\alpha'$

$$\approx> X^M(\tau,\sigma)$$

$$+\text{susy} \Rightarrow \left.\begin{array}{l}\theta^1(\tau,\sigma)\\ \theta^2(\tau,\sigma)\end{array}\right\}\text{Grassman variables}$$

Set D = 10 and $\theta^{1,2}$ being M-W fermions

→ 2D conformal field theory : $X^M(\tau,\sigma)$, $\theta^{1,2}(\tau,\sigma)$

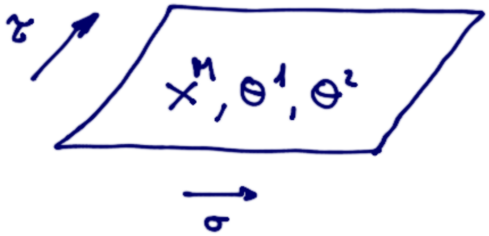
$$S_{2D} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \underbrace{\eta^{\alpha\beta}}\, \partial_\alpha X^M \partial_\beta X^M + \text{fermion terms}$$

with $\sigma^\alpha = (\tau,\sigma)$

$\eta^{\alpha\beta} = (-1,1)$

↳ gauge fixing : diff + Weyl in 2D

→ Mode expansion and states



$$\Rightarrow$$

$$X_M \supset \sum_n a_M^{(n)} e^{-2in(\tau-\sigma)} + \tilde{a}^{(n)} e^{-2in(\tau+\sigma)}$$

$$\theta^1 \supset \sum_n b^{(n)} e^{-2in(\tau-\sigma)} ; \theta^2 \supset \sum_n e^{-2in(\tau+\sigma)}$$

Promote a's, ã's, b's, $\tilde{b}$'s to operators with [,] or { , } relations :

$$|\text{state}\rangle = a^{\dagger}_{M}\, a^{\dagger}_{N}\, |0\rangle \Rightarrow \underbrace{G_{MN}} \oplus \underbrace{B_{MN}} \oplus \underbrace{\Phi}$$

$D=10$: metric antisym scalar

→ Mass of a state :

$$M^2 = \frac{1}{\ell_s^2}\Big[\underbrace{N(a,b) + \tilde{N}(\tilde{a},\tilde{b})}_{\text{ocupation numbers}}\Big] \Rightarrow \underset{M^2\to\infty}{\overset{\ell_s\to 0}{\bigcirc \to \bullet}} \Rightarrow \overbrace{\text{Keep only massless states !!}}^{\text{"low energy"}}$$

→ $\mathcal{N}=2$, $D=10$ massles spectrum : $\overbrace{G_{MN},\ B_{(2)},\ \Phi,\ \underbrace{C_{(p)}}}^{\text{Bosons}}\ ;\ \overbrace{\chi^{1,2}_{\alpha},\ \Psi^{1,2}_{M\alpha}}^{\text{Fermions}}$

($\text{ch}\,\Psi^1 \neq \text{ch}\,\Psi^2$) IIA : $p = 1, 3$

($\text{ch}\,\Psi^1 = \text{ch}\,\Psi^2$) IIB : $p = 0, 2, 4$

- Lagrangian : a candidate

$$\mathcal{L}_{10D} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \sqrt{G}\Big[R - \frac{1}{2}\partial_M\Phi\,\partial^M\Phi - \frac{1}{2\times 3!}e^{-\Phi}\underbrace{H_{MNP}}H^{MNP} + \ldots + \text{fermi}$$

$$H_{(3)} \equiv H_{MNP} = \partial_{[M}B_{NP]}$$

with $2\kappa_{10}^2 = \frac{1}{2\pi}(2\pi\ell_s)^8$

→ We can also study a probe string propagating in a background $\{ G_{MN}, B_{MN}, \Phi, C_{(p)} \}$ generated by other strings around:

$$S_{\substack{\text{probe}\\ \text{string}}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \Big[(\partial_\alpha X^M)(\partial^\alpha X^N)\, \underline{G_{MN}(X)} + \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N\, \underline{B_{MN}(X)} + \dots$$

G_{MN}, $B_{(2)}$, etc can be viewed as couplings in the 2D field theory !!

Conformal invariance $\Rightarrow$ $\underbrace{\beta_G^{MN} = \beta_B^{MN} = \dots = 0}$

At lowest order in $\frac{\sqrt{\alpha'}}{L}$ ~ system scale

$\Rightarrow$ E.O.M of an action !!

→ $\mathcal{N}=2$, $D=10$ Supergravity action:

$$\boxed{\begin{aligned} S_{\substack{\text{SUGRA}\\ \mathcal{N}=2,\, D=10}} &= \frac{1}{2K_{10}^2}\int d^{10}x \sqrt{G} \Big[R - \frac{1}{2}\partial^M\Phi\,\partial_M\Phi - \frac{1}{2\times 3!} e^{-\Phi} H_{MNP} H^{MNP} \\ &- \frac{1}{4K_{10}^2}\int d^{10}x\sqrt{G} \begin{cases} \text{IIA}: \frac{1}{2!} e^{3/2\,\Phi}\hat{F}_{MN}\hat{F}^{MN} + \frac{1}{4!} e^{1/2\,\Phi}\hat{F}_{M_1\dots M_4}\hat{F}^{M_1\dots M_4} \\ \text{IIB}: e^{2\Phi}\partial_M C_{(0)}\partial^M C_{(0)} + \frac{1}{3!} e^{\Phi}\hat{F}_{MNP}\hat{F}^{MNP} + \frac{1}{5!}\hat{F}_{M_1\dots M_5}\hat{F}^{M_1\dots M_5} \end{cases} \end{aligned}}$$

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x \begin{cases} \text{IIA: } \epsilon^{M_1\ldots M_{10}}\, B_{M_1M_2}\, F_{M_3\ldots M_6}\, F_{M_7\ldots M_{10}} \\ \text{IIB: } \epsilon^{M_1\ldots M_{10}}\, C_{M_1\ldots M_4}\, H_{M_5M_6M_7}\, F_{M_8M_9M_{10}} \end{cases}$$

$$+\ S_{\mathrm{Fermi}}(\chi^{1/2}, \Psi^{1/2})$$

where the gauge invariant field strengths are given by:

IIA: $\hat{F}_{(2)} = F_{(2)} = dC_{(1)}$

$$\hat{F}_{(4)} = \underbrace{F_{(4)}}_{dC_{(3)}.} + C_{(1)} \wedge H_{(3)}$$

IIB: $\hat{F}_{(3)} = \underbrace{F_{(3)}}_{dC_{(2)}} - H_{(3)} \wedge C_{(0)}$

$$\hat{F}_{(5)} = \underbrace{F_{(5)}}_{dC_{(4)}} + \frac{1}{2}\left[B_{(2)} \wedge F_{(3)} - C_{(2)} \wedge H_{(3)}\right]$$

=> Starting from closed superstrings we have obtained $\mathcal{N}=2$, $D=10$ Supergravities as the low-energy limit !!

=> Superstrings live in a ten-dimensional space-time ...

... so what about $10-4=6$ extra dimensions ?

→ The type IIA supergravity can be connected to the one and unique $\mathcal{N}=1$, $D=11$ Supergravity conjectured to be the low-energy limit of a mysterious theory of membranes called "M-theory"

$$\boxed{\begin{aligned} S_{\substack{\text{SUGRA}\\ \mathcal{N}=1,\, D=11}} = & \frac{1}{2K_{11}^2} \int d^{11}x \sqrt{G} \left[R - \frac{1}{2\times 4!} F_{M_1 \dots M_4} F^{M_1 \dots M_4} \right] \\ & - \frac{1}{12 K_{11}^2} \int d^{11}x \, \epsilon^{M_1 \dots\dots M_{11}} A_{M_1 M_2 M_3} F_{M_4 \dots M_7} F_{M_8 \dots M_{11}} \\ & + S_{\text{fermi}} (\Psi) \end{aligned}}$$

with $2K_{11}^2 = \frac{1}{2\pi} (2\pi \ell_p)^9$

↳ Planck scale

* The field content of the theory is $G_{MN} \oplus A_{MNP} \oplus \Psi_{M\alpha}$ with $F_{(4)} \equiv F_{M_1 \dots M_4} = \partial_{[M_1} A_{M_2 M_3 M_4]} \equiv dA_{(3)}$. It is invariant under local susy:

$$\delta_\epsilon e_M{}^A = \bar{\epsilon} \Gamma^A \Psi_M$$

$$\delta_\epsilon A_{MNP} = -3 \bar{\epsilon} \Gamma_{[MN} \Psi_{P]}$$

$$\delta_\epsilon \Psi_M = D_M \epsilon + \frac{1}{12} \left[\Gamma_M \frac{1}{4!} F_{QRST} \Gamma^{QRST} - 3 \frac{1}{3!} F_{MNPQ} \Gamma^{NPQ} \right] \epsilon$$

Important : Note that there is no coupling to be tuned !!

- M-theory $\Rightarrow$ IIA $\left\{ \begin{array}{l} G_{\hat{M}\hat{N}} \Rightarrow G_{MN} \oplus G_{M\,11} \equiv C_M \oplus G_{11\,11} \equiv \Phi \\ A_{\hat{M}\hat{N}\hat{P}} \Rightarrow A_{MNP} \equiv C_{MNP} \oplus A_{MN\,11} \equiv B_{MN} \end{array} \right.$

$\hat{M} = (M, 11)$

D=11 ↩ ↳ D=10